Nonlinear waves in coherently coupled Bose-Einstein condensates

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We consider a quasi-one-dimensional two-component Bose-Einstein condensate subject to a coherent coupling between its components, such as realized in spin-orbit coupled condensates. We study how nonlinearity modifies the dynamics of the elementary excitations. The spectrum has two branches, which are affected in different ways. The upper branch experiences a modulational instability, which is stabilized by a long-wave–short-wave resonance with the lower branch. The lower branch is stable. In the limit of weak nonlinearity and small dispersion it is described by a Korteweg–de Vries equation or by the Gardner equation, depending on the value of the parameters of the system.

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I. INTRODUCTION

The Bose-Einstein condensation of a mixture of different hyperfine states of the same element (first realized by the JILA group [1]) offers the possibility to transfer atoms from an internal state to another one in a macroscopic matter wave. This feature has driven a rich body of experimental studies of phenomena such as the formation of spin domains, vortices, and other nonlinear structures [2], internal Josephson effect [3], formation of squeezed and entangled states [4], motion of spin impurities [5], persistent currents [6], effective gauge potentials [7], and spin-orbit coupled systems [8], which has itself opened an avenue of new research: observation of a superfluid Hall effect [9], of Zitterbewegung [10], of spin Hall effect [11], of tunable Landau-Zener transitions [12], of a Dicke-type phase transition [13], of the softening of a rotonlike dispersion relation, etc. [14]. In some of the above cited works, the change of internal state is only due to spin-dependent collisions, but in others the coupling is externally driven by a combination of radio frequency and microwave fields [15] or by using Raman coupling lasers [16]. In the present study we concentrate on an effective spin $1/2$ system in which two internal states are coherently coupled by an external potential. In such a system, the coupling explicitly breaks the $U(1)\times U(1)$ symmetry originating from the irrelevance of global phase factors of each the two components: the relative phase is no longer free and only remains a $U(1)$ symmetry for the global phase of the spinor. As a consequence, the two-branched spectrum of the system has a single Goldstone mode and the other branch is gapped. The mean-field dynamics of the system is described by two coupled Gross-Pitaevskii equations accounting for intraspecies and interspecies collisions for the external coupling field and also possibly for a spin-orbit term. The ground state of the system and the associated possible phase transitions and the elementary excitations have been theoretically studied in Refs. [17] and [18], as well as a rich variety of nonlinear structures (Refs. [19] and [20]).

The reason for the protean aspect of the theoretical approaches of the system lies in the fact that its dynamics is described by a nonintegrable set of coupled Gross-Pitaevskii equations, which do not admit simple integrable equations as limiting cases. Even in the simpler case of a spinor condensate in the absence of spin-orbit and Raman coupling, the integrable limit is the so-called Manakov system (obtained when all the nonlinear interaction constants are equal), which does not pertain to the well-studied Ablowitz-Kaup-Newell-Segur hierarchy and for which all the types of solutions are not yet fully classified (see, e.g., Refs. [21]). The aim of the present work is to partially clarify the rich nonlinear behavior of the system by presenting a systematic study revealing how nonlinear effects modify the elementary excitations of the system.

The paper is organized as follows. The model, its ground state, and linear excitations are described in Sec. II. We then use a singular perturbation theory to describe in Sec. III how excitations in the upper branch of the dispersion relation are affected by nonlinear effects. The method is exposed in Secs. III A, III B, and III C and the results are summarized and discussed in Sec. III D. The technique used in Sec. III can also be employed for describing the effects of nonlinearity on the lower branch of the spectrum. However, for this branch another approach can be used, which is more appropriate in the long wave length limit. This is explained in Sec. IV and we show in Secs. IV A and IV B how to deal with this issue. The general doctrine is presented in Sec. IV C where we also discuss the different regimes accessible in present-day experiments. Our conclusions are summarized in Sec. V and some technical aspects are detailed in Appendixes A and B.

II. MODEL AND ELEMENTARY EXCITATIONS

We consider a one-dimensional (1D) system described by a two-component spinor order parameter $\Psi(x,t)=(\psi_\uparrow,\psi_\downarrow)^t$ (where the superscript ^{t} denotes the transposition) obeying the following coupled Gross-Pitaevskii equations

$$i\,\hbar\,\partial_t\Psi = H_0\,\Psi + \begin{pmatrix}\alpha_1|\psi_\uparrow|^2 & \alpha_2\psi_\downarrow^*\psi_\uparrow\\ \alpha_2\psi_\uparrow^*\psi_\downarrow & \alpha_1|\psi_\downarrow|^2\end{pmatrix}\Psi, \tag{1}$$

where H_0 is the single-particle Hamiltonian:

$$H_0=\frac{1}{2m}\left(\frac{\hbar}{i}\,\partial_x-\hbar\,k_0\,\sigma_z\right)^2+\frac{\hbar\Omega}{2}\,\sigma_x, \tag{2}$$

σ_x and σ_z being Pauli matrices. This corresponds to a system with equal contribution of Rashba and Dresselhaus coupling, as realized in spin-orbit coupled condensates (see, e.g., Ref. [22]). In Eq. (1), $\alpha_2 = \alpha_{\uparrow\downarrow}$ is the interspecies interaction coefficient, and for simplicity we have assumed equal intraspecies interaction: $\alpha_{\uparrow\uparrow} = \alpha_{\downarrow\downarrow} \equiv \alpha_1$. In the following we will consider the case of repulsive intraspecies interaction: $\alpha_1 > 0$.

It is convenient to reparametrize the spinor wave function [23] :

$$\Psi(x,t) = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \sqrt{\rho}\, e^{i\Phi/2} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}. \tag{3}$$

Here $\rho(x,t) = |\psi_\uparrow|^2 + |\psi_\downarrow|^2$ denotes the total density of the condensate and $\Phi(x,t)$ has the meaning of the velocity potential of its in-phase motion; the angle $\theta(x,t)$ is the variable describing the relative density of the two components [$\cos\theta = (|\psi_\uparrow|^2 - |\psi_\downarrow|^2)/\rho$] and the phase $\varphi(x,t)$ is the potential of their relative (counterphase) motion. Accordingly, the densities of the components of the condensate are given by

$$\begin{aligned} \rho_\uparrow(x,t) &= |\psi_\uparrow|^2 = \rho\cos^2(\theta/2), \\ \rho_\downarrow(x,t) &= |\psi_\downarrow|^2 = \rho\sin^2(\theta/2). \end{aligned} \tag{4}$$

Their velocities are defined as

$$\begin{aligned} v_\uparrow(x,t) &= \tfrac{1}{2}(\Phi_x - \varphi_x) - k_0, \\ v_\downarrow(x,t) &= \tfrac{1}{2}(\Phi_x + \varphi_x) + k_0. \end{aligned} \tag{5}$$

It will also be convenient to define the following velocity fields

$$U(x,t) = \Phi_x \quad \text{and} \quad v(x,t) = \varphi_x. \tag{6}$$

Equation (1), expressed in terms of the real fields Φ, ρ, θ, and φ, reads

$$\rho_t = \frac{1}{2}[\rho\,(\varphi_x + 2k_0)\cos\theta]_x - \frac{1}{2}(\rho\,\Phi_x)_x, \tag{7a}$$

$$\begin{aligned} -\Phi_t = &-\frac{1}{2}\cot\theta\frac{(\rho\,\theta_x)_x}{\rho} + \frac{1}{2}\left(\frac{\rho_x^2}{2\rho^2} - \frac{\rho_{xx}}{\rho}\right) \\ &+ \frac{1}{4}\left[\Phi_x^2 + \theta_x^2 + (\varphi_x + 2k_0)^2\right] \\ &+ (\alpha_1 + \alpha_2)\rho + \Omega\,\frac{\cos\varphi}{\sin\theta}, \end{aligned} \tag{7b}$$

$$-\theta_t = \frac{1}{2}\Phi_x\theta_x + \frac{1}{2\rho}[\rho(\varphi_x + 2k_0)\sin\theta]_x + \Omega\,\sin\varphi, \tag{7c}$$

$$\begin{aligned} \varphi_t = &\frac{1}{2\sin\theta}\frac{(\rho\,\theta_x)_x}{\rho} - \frac{1}{2}\Phi_x(\varphi_x + 2k_0) \\ &+ (\alpha_1 - \alpha_2)\rho\cos\theta - \Omega\,\cos\varphi\cot\theta, \end{aligned} \tag{7d}$$

where we have used units such that $\hbar = 1 = m$.

In all the following we will assume that the different parameters of the Hamiltonian are fixed in such a way that the ground state of the system corresponds to a configuration in which both components are homogeneous ($\rho = \rho_0$ and $\theta = \theta_0$), in phase ($\varphi = 0$), stationary (ρ, Φ_x, θ, and φ_x are time independent) with equal densities ($\theta_0 = -\pi/2$ [24]). In this case, one obtains $\Phi = -2\,\mu\,t$, where

$$\mu = \frac{k_0^2}{2} + \frac{g_1}{2} - \frac{\Omega}{2} \tag{8}$$

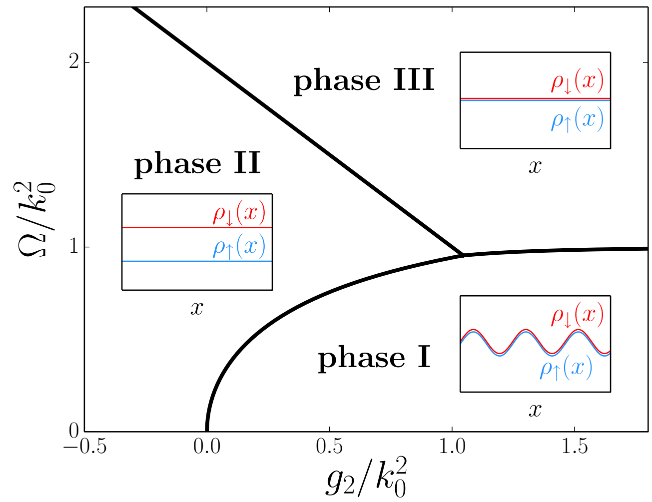

FIG. 1. Schematic phase space of the spin-orbit coupled system as a function of the parameters Ω/k_0^2 and g_2/k_0^2. For each phase the inset represents a typical density pattern. The boundary between phases III and II corresponds to $\Omega + g_2 = 2\,k_0^2$.

is the chemical potential. In this expression we have used the notation $g_1 = (\alpha_1 + \alpha_2)\rho_0$. It will also be convenient to define $g_2 = (\alpha_1 - \alpha_2)\rho_0$ and to introduce a rescaled density $n(x,t) = \rho(x,t)/\rho_0$.

In the absence of spin-orbit coupling ($k_0 = 0$) this ground state is stable provided $\Omega + g_2 > 0$ [17]. For a spin-orbit coupled system, this ground state is denoted as the "single minimum" or "zero momentum" or "phase III" ground state. It is the true ground state of the system in a region of parameters, which is schematically depicted in Fig. 1 (adapted from Ref. [22]).

Although the present work is devoted to the study of nonlinear effects in phase III, we note that the methods we use also apply—with unessential modifications—in phase II, which is a spin polarized phase where the system condensates in a single plane-wave state with nonzero momentum. Phase I (the so-called striped phase), which has a modulated ground-state density deserves a special treatment.

A first insight in the dynamics of the system can be obtained by linearizing Eqs. (7). For simplifying the notations we introduce the column vector

$$\Xi(x,t) = \begin{pmatrix} n \\ \Phi \\ \theta \\ \varphi \end{pmatrix}, \quad \text{with} \quad \Xi^{(0)}(t) = \begin{pmatrix} 1 \\ -2\mu t \\ -\pi/2 \\ 0 \end{pmatrix} \tag{9}$$

being the ground state value of $\Xi(x,t)$. We write

$$\Xi(x,t) = \Xi^{(0)}(x,t) + \Xi'(x,t), \tag{10}$$

where $\Xi'(x,t)$ describes a small departure of the fields n, Φ, θ, and φ from their ground-state values. Inserting this *Ansatz* into (7) one obtains at first order in Ξ' a system of the form

$$\mathbb{M}(\partial_x,\partial_t)\,\Xi' = 0, \tag{11}$$

where

$$\mathbb{M} = \begin{pmatrix} \partial_t & \frac{\partial_x^2}{2} & -k_0\partial_x & 0 \\ -\frac{\partial_x^2}{2}+g_1 & \partial_t & 0 & k_0\partial_x \\ -k_0\partial_x & 0 & \partial_t & -\frac{\partial_x^2}{2}+\Omega \\ 0 & k_0\partial_x & \frac{\partial_x^2}{2}-\Omega-g_2 & \partial_t \end{pmatrix}. \tag{12}$$

Equation (11) being linear one can expand $\Xi'(x,t)$ on a basis of plane waves of wave-vector k and angular frequency ω. This amounts to look for solutions of the form $\Xi'(x,t) = \hat{\Xi}' \exp[i(kx-\omega t)] + \text{c.c.}$, where c.c. stands for complex conjugate and $\hat{\Xi}'$ is a constant column vector whose entries are possibly complex. One then obtains a system of linear equations, which reads

$$\mathbb{M}_1\,\hat{\Xi}' = 0, \tag{13}$$

where

$$\mathbb{M}_1 = \mathbb{M}(ik,-i\omega) = \begin{pmatrix} -i\omega & -\frac{k^2}{2} & -ik_0k & 0 \\ \frac{k^2}{2}+g_1 & -i\omega & 0 & ik_0k \\ -ik_0k & 0 & -i\omega & \frac{k^2}{2}+\Omega \\ 0 & ik_0k & -\frac{k^2}{2}-\Omega-g_2 & -i\omega \end{pmatrix}. \tag{14}$$

The system (13) has nontrivial solutions only if the determinant of $\mathbb{M}_1$ vanishes. This fixes the dispersion relation of the elementary excitations, with two branches $\omega = \omega_\pm(k)$, which are represented in Fig. 2. They are solutions of

$$0 = \omega^4 - \omega^2\left[\frac{k^4}{2} + 2k_0^2k^2 + \Omega k^2 + \Omega^2 + (g_1+g_2)\frac{k^2}{2} + \Omega g_2\right] + \frac{k^2}{2}\left[\frac{k^2}{2}+\Omega+g_2-2k_0^2\right] \times \left[\left(\frac{k^2}{2}+\Omega\right)\left(\frac{k^2}{2}+g_1\right) - k^2k_0^2\right]. \tag{15}$$

The upper branch $\omega = \omega_+(k)$ is gapped, with a dispersion relation of the form

$$\omega_+(k) = \sqrt{\Omega(\Omega+g_2)} + O(k^2). \tag{16}$$

The lower branch $\omega = \omega_-(k)$ is not gapped: it accounts for the Goldstone mode corresponding to the spontaneous breaking of the global $U(1)$ symmetry of the system. One sees in Fig. 2 that the upper branch is not qualitatively affected by interaction effects, contrarily to the lower branch whose long wavelength dispersion relation would be quadratic in the absence of interaction and becomes linear in its presence. The lower branch admits, for the positive k portion of the spectrum, the following expansion (corresponding to linear waves propagating in the positive-x direction):

$$\omega_-(k) = c\,k + c_3\,k^3 + O(k^5), \tag{17}$$

where

$$c = \sqrt{\frac{g_1}{2}\left(1 - \frac{2k_0^2}{\Omega+g_2}\right)} \tag{18}$$

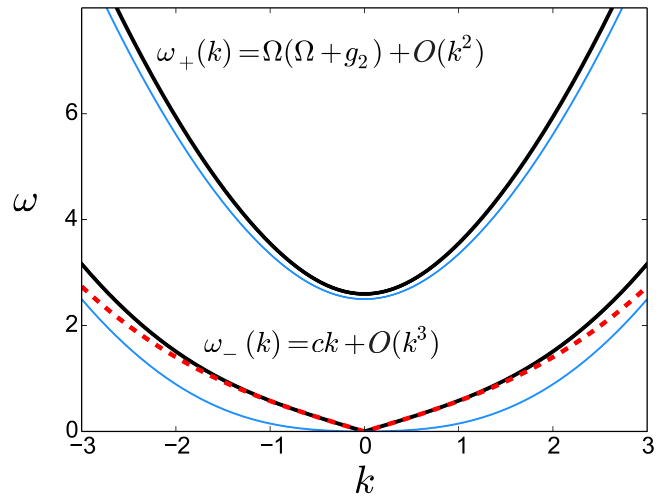

FIG. 2. The black solid lines represent the exact dispersion relations $\omega_+(k)$ (upper branch) and $\omega_-(k)$ (lower branch), solutions of Eq. (15). The figure is drawn in the case $\alpha_1\rho_0 = 1.2$, $\alpha_2\rho_0 = 1.0$, $k_0 = 1.0$, and $\Omega = 2.5$. The (red) dashed line represent the long wavelength expansion (17). The thin (blue) lines represent the spectrum of the single particle Hamiltonian H_0 [cf. Eq. (2)]. They are obtained by taking $g_1 = g_2 = 0$ in Eq. (15): in this case one obtains $\omega_\pm(k) = k^2/2 + \Omega/2 \pm [k_0^2k^2 + \Omega^2/4]^{1/2}$.

is the sound velocity, and the parameter c_3 verifies

$$4cc_3 = \frac{1}{2} - \frac{2k_0^2}{\Omega(\Omega+g_2)}\left[2\Omega+g_1+g_2 - \frac{(\Omega+g_1+g_2)(2\Omega-g_1+g_2)}{2(\Omega+g_2)} - k_0^2\frac{(\Omega+g_1+g_2)^2}{(\Omega+g_2)^2}\right]. \tag{19}$$

III. NONLINEAR PERTURBATION THEORY FOR EXCITATIONS PROPAGATING IN THE UPPER BRANCH

We study in the present section how nonlinear effects modify the structure of an excitation propagating in the upper branch of the spectrum. For instance, one can anticipate that nonlinear terms cause some modulations or anharmonicities of this wave, and make it interact with the other branch of the spectrum. Instead of the simple linear analysis of Sec. II [Eqs. (9), (10) and following], we perform here a singular perturbative expansion by writing the term $\Xi'(x,t)$ in Eq. (10) under the form (see, e.g., Refs. [25–29]):

$$\Xi'(x,t) = \sum_{n\geqslant 1} \epsilon^n\,\Xi^{(n)}(x,t,X,T_1,T_2). \tag{20}$$

In this expansion ϵ is a small parameter.

$$X = \epsilon x \quad \text{and} \quad T_2 = \epsilon T_1 = \epsilon^2 t, \tag{21}$$

are multiscale coordinates aiming at describing the slow spatial and temporal modulations of a wave packet of finite amplitude.

$\Xi^{(0)}$ in (20) is the same as in (9) and we make the following *Ansatz* for the form of the $O(\epsilon)$ term:

$$\Xi^{(1)}(x,t,X,T_1,T_2) = \overline{\Xi}_0^{(1)}(X,T_1,T_2) + \big(\widetilde{\Xi}_1^{(1)}(X,T_1,T_2)e^{i\beta(x,t)} + \text{c. c.}\big), \tag{22}$$

where

$$\beta(x,t) = kx - \omega t. \tag{23}$$

This means that we assume that the $O(\epsilon)$ solution of (7) consists in a slowly varying contribution ($\overline{\Xi}_0^{(1)}$, [30]) plus an oscillating term with a smoothly varying amplitude $\widetilde{\Xi}_1^{(1)}$. We will see below that the nonoscillating contribution $\overline{\Xi}_0^{(1)}$ is necessary for the consistency of the approach, meaning that nonlinearity not only modifies the shape of a finite amplitude wave but also affects the background on top of which the wave propagates.

We enforce a behavior of type (22) only at order ϵ; it then will be automatically verified at higher orders, with also possible contributions from higher harmonics: see (34) for the form of the $O(\epsilon^2)$ solution.

The multiscale analysis consists in considering that the time variables t, T_1, and T_2 (and also the spatial coordinates x and X) are independent. One thus writes

$$\partial_x = k\,\partial_\beta + \epsilon\partial_X,$$
$$\text{and}\quad \partial_t = -\omega\,\partial_\beta + \epsilon\partial_{T_1} + \epsilon^2\partial_{T_2}. \tag{24}$$

The method applies for any value of k_0, provided one remains in phase III, but the general expressions are quite cumbersome: For legibility we present the computation in the simpler case $k_0 = 0$.

A. Order ϵ

At this order, Eq. (7) reads [as already obtained in Eq. (11)]

$$\mathbb{M}(k\partial_\beta, -\omega\partial_\beta)\,\Xi^{(1)} = 0, \tag{25}$$

where $\mathbb{M}$ is defined in (12). Using the matrix $\mathbb{M}_1$ of Eq. (14) and defining $\mathbb{M}_0$ by

$$\mathbb{M}_0 = \mathbb{M}(0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & -\Omega - g_2 & 0 \end{pmatrix}, \tag{26}$$

one can rewrite Eq. (25) as

$$\mathbb{M}_0\,\overline{\Xi}_0^{(1)}(X,T_1,T_2) = 0,$$
$$\text{and}\quad \mathbb{M}_1\,\widetilde{\Xi}_1^{(1)}(X,T_1,T_2) = 0. \tag{27}$$

For (27) to have nontrivial solutions we need to impose $\det\mathbb{M}_1 = 0$ (we already have $\det\mathbb{M}_0 = 0$). As was seen in Sec. II, this determines the dispersion relation. We study here a wave propagating in the upper branch of the spectrum, that is, in the expression (23) for $\beta(x,t)$, one has $\omega = \omega_+(k)$. We then obtain for the solutions $\overline{\Xi}_0^{(1)}$ and $\widetilde{\Xi}_1^{(1)}$ of Eqs. (27) expressions of the form:

$$\overline{\Xi}_0^{(1)}(X,T_1,T_2) = \begin{pmatrix} \overline{n}^{(1)} \\ \overline{\Phi}^{(1)} \\ \overline{\theta}^{(1)} \\ \overline{\varphi}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \overline{\Phi}^{(1)} \equiv \overline{R}_0\,\overline{\Phi}^{(1)}(X,T_1,T_2), \tag{28}$$

and

$$\widetilde{\Xi}_1^{(1)}(X,T_1,T_2) = \begin{pmatrix} \widetilde{n}^{(1)} \\ \widetilde{\Phi}^{(1)} \\ \widetilde{\theta}^{(1)} \\ \widetilde{\varphi}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\,\Delta \end{pmatrix} \widetilde{\theta}^{(1)} \equiv \widetilde{R}_1\,\widetilde{\theta}^{(1)}(X,T_1,T_2), \tag{29}$$

where $\Delta = (1 + \frac{2g_2}{k^2+2\Omega})^{1/2}$. At this point $\overline{\Phi}^{(1)}(X,T_1,T_2)$ and $\widetilde{\theta}^{(1)}(X,T_1,T_2)$ in expressions (28) and (29) are still unknown, but we already collected some useful pieces of information on the form of the wave: we see that $\overline{n}^{(1)} = \overline{\theta}^{(1)} = \overline{\varphi}^{(1)} = \widetilde{n}^{(1)} = \widetilde{\Phi}^{(1)} = 0$ and that $\widetilde{\varphi}^{(1)}$ is proportional to $\widetilde{\theta}^{(1)}$. In the case $k_0 \neq 0$, $\overline{n}^{(1)}$ and $\widetilde{\Phi}^{(1)}$ are nonzero, but both are proportional to $\widetilde{\theta}^{(1)}$, as well as $\widetilde{\varphi}^{(1)}$.

B. Order ϵ^2

At this order Eq. (7) reads

$$\begin{aligned} \mathbb{M}(k\partial_\beta, -\omega\partial_\beta)[\Xi^{(2)}(x,t,X,T_1,T_2)] &= \overline{C}_0(X,T_1,T_2) + [\widetilde{C}_1(X,T_1,T_2)e^{i\beta(x,t)} + \text{c. c.}] \\ &\quad + [\widetilde{C}_2(X,T_1,T_2)e^{2i\beta(x,t)} + \text{c. c.}]. \end{aligned} \tag{30}$$

with

$$\overline{C}_0(X,T_1,T_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \partial_{T_1}\overline{\Phi}^{(1)} + \begin{pmatrix} 0 \\ g_2 \\ 0 \\ 0 \end{pmatrix} |\widetilde{\theta}^{(1)}|^2, \tag{31}$$

$$\widetilde{C}_1(X,T_1,T_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\,\Delta \end{pmatrix} \partial_{T_1}\widetilde{\theta}^{(1)} + \begin{pmatrix} 0 \\ 0 \\ \Delta\,k \\ i\,k \end{pmatrix} \partial_X\widetilde{\theta}^{(1)}, \tag{32}$$

and

$$\widetilde{C}_2(X,T_1,T_2) = \begin{pmatrix} i\,\Delta\,k^2 \\ \frac{1}{2}\big(\frac{k^2-2\Omega}{k^2+2\Omega}\,g_2 - k^2 - 2\,\Omega\big) \\ 0 \\ 0 \end{pmatrix} (\widetilde{\theta}^{(1)})^2. \tag{33}$$

In expressions (32) and (33) we have used the same notation Δ as in (29). The precise expressions (31), (32), and (33) for $\overline{C}_0$, $\widetilde{C}_1$, and $\widetilde{C}_2$ result from the formulas (28) and (29) for $\overline{\Xi}_0^{(1)}$ and $\widetilde{\Xi}_1^{(1)}$.

Since the operator $\mathbb{M}(\partial_x,\partial_t)$ is linear, the solution of equation (30) consists of three contributions, one for each of the source terms. Hence $\Xi^{(2)}$ is of the form

$$\Xi^{(2)} = \overline{\Xi}_0^{(2)}(X,T_1,T_2) + \big[\widetilde{\Xi}_1^{(2)}(X,T_1,T_2)e^{i\beta} + \text{c. c.}\big] + \big[\widetilde{\Xi}_2^{(2)}(X,T_1,T_2)e^{2i\beta} + \text{c. c.}\big], \tag{34}$$

the different components being solutions of

$$\mathbb{M}_0\ \overline{\Xi}_0^{(2)}(X,T_1,T_2) = \overline{C}_0(X,T_1,T_2), \tag{35}$$

$$\mathbb{M}_1\ \widetilde{\Xi}_1^{(2)}(X,T_1,T_2) = \widetilde{C}_1(X,T_1,T_2), \tag{36}$$

and

$$\mathbb{M}_2\ \widetilde{\Xi}_2^{(2)}(X,T_1,T_2) = \widetilde{C}_2(X,T_1,T_2); \tag{37}$$

where $\mathbb{M}_2$ is defined similarly to $\mathbb{M}_1$ in Eq. (14) and $\mathbb{M}_0$ in Eq. (26):

$$\mathbb{M}_2 = \mathbb{M}(2ik, -2i\omega_+(k)). \tag{38}$$

Equation (37) is easily solved because $\det \mathbb{M}_2 \neq 0$ [31]. We do not write its solution explicitly, but it is necessary for next order in ϵ: it contributes to the right-hand side of Eq. (44), in particular to the expression (46) for the coefficient $\widetilde{D}_1$.

Solving Eqs. (35) and (36) is more complicated than solving Eq. (37) because $\det \mathbb{M}_0 = 0$ and $\det \mathbb{M}_1 = 0$. Hence, if $\overline{C}_0$ is not in the image space of $\mathbb{M}_0$ (or if $\widetilde{C}_1$ is not in the image space of $\mathbb{M}_1$) one cannot find a solution. One must thus impose that $\overline{C}_0$ is in the image space of $\mathbb{M}_0$ and that $\widetilde{C}_1$ is in the image space of $\mathbb{M}_1$. This can be done conveniently through the following technique. Let us define $\overline{L}_0$ and $\widetilde{L}_1$ such that

$$\begin{aligned} &\mathbb{M}_0^t\,\overline{L}_0 = 0 \quad \Rightarrow \quad \overline{L}_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\\ &\mathbb{M}_1^t\,\widetilde{L}_1 = 0 \quad \Rightarrow \quad \widetilde{L}_1 = \begin{pmatrix}0\\0\\1\\-i/\Delta\end{pmatrix}. \end{aligned} \tag{39}$$

Multiplying (35) by the transposed row vector $\overline{L}_0^t$ and (36) by $\widetilde{L}_1^t$ one obtains [32]

$$\overline{L}_0^t \cdot \overline{C}_0(X,T_1,T_2) = 0, \quad \text{and} \quad \widetilde{L}_1^t \cdot \widetilde{C}_1(X,T_1,T_2) = 0. \tag{40}$$

The first of these equations is trivially satisfied. The second imposes that

$$\partial_{T_1}\widetilde{\theta}^{(1)} + \omega_+'(k)\partial_X\widetilde{\theta}^{(1)} = 0, \tag{41}$$

which implies the important physical result that the envelope of the wave packet propagates with the group velocity $\omega_+'(k) = d\omega_+/dk$.

Once Eq. (41) is satisfied, the compatibility condition (40) is fulfilled and one can solve Eqs. (35) and (36). One obtains

$$\overline{\Xi}_0^{(2)}(X,T_1,T_2) = \begin{pmatrix} \frac{1}{g_1}\partial_{T_1}\overline{\Phi}^{(1)} + \frac{g_2}{g_1}|\widetilde{\theta}^{(1)}|^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{42}$$

and

$$\widetilde{\Xi}_1^{(2)}(X,T_1,T_2) = \begin{pmatrix} 0 \\ 0 \\ \frac{i\,k\,g_2}{\omega_+^2(k)}\partial_X\widetilde{\theta}^{(1)} \\ 0 \end{pmatrix}. \tag{43}$$

C. Order ϵ^3

At this order one obtains an equation whose form is quite similar to that of Eq. (30) with additional harmonics:

$$\begin{aligned} &\mathbb{M}(k\partial_\beta, -\omega\partial_\beta)[\Xi^{(3)}(x,t,X,T_1,T_2)]\\ &\quad = \overline{D}_0(X,T_1,T_2) + [\widetilde{D}_1(X,T_1,T_2)e^{i\beta(x,t)} + \text{c. c.}]\\ &\quad + [\widetilde{D}_2(X,T_1,T_2)e^{2i\beta(x,t)} + \text{c. c.}]\\ &\quad + [\widetilde{D}_3(X,T_1,T_2)e^{3i\beta(x,t)} + \text{c. c.}]. \end{aligned} \tag{44}$$

We need not write the expressions for $\widetilde{D}_2$ and $\widetilde{D}_3$ because they are not necessary to determine the dynamic of $\widetilde{\theta}^{(1)}$. The terms $\overline{D}_0$ and $\widetilde{D}_1$ read (remember that for legibility we give the explicit expressions only in the case $k_0 = 0$)

$$\begin{aligned} \overline{D}_0 = &\begin{pmatrix} \frac{k(g_1(k^2+2\,\Omega+2\,g_2)+g_2(k^2+2\,\Omega+g_2))}{2\,g_1\,\omega_+(k)} \\ \frac{i\,g_2^2\,k}{2\,\omega_+^2(k)} \\ 0 \\ 0 \end{pmatrix}\partial_X|\widetilde{\theta}^{(1)}|^2\\ &+ \begin{pmatrix} \frac{1}{2}\,\partial_X^2\overline{\Phi}^{(1)} - \frac{1}{g_1}\partial_{T_1}^2\overline{\Phi}^{(1)} \\ \partial_{T_2}\overline{\Phi}^{(1)} \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{45}$$

and

$$\begin{aligned} \widetilde{D}_1 = &\begin{pmatrix}0\\0\\1\\i\,\Delta\end{pmatrix}\partial_{T_2}\widetilde{\theta}^{(1)} + \begin{pmatrix}0\\0\\i\,P(k)\\Q(k)\end{pmatrix}|\widetilde{\theta}^{(1)}|^2\widetilde{\theta}^{(1)}\\ &+ \begin{pmatrix}0\\0\\-i\,\frac{8\,g_2\,\Omega(g_2+k^2+2\,\Omega)+(k^2+2\,\Omega)^3}{16\,\omega_+^3(k)}\\ \frac{1}{2} + \frac{k^2 g_2}{2\,\omega_+^2(k)}\end{pmatrix}\partial_X^2\widetilde{\theta}^{(1)}\\ &+ \begin{pmatrix}0\\0\\\frac{i\,k}{2}\\-\frac{\Delta\,k}{2}\end{pmatrix}\widetilde{\theta}^{(1)}\partial_X\overline{\Phi}^{(1)} + \begin{pmatrix}0\\0\\0\\\frac{g_2}{g_1}\end{pmatrix}\widetilde{\theta}^{(1)}\partial_{T_1}\overline{\Phi}^{(1)}. \end{aligned} \tag{46}$$

In the above expression for $\widetilde{D}_1$ the quantities $P(k)$ and $Q(k)$ are defined as

$$\begin{aligned} P(k) = &-\frac{\sqrt{2g_2+k^2+2\Omega}}{4(k^2+2\Omega)^{3/2}F(k)}\\ &\times [(k^2+2\Omega)4g_2(-2g_2\Omega+k^4-4k^2\Omega-4\Omega^2)\\ &+ (k^2+2\Omega)^2(3k^2+2\Omega)(k^2-2\Omega)\\ &+ 2g_1k^2(4g_2\Omega+(k^2+2\Omega)^2)], \end{aligned} \tag{47}$$

and

$$\begin{aligned} Q(k) = &\frac{1}{4g_1(k^2+2\Omega)F(k)}\{-8g_2^3(k^2+2\Omega)^2\\ &- g_1(k^2+2\Omega)^2[2g_1k^2+(3k^2+2\Omega)(k^2-2\Omega)]\\ &+ 4g_2^2g_1(5k^4-2k^2\Omega-8\Omega^2) \end{aligned}$$

$$+ 4g_2^2(k^2+2\Omega)(3k^2+2\Omega)(k^2-2\Omega) + 2g_1^2 g_2(6k^4+8k^2\Omega) + 2g_1g_2(k^2+2\Omega)(7k^4-4k^2\Omega-4\Omega^2)\big\}, \quad (48)$$

where

$$F(k) = -2g_2(k^2+2\Omega) + 2g_1k^2 + (3k^2+2\Omega)(k^2-2\Omega). \quad (49)$$

Following the same method as in Sec. III B, we write

$$\Xi^{(3)} = \overline{\Xi}_0^{(3)}(X,T_1,T_2) + \big[\widetilde{\Xi}_1^{(3)}(X,T_1,T_2)e^{i\beta(x,t)} + \text{c. c.}\big] + \big[\widetilde{\Xi}_2^{(3)}(X,T_1,T_2)e^{2i\beta(x,t)} + \text{c. c.}\big] + \big[\widetilde{\Xi}_3^{(3)}(X,T_1,T_2)e^{3i\beta(x,t)} + \text{c. c.}\big]. \quad (50)$$

We will not need to consider the contribution of the second and third harmonics in (44) and (50). However, the contributions of the first harmonic [$\widetilde{D}_1(X,T_1,T_2)$ and $\widetilde{\Xi}_1^{(3)}$] and of the zero harmonic [$\overline{D}_0(X,T_1,T_2)$ and $\overline{\Xi}_0^{(3)}$] are important. Reinserting expression (50) in (44) yields:

$$\mathbb{M}_0\, \overline{\Xi}_0^{(3)}(X,T_1,T_2) = \overline{D}_0(X,T_1,T_2), \quad (51)$$

and

$$\mathbb{M}_1\, \widetilde{\Xi}_1^{(3)}(X,T_1,T_2) = \widetilde{D}_1(X,T_1,T_2). \quad (52)$$

Again, for solving Eq. (51) one must make sure that $\overline{D}_0$ is in the image space of $\mathbb{M}_0$: this yields

$$\overline{L}_0^{\,t} \cdot \overline{D}_0(X,T_1,T_2) = 0, \quad (53)$$

which writes

$$\partial_{T_1}^2 \overline{\Phi}^{(1)} - c^2\,\partial_X^2 \overline{\Phi}^{(1)} = S(k)\,\partial_X|\widetilde{\theta}^{(1)}|^2, \quad (54)$$

where c is the speed of sound [cf. Eq. (18)] and

$$S(k) = \frac{(g_1+g_2)(k^2+2\,\Omega+g_2)+g_1g_2}{2\,\omega_+(k)/k}. \quad (55)$$

The solution of (54) reads (computations are explained in Appendix A):

$$\overline{\Phi}^{(1)}(X,T_1,T_2) = W(k)\int^X dX|\widetilde{\theta}^{(1)}|^2, \quad (56)$$

where

$$W(k) = \frac{S(k)}{[\omega'_+(k)]^2 - c^2}. \quad (57)$$

Expression (56) combined with Eq. (41) shows that

$$\partial_{T_1}\overline{\Phi}^{(1)} + \omega'_+(k)\,\partial_X\overline{\Phi}^{(1)} = 0. \quad (58)$$

This result shows that the deformation of the background propagates with the group velocity, as does the envelope of the wave [which obeys the same equation, cf. (41)].

Finally, for being able to solve Eq. (52) we need $\widetilde{D}_1$ be in the image space of $\mathbb{M}_1$:

$$\widetilde{L}_1^{\,t} \cdot \widetilde{D}_1(X,T_1,T_2) = 0. \quad (59)$$

This reads

$$i\,\partial_{T_2}\widetilde{\theta}^{(1)} = -\frac{\omega''_+(k)}{2}\,\partial_X^2\widetilde{\theta}^{(1)} + \Bigg[\bigg(P(k) - \frac{Q(k)}{\Delta}\bigg)|\widetilde{\theta}^{(1)}|^2 + k\bigg(1+\frac{g_2}{g_1}\frac{k^2+2\,\Omega+g_2}{k^2+2\,\Omega+2\,g_2}\bigg)\partial_X\overline{\Phi}^{(1)}\Bigg]\frac{\widetilde{\theta}^{(1)}}{2}, \quad (60)$$

where $\omega''_+(k) = d^2\omega_+/dk^2$. One can reexpress the term $\partial_X\overline{\Phi}^{(1)}$ using Eq. (56). One then obtains a nonlinear Schrödinger equation (NLS) for $\widetilde{\theta}^{(1)}(X,T_1,T_2)$:

$$i\,\partial_{T_2}\widetilde{\theta}^{(1)} = -\frac{\omega''_+(k)}{2}\,\partial_X^2\widetilde{\theta}^{(1)} + g_{\text{eff}}(k)|\widetilde{\theta}^{(1)}|^2\widetilde{\theta}^{(1)}, \quad (61)$$

with

$$g_{\text{eff}}(k) = \frac{1}{2}\bigg(P(k) - \frac{Q(k)}{\Delta}\bigg) + \frac{k}{2}\bigg(1+\frac{g_2}{g_1}\frac{k^2+2\,\Omega+g_2}{k^2+2\,\Omega+2\,g_2}\bigg)W(k). \quad (62)$$

One has reached a point where the approach is self-contained, as far as the first-order term $\Xi^{(1)}$ in expansion (20) is concerned. One just needs to return to the actual variables x and t using the reverse of transformations (24) [33]. We give below final formulas valid even when $k_0 \neq 0$.

D. Final formulas and discussion

A nonlinear wave packet propagating in the upper branch is described by a set of fields $\Xi(x,t)$ of the form (10) with

$$\Xi'(x,t) = \overline{\Xi}(x,t) + [\widetilde{\Xi}(x,t)e^{i(kx-\omega_+(k)t)} + \text{c. c.}]. \quad (63)$$

The component $\widetilde{\theta}(x,t)$ of the envelope $\widetilde{\Xi}(x,t)$ is solution of

$$i\,\partial_t\widetilde{\theta} = -\frac{\omega''_+(k)}{2}\,\partial_y^2\widetilde{\theta} + g_{\text{eff}}(k)|\widetilde{\theta}|^2\widetilde{\theta}, \quad (64)$$

where $y = x - \omega'_+(k)t$ is the space coordinate in a frame moving at the group velocity.

Once $\widetilde{\theta}(x,t)$ has been determined, the component $\overline{\Phi}(x,t)$ of the background deformation is obtained as

$$\overline{\Phi}(x,t) = W(k)\int^x dx|\widetilde{\theta}|^2. \quad (65)$$

The other components of the background and of the envelope are given by

$$\overline{\Xi}(x,t) = \begin{pmatrix}\overline{n}(x,t)\\ \overline{\Phi}(x,t)\\ \overline{\theta}(x,t)\\ \overline{\varphi}(x,t)\end{pmatrix} = \overline{R}\,\overline{\Phi}(x,t), \quad (66)$$

and

$$\widetilde{\Xi}(x,t) = \begin{pmatrix}\widetilde{n}(x,t)\\ \widetilde{\Phi}(x,t)\\ \widetilde{\theta}(x,t)\\ \widetilde{\varphi}(x,t)\end{pmatrix} = \widetilde{R}\,\widetilde{\theta}(x,t), \quad (67)$$

where $\overline{R}^t = (0,1,0,0)$ and

$$\widetilde{R} = \begin{pmatrix} \frac{k(k^2+2\Omega)(2g_2+k^2-4k_0^2+2\Omega)-4k\,\omega_+^2(k)}{8k_0(k^2+\Omega)\omega_+(k)} \\ -\frac{i((2g_2+k^2+2\Omega)(k^2+2\Omega)+4k^2k_0^2-4\omega_+(k)^2)}{4kk_0(k^2+\Omega)} \\ 1 \\ \frac{i(k^2(2g_2+k^2-4k_0^2+2\Omega)+4\omega_+^2(k))}{4(k^2+\Omega)\omega_+(k)} \end{pmatrix}. \qquad (68)$$

We do not write here the explicit forms of $W(k)$ and $g_{\rm eff}(k)$ for $k_0 \neq 0$ because they are too cumbersome. However, it is important for subsequent discussions to stress that Eq. (57) still holds for $k_0 \neq 0$, but with a numerator $S(k)$ whose expression is different from the one given in Eq. (55) for the case $k_0 = 0$. On the other hand, the formulas (66), (67), and (68) are valid even when $k_0 \neq 0$. Note that $\overline{R}$ is identical to $\overline{R}_0$ defined in (28) and that $\widetilde{R}$ reduces to $\widetilde{R}_1$ defined in (29) when $k_0 = 0$. In this case, the first two components of $\widetilde{R}$ cancel and the nonlinear structure corresponds to a polarization signal, with oscillations of $\rho_\uparrow$ and $\rho_\downarrow$ preserving a fixed total density.

The nonlinear Schrödinger equation (64) describes the spatiotemporal evolution of the envelope wave, which is advected by the group velocity $\omega'_+(k)$ while dispersion and nonlinearity give corrections to the dynamics of the wave train, in particular for large times. It has been obtained through a multiscale expansion assuming the existence of well-separated spatial and temporal scales. The two spatial scales are the wavelength $\sim k^{-1}$ [associated with the x dependence of the phase $\beta(x,t)$ (23)] and the length a characteristic of the spatial variations of the envelope of the wave packet (associated to coordinate X). These length scales should be widely different and this corresponds to defining our small parameter as

$$\epsilon = \frac{1}{ak} \ll 1. \qquad (69)$$

The three time scales legitimating the introduction of the three different time coordinates t, T_1, and T_2 are:

$$\tau \ll \tau_1 \ll \tau_2, \qquad (70)$$

where $\tau \sim 1/\omega_+(k)$ is the period of the carrier wave. The characteristic time τ_1 is associated to the group motion of the envelope. The time τ_2 accounts for the fact that the envelope not only propagates with the group velocity, but also changes form because of higher-order dispersive effects and of nonlinearity (both effects typical balance in a nonlinear wave such as the soliton solutions discussed below). From Eqs. (41) and (61) one can check that when the condition (69) is fulfilled one has $\tau/\tau_1 \sim \tau_1/\tau_2 \sim \epsilon$, thus legitimating *a posteriori* the introduction of the three time coordinates (21).

When $g_{\rm eff}(k)$ is positive, periodic wave trains with constant amplitude formed in the upper branch of the spectrum are dynamically stable. They can support nonlinear excitations such as dark solitons. In this case $\widetilde{\theta}$—solution of (64)—is of the form

$$\widetilde{\theta}(y,t) = \Theta_0\, e^{-i\, g_{\rm eff}(k)\,\Theta_0^2\, t}[\cos\alpha \tanh(Y) + i\,\sin\alpha], \qquad (71)$$

where $\Theta_0 \in \mathbb{R}^+$ is the amplitude of the wave train and $\alpha \in [0,\pi/2]$; $\sin\alpha$ is the dimensionless velocity of the dark soliton,

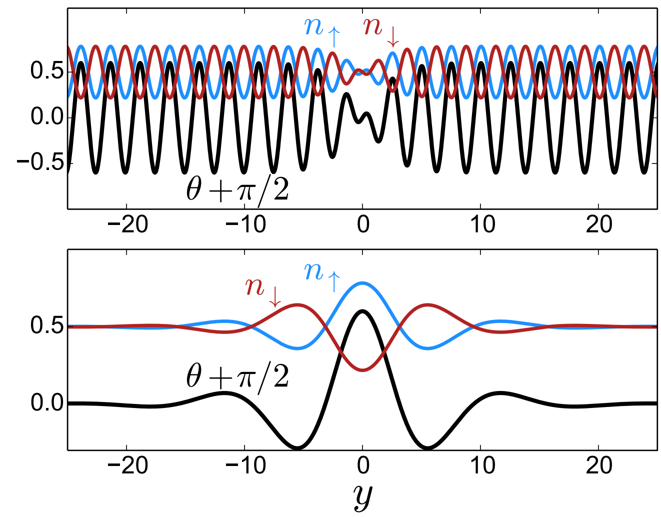

FIG. 3. θ, $n_\uparrow$, and $n_\downarrow$ as functions of y for a dark envelope soliton (71) (top) and a bright envelope one (73) (bottom). The system's parameters are the same as in Fig. 4. For both plots $\Theta_0 = 0.3$. The dark soliton is plotted for $k = 2.5$ [$g_{\rm eff}(k) = 1.4$] and $V_{\rm sol} = 0$ (black soliton) and the bright soliton for $k = 0.5$ [$g_{\rm eff}(k) = -0.6$].

cf. Eq. (72). The argument Y in (71) is

$$Y = \frac{y - V_{\rm sol}\, t}{\xi_{\rm eff}(k)} \cos\alpha, \text{ where } V_{\rm sol} = \sin\alpha\, c_{\rm eff}(k), \qquad (72)$$

and $c_{\rm eff}(k) = \Theta_0\sqrt{g_{\rm eff}(k)\omega''_+(k)} = \omega''_+(k)/\xi_{\rm eff}(k)$.

In the case where $g_{\rm eff}(k) < 0$, wave trains in the upper branch are dynamically unstable (they experience a modulational instability, see below), but one may observe stable bright envelope solitons, for which the solution of (64) is of the form

$$\widetilde{\theta}(y,t) = \frac{\Theta_0 \exp\left(-i\,\frac{g_{\rm eff}(k)}{2}\,\Theta_0^2\, t\right)}{\cosh\left(\Theta_0\sqrt{\frac{-g_{\rm eff}(k)}{\omega''_+(k)}}\,y\right)}, \qquad (73)$$

where Θ_0 is a positive real parameter governing the amplitude of the soliton. Once $\widetilde{\theta}$ is known, the corresponding values of the other fields describing the system are then given by (65), (66), and (67). The density profiles of typical envelope solitons are plotted in Fig. 3.

In order to get better insight on the type of dynamics described by the envelope NLS equation (64), we show in Fig. 4 how the effective nonlinear constant $g_{\rm eff}$ depends on k.

We see that $g_{\rm eff}(k)$ starts at low k with a negative value, and since $\omega''_+(k) > 0$, this means that wave trains in the upper branch experience a modulational instability, see, e.g., Ref. [34] and references therein. Modulational instability in Bose-Einstein condensates with repulsive interaction have already been studied in the presence of an external optical lattice potential [35] and for the counterflow of two miscible species [36]. Here we consider a scenario closer to the original Bejamin-Feir configuration [37], where nonlinearity destabilizes a periodic wave train through generation of spectral sidebands [see the discussion below, around Eqs. (76) and (77)].

When discussing the physical origin of the modulational instability in the present context, it is interesting to note that $g_{\rm eff}(k)$ diverges and changes sign for a value of k, which is

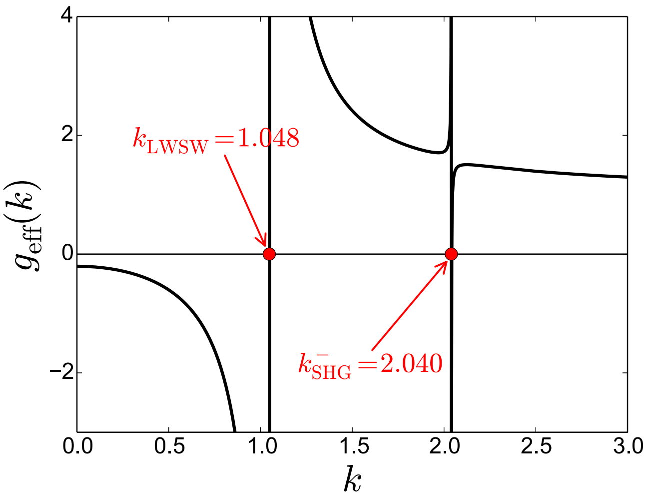

FIG. 4. The solid line represents $g_{\rm eff}(k)$ for the choice of parameters $k_0 = 0$, $g_1 = 2.2$, $g_2 = 0.2$, and $\Omega = 2.5$.

denoted as $k_{\rm LWSW}$ in Fig. 4. For this value of k, the system displays a so-called long-wave–short-wave resonance [38]. In the present configuration this corresponds to a case where the wave in the upper branch (with wave vector k) decays into two waves, one in the same branch with a similar wave vector (k'), and another one in the lower branch, with a small wave vector (q, the long wave). The conditions of conservation of momentum and energy read $k = k' + q$ and

$$\omega_+(k) = \omega_+(k-q) + \omega_-(q). \tag{74}$$

Since q is small one can expand the first term of the right-hand side of (74) as: $\omega_+(k-q) \simeq \omega_+(k) - q\,\omega'_+(k)$, and also write $\omega_-(q) \simeq c\,q$. Hence, the phenomenon occurs at $k = k_{\rm LWSW}$ such that

$$\omega'_+(k_{\rm LWSW}) = c, \tag{75}$$

meaning that the condition of resonance is that the group velocity of the short wave is equal to the phase velocity of the long wave.

The location of the resonance is clearly seen in Fig. 4, at a value of k in exact agreement with the value $k_{\rm LWSW}$ determined by (75). From the derivation leading to the NLS Eq. (64), one can locate the mathematical origin of the resonance phenomenon in Eq. (65), where $W(k)$ as given by (57) clearly diverges exactly at resonance. The phenomenological analysis just presented assumes that this divergence corresponds to a transfer of excitation from the upper branch to the lower one, but one should ascertain that this is indeed the case in our mathematical treatment. Indeed, it might seem from Eq. (54) that the divergence is connected to a resonance with a deformation of the background (of $\overline{\Phi}^{(1)}$), which might not be exactly connected to the lower branch of excitation. A first clue of this connection comes from the left-hand side of Eq. (54) itself: in this equation, the zero mode of the operator acting on $\overline{\Phi}^{(1)}$ corresponds to a dispersion relation, which is the long wave length approximation of the lower branch: $\omega'_-(k) \simeq c\,k$. The second and final reason explaining why in this context $\overline{\Phi}^{(1)}$ indeed represents the lower branch excitation comes from the very reason for its appearance in (54): it originates from Eq. (28), more precisely, from the specific form of $\overline{\Xi}_0^{(1)}$, which is tailored to be representative of the kernel of $\mathbb{M}_0$. Additionally, as can be checked by a comparison of the forms and definitions of $\mathbb{M}_1$ (14) and $\mathbb{M}_0$ (26), $\mathbb{M}_0$ is the $k \to 0$ limit of $\mathbb{M}_1$ when $\omega = \omega_-(k)$: hence the background contributions in the *Ansatz* (22) (and in the higher-order terms) is indeed a low-k contribution in the lowest branch and the divergence of $W(k)$ in (57) indeed corresponds to a resonance between the upper branch and the (long wavelength limit of) the lower branch.

It is remarkable that the occurrence of the long-wave–short-wave resonance is connected to a disappearance of the modulational instability of the upper branch: as one can see from Fig. 4 the nonlinear parameter $g_{\rm eff}(k)$ is positive when k is larger than $k_{\rm LWSW}$ and wave trains in the upper branch are thus stable when their wave vector is larger than the one of the long-wave–short-wave resonance. In order to appreciate the origin of this phenomenon one first needs to get some physical insight on the cause of the modulational instability. Since the reasoning presented below is quite general, and for simplifying the notations, we will here for a moment denote the dispersion relation as $\omega(k)$ instead of $\omega_+(k)$.

If one studies a wave train with wave vector k and constant (real) amplitude Θ_0, one finds from (64) that the corresponding $\widetilde{\theta}(x,t)$ is equal to $\Theta_0 \exp[-i\,g_{\rm eff}(k)\,\Theta_0^2\,t]$. Then, a perturbative treatment of Eq. (64) readily shows (see, e.g., Refs. [39–41]) that small amplitude modulations of the carrier wave with relative wave vector q and angular frequency ϖ obey the dispersion relation

$$(\varpi - \omega'(k)\,q)^2 = \left(\frac{\omega''(k)\,q^2}{2}\right)^2 + g_{\rm eff}(k)\,\Theta_0^2\,\omega''(k)\,q^2. \tag{76}$$

If $g_{\rm eff}(k)$ is negative, ϖ will be imaginary (for low enough values of q), meaning that the wave train is dynamically unstable. The value q^* of q corresponding to the largest imaginary part of ϖ, i.e., to the greatest growth rate of the perturbations, verifies

$$\frac{\omega''(k)}{2}(q^*)^2 = -g_{\rm eff}(k)\,\Theta_0^2. \tag{77}$$

One gets here a confirmation that the wave train is unstable when $\omega''(k)\,g_{\rm eff}(k)$ is negative. This corresponds to the so-called Lighthill-Benjamin-Feir criterion of modulational instability [34], which can be given the following intuitive interpretation: one assumes that a wave train of finite amplitude Θ_0 corresponds to the renormalized dispersion relation

$$\omega_{\rm ren}(k) = \omega(k) + g_{\rm eff}(k)\,\Theta_0^2. \tag{78}$$

The initial carrier wave at angular frequency $\omega(k)$ and wave vector k may decay into two side bands according to the following process:

$$\begin{aligned} \omega(k) + \omega(k) &\to \omega_{\rm ren}(k-q^*) + \omega_{\rm ren}(k+q^*), \\ k + k &\to (k-q^*) + (k+q^*), \end{aligned} \tag{79}$$

where q^* as given by (77) enforces the energy and momentum conservation relations in the process (79), as can be checked analytically (by an expansion in q^*) and is graphically demonstrated in Fig. 5. It is clear that, when $\omega''(k) > 0$,

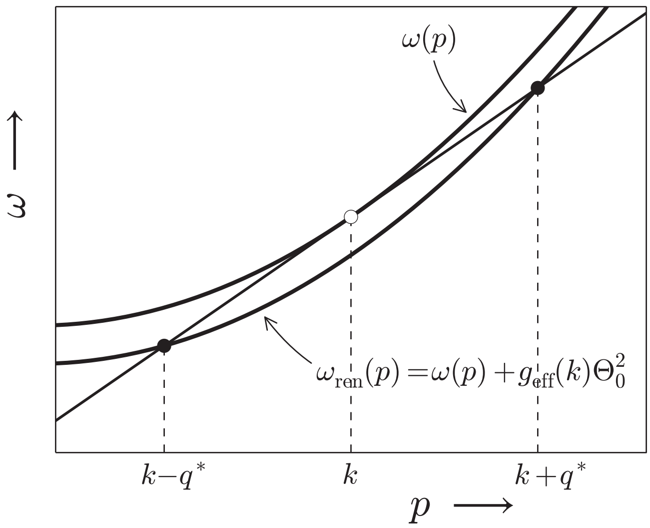

FIG. 5. Illustration of the modulational instability process. The straight solid line is the tangent to $\omega(p)$ at $p = k$. Two initial elementary excitations $[k,\omega(k)]$ can decay into $[k-q^*,\omega_{\rm ren}(k-q^*)]$ and $[k+q^*,\omega_{\rm ren}(k+q^*)]$ provided q^* verifies the above construction, i.e., that the two black points are the intersections of the straight solid line with the renormalized dispersion relation $\omega_{\rm ren}(p)$. An expansion of $\omega(p)$ at second order in the vicinity of k shows that (i) the white point is the middle of the two black points [and this automatically implies momentum and energy conservation in the process (79)] and that (ii) q^* defined by the above construction verifies Eq. (77).

the geometrical construction of Fig. 5 is only possible if $g_{\rm eff}(k) < 0$: in this case the wave train is modulationally unstable.

Hence, we understand why the change of sign of $g_{\rm eff}(k)$ observed in Fig. 4 when k crosses $k_{\rm LWSW}$ changes the stability of the wave train. Now it remains to understand the physical reason for this change of sign. Actually, the reason for it becomes clear when one focuses on the nonlinear term in brackets in Eq. (60). The first part of this term [with the $P(k)$ and $Q(k)$ contributions] is a genuine nonlinear self-interaction, but the second part is proportional to $k\,\partial_X\overline{\Phi}^{(1)}$, i.e., to $k\,\overline{U}^{(1)}$, which is a Doppler contribution to the energy of an excitation moving over a background of velocity $\overline{U}^{(1)}$. For $k < k_{\rm LWSW}$, the momentum q imparted to the lower branch is negative, and the corresponding value of $\overline{U}^{(1)}$ is also, as physically clear and mathematically demonstrated by the fact that in this case $W(k) < 0$ [see Eq. (57)]. It so happens that this Doppler contribution is dominant over the self-interaction terms, and, as a result, $g_{\rm eff}(k < k_{\rm LWSW}) < 0$. On the contrary, for $k > k_{\rm LWSW}$ the momentum imparted to the lower branch is positive, $\overline{U}^{(1)} > 0$ and $g_{\rm eff}(k > k_{\rm LWSW}) > 0$. This ends our discussion of the behavior of $g_{\rm eff}(k)$ around $k \simeq k_{\rm LWSW}$ and the explanation for the disappearance of the modulational instability when $k \gtrsim k_{\rm LWSW}$.

Besides the long-wave–short-wave resonance, one can notice another resonantlike structure in Fig. 4. It corresponds to a generation of second harmonic according to the three waves process

$$k + k \to 2k, \quad \omega_+(k) + \omega_+(k) \to \omega_-(2k). \qquad (80)$$

The condition of conservation of momentum and energy in the above process determines the value of the resonant wave vector $k^-_{\rm SHG}$ in excellent agreement with the location of the divergence of $g_{\rm eff}(k)$ observed in Fig. 4. In the vicinity of $k^-_{\rm SHG}$ our approach fails [and the envelope NLS equation (64) is not relevant] because the determinant of $\mathbb{M}_2$ vanishes, contrarily to what has been stated after Eq. (38), and the procedure that has been used for determining $\widetilde{\Xi}_2^{(2)}$ from Eq. (37) is incorrect. In this case the assumption that higher-order harmonics have a very small contribution is wrong. The fact that second harmonic generation is associated with vanishing of the determinant of $\mathbb{M}_2$ is an immediate result of the definition (38) and of energy conservation in the process (80): at resonance one has $\mathbb{M}_2 \equiv \mathbb{M}[2i\,k^-_{\rm SHG}, -2i\,\omega_+(k^-_{\rm SHG})] = \mathbb{M}[2i\,k^-_{\rm SHG}, -i\omega_-(2k^-_{\rm SHG})]$. The determinant of this last matrix is zero, because, for any p, $\det\mathbb{M}[ip,-i\omega_-(p)] = 0$, since $\omega = \omega_-(p)$ is one of the dispersion relations of the system.

For concluding the discussion, it is interesting to notice that, besides the second harmonic generation identified in Fig. 4, there exists another possible generation of second harmonics, which only involves excitations of the upper branch:

$$\begin{gathered} k + k \to 2k, \\ \omega_+(k) + \omega_+(k) \to \omega_+(2k). \end{gathered} \qquad (81)$$

This new process should induce a divergence of $g_{\rm eff}(k)$ at the wave vector $k = k^+_{\rm SHG}$, which ensures energy conservation in (81). Indeed, in this case we have a linear system $\mathbb{M}[2i\,k^+_{\rm SHG}, -2i\,\omega_+(k^+_{\rm SHG})] = \mathbb{M}[2i\,k^+_{\rm SHG}, -i\omega_+(2k^+_{\rm SHG})]$ that has a zero determinant because $\omega = \omega_+(p)$ is one of the dispersion relations of the system. For the set of parameters corresponding to Fig. 4, this second harmonic generation should occur at $k^+_{\rm SHG} = 1.612$. It is then surprising that this resonance is not seen in this figure. However, it is clearly seen when $k_0 \neq 0$ (see Fig. 6), at the value predicted by the conservation of energy in (81).

Actually, in the case where $k_0 = 0$, at $k = k^+_{\rm SHG}$ one has $\det\mathbb{M}_2 = 0$, and the divergent factor involved in the

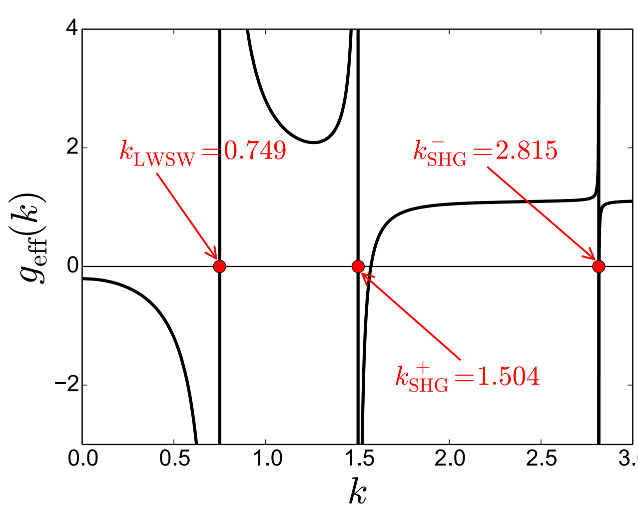

FIG. 6. The solid line represents $g_{\rm eff}(k)$ for the choice of parameters $k_0 = 0.5$, $g_1 = 2.2$, $g_2 = 0.2$, and $\Omega = 2.5$. The location of the resonances is determined by momentum and energy conservation in the processes (74) (for $k_{\rm LWSW}$), (80) (for $k^-_{\rm SHG}$), and (81) (for $k^+_{\rm SHG}$).

determination of $\widetilde{\Xi}_2^{(2)}$ from Eq. (37) [and which results in a divergence in the expression of $g_{\rm eff}(k)$] is canceled by another contribution. This can be easily understood by noticing that $\mathbb{M}$ defined in Eq. (12) is a block matrix when $k_0 = 0$:

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_- & 0 \\ 0 & \mathbb{M}_+ \end{pmatrix}, \tag{82}$$

where $\mathbb{M}_-$ and $\mathbb{M}_+$ are 2×2 matrices accounting for the lower and the upper excitation branches. We are here interested in second harmonic generation, i.e., in the specific matrix $\mathbb{M}_2 = \mathbb{M}(2ik, -2i\omega_+(k))$. In this case, we denote the matrices $\mathbb{M}_-$ and $\mathbb{M}_+$ as $\mathbb{M}_{2-}$ and $\mathbb{M}_{2+}$ and their inverses are

$$\mathbb{M}_{2-}^{-1} = \frac{1}{\omega_-^2(2k) - [2\omega_+(k)]^2}\,\text{adj}(\mathbb{M}_{2-}), \tag{83a}$$

$$\mathbb{M}_{2+}^{-1} = \frac{1}{\omega_+^2(2k) - [2\omega_+(k)]^2}\,\text{adj}(\mathbb{M}_{2+}), \tag{83b}$$

where "adj" denotes the adjugate matrix. The divergence of $g_{\rm eff}(k^+_{\rm SHG})$ is associated with the divergence of the denominator in (83b), corresponding to energy conservation in the process (81). In the special case $k_0 = 0$, the solution of Eq. (37) reads:

$$\widetilde{\Xi}_2^{(2)} = \begin{pmatrix} \mathbb{M}_{2-}^{-1} & 0 \\ 0 & \mathbb{M}_{2+}^{-1} \end{pmatrix} \widetilde{C}_2, \tag{84}$$

where $\widetilde{C}_2$ is given by Eq. (33) when $k_0 = 0$: in this case its last two components are zero. Eq. (84) then reads

$$\widetilde{\Xi}_2^{(2)} = \begin{pmatrix} \mathbb{M}_{2-}^{-1} \begin{pmatrix} i\,\Delta\,k^2 \\ \frac{1}{2}\left(\frac{k^2-2\Omega}{k^2+2\Omega}\,g_2 - k^2 - 2\,\Omega\right) \end{pmatrix} \\ \mathbb{M}_{2+}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} (\widetilde{\theta}^{(1)})^2 \tag{85}$$

and the possible divergence of the denominator of $\mathbb{M}_{2+}^{-1}$ is masked. This is the reason for the inhibition of the second harmonic generation process (81) when $k_0 = 0$.

IV. NONLINEAR PERTURBATION THEORY FOR EXCITATIONS PROPAGATING IN THE LOWER BRANCH

We now study the propagation of a sound pulse which, in a linear approximation, would lie on the lower excitation branch. The method used in Sec. III can be employed in the present case. It yields for the nonlinear coefficient $g_{\rm eff}(k)$ a behavior represented in Fig. 7.

The nonlinear coefficient diverges at large wavelength. This is due to the fact that, for the lower branch, the analog of the coefficient $W(k)$ (57) diverges when $k \to 0$ since $\omega_-'(0) = c$. In this case the nonlinear time $t_{\rm NL} \propto g_{\rm eff}(k)^{-1}$ associated to Eq. (64) diverges indicating that nonlinear structures form extremely rapidly. $t_{\rm NL}$ may become even smaller that the period of the wave (except for waves of extremely small amplitude) and in this case the technique of the envelope NLS fails.

In this long wavelength limit one can suggest an alternative method consisting in deriving equations for the interacting fields themselves instead of an effective equation for the envelope. This method is based on the following reasoning: In the linear regime and at the level of accuracy at which the expansion (17) holds, any of the components of $\Xi'(x,t)$—n',

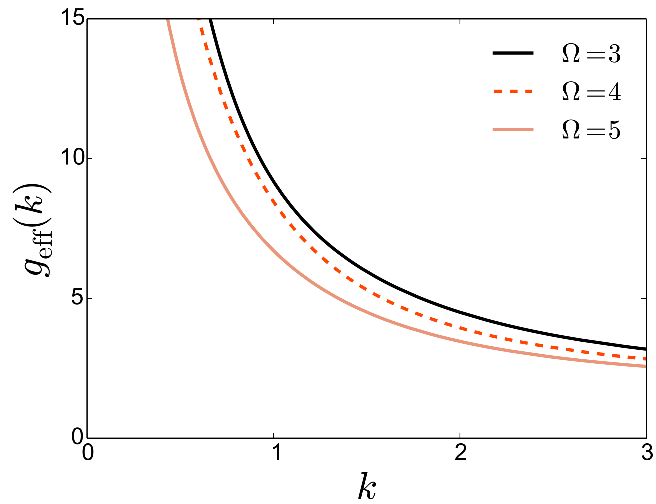

FIG. 7. Nonlinear coefficient $g_{\rm eff}(k)$ for the envelope NLS equation describing a wave packet propagating in the lower excitation branch. The curves are drawn for different values of Ω; the other parameters are $k_0 = 1$, $g_1 = 2.2$, and $g_2 = 0.2$.

say—satisfies the linear equation

$$n'_t + c\,n'_x - c_3\,n'_{xxx} = 0, \tag{86}$$

where the last term describes a small dispersive correction to the propagation with constant velocity c. If the amplitude n' is small but finite and such that this term has the same order of magnitude as the leading nonlinear correction to (86) (which is typically quadratic in n'), then nonlinear effects cannot be omitted for correctly describing the propagation of the pulse. In this regime one can try to derive an equation of the type (86) with additional terms taking into account weak nonlinear effects. The most natural extension of (86) is a Korteweg–de Vries (KdV) equation in which a nonlinear term of the form $n'n'_x$ accounts for a dependence in density fluctuations ($\propto n'$) of the velocity of sound.

A. Quadratic nonlinearity: KdV regime

It now is appropriate to work in a reference frame moving at the speed of sound c, and to use $x - ct$ and t as coordinates. In order that the derivatives in equations of type (86) appear at the same order, we define

$$\xi = \epsilon^{1/2}(x - ct), \quad \text{and} \quad \tau = \epsilon^{3/2}t. \tag{87}$$

where ϵ will henceforth be a small positive parameter. The choice of the specific powers $\epsilon^{1/2}$ and $\epsilon^{3/2}$ in (87) (instead of ϵ and ϵ^3 for instance) will make sure that the derivatives in equations of type (86) appear at the same order as the quadratic nonlinear contribution [$\propto n'n'_x$, see Eq. (104) below]. In terms of the new variables ξ and τ and of the velocities U and v defined in Eq. (6), the system (7) reads

$$\epsilon^{3/2}n_\tau = \epsilon^{1/2}c\,n_\xi + \epsilon^{1/2}k_0[n\cos\theta]_\xi - \frac{1}{2}\epsilon^{1/2}[n(U - v\cos\theta)]_\xi, \tag{88a}$$

$$\begin{aligned}\epsilon^{3/2}U_\tau &= \epsilon^{1/2}c\,U_\xi - \epsilon^{1/2}k_0\,v_\xi \\ &\quad - \epsilon^{1/2}\left[\frac{\epsilon}{4}\frac{n_\xi^2}{n^2} - \frac{\epsilon}{2}\frac{n_{\xi\xi}}{n} - \epsilon\frac{\cot\theta}{2}\frac{[n\,\theta_\xi]_\xi}{n}\right]_\xi \\ &\quad - \epsilon^{1/2}\left[\frac{\epsilon\theta_\xi^2 + U^2 + v^2}{4} + g_1 n\right]_\xi \\ &\quad + \epsilon^{1/2}\theta_\xi\frac{\Omega\cos\theta}{\sin^2\theta}\cos\varphi + \frac{\Omega}{\sin\theta}\,v\,\sin\varphi, \end{aligned} \tag{88b}$$

$$\begin{aligned}\epsilon^{3/2}\theta_\tau &= \epsilon^{1/2}c\,\theta_\xi - \Omega\,\sin\varphi \\ &\quad - \frac{1}{2}\epsilon^{1/2}\left(U\theta_\xi + \frac{[n\,(v+2k_0)\sin\theta]_\xi}{n}\right),\end{aligned} \tag{88c}$$

$$\begin{aligned}\epsilon^{3/2}v_\tau &= \epsilon^{1/2}c\,v_\xi - \epsilon^{1/2}k_0\,U_\xi \\ &\quad + \epsilon^{1/2}\left[\frac{\epsilon}{2\sin\theta}\frac{[n\,\theta_\xi]_\xi}{n} - \frac{U\,v}{2} + g_2 n\cos\theta\right]_\xi \\ &\quad + \epsilon^{1/2}\theta_\xi\frac{\Omega\cos\varphi}{\sin^2\theta} + \Omega\,v\,\cot\theta\,\sin\varphi.\end{aligned} \tag{88d}$$

We perform a multiscale analysis by expanding (n,U,θ,v) in the following way:

$$\begin{pmatrix} n(\xi,\tau)\\ U(\xi,\tau)\\ \theta(\xi,\tau)\\ v(\xi,\tau)\end{pmatrix} = \begin{pmatrix}1\\0\\-\frac{\pi}{2}\\0\end{pmatrix} + \epsilon\begin{pmatrix}n^{(1)}\\U^{(1)}\\\theta^{(1)}\\v^{(1)}\end{pmatrix} + \epsilon^2\begin{pmatrix}n^{(2)}\\U^{(2)}\\\theta^{(2)}\\v^{(2)}\end{pmatrix} + \cdots. \tag{89}$$

In agreement with the definitions (6) and (87) we have

$$U = \epsilon^{1/2}\Phi_\xi, \quad \text{and} \quad v = \epsilon^{1/2}\varphi_\xi. \tag{90}$$

Taking into account expansion (89) and the leading order (9) for Φ and φ we thus have

$$\begin{aligned}\Phi &= -2\epsilon^{-3/2}\mu\tau + \epsilon^{1/2}\Phi^{(1)} + \epsilon^{3/2}\Phi^{(2)} + \cdots \\ \varphi &= \epsilon^{1/2}\varphi^{(1)} + \epsilon^{3/2}\varphi^{(2)} + \cdots.\end{aligned} \tag{91}$$

Once the *Ansätze* (89) and (91) are inserted back into (88), the leading term is $O(\epsilon^{1/2})$ and simply yields $\varphi^{(1)} = 0$ (and thus $v^{(1)} = 0$). At next order $(\epsilon^{3/2})$ one obtains

$$\mathbb{K}\begin{pmatrix}n_\xi^{(1)}\\U_\xi^{(1)}\\\theta_\xi^{(1)}\\\varphi^{(2)}\end{pmatrix} = 0, \tag{92}$$

where

$$\mathbb{K} = \begin{pmatrix} c & -\frac{1}{2} & k_0 & 0\\ -g_1 & c & 0 & 0\\ k_0 & 0 & c & -\Omega\\ 0 & -k_0 & \Omega+g_2 & 0\end{pmatrix}. \tag{93}$$

Since $\det\mathbb{K} = 0$, Eq. (92) has nontrivial solutions. The kernel of $\mathbb{K}$ is one dimensional; as a result, the solution of Eq. (92) is of the form:

$$\begin{pmatrix}n_\xi^{(1)}\\U_\xi^{(1)}\\\theta_\xi^{(1)}\\\varphi^{(2)}\end{pmatrix} = \begin{pmatrix}1\\ \frac{g_1}{c}\\ \frac{1}{c}\frac{k_0 g_1}{\Omega+g_2}\\ \frac{k_0}{\Omega}\frac{\Omega+g_1+g_2}{\Omega+g_2}\end{pmatrix} n_\xi^{(1)} \equiv R\,n_\xi^{(1)}. \tag{94}$$

We also need (for later use) to determine the column vector L such that

$$\mathbb{K}^t\,L = 0 \Leftrightarrow L^t\,\mathbb{K} = 0, \tag{95}$$

This fixes

$$L^t \propto \left(1,\frac{c}{g_1},0,-\frac{k_0}{\Omega+g_2}\right). \tag{96}$$

At order $\epsilon^{5/2}$ we obtain

$$\mathbb{K}\begin{pmatrix}n_\xi^{(2)}\\U_\xi^{(2)}\\\theta_\xi^{(2)}\\\varphi^{(3)}\end{pmatrix} = \begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix}, \tag{97}$$

where

$$\begin{aligned} A_1 &= n_\tau^{(1)} + \frac{1}{2}[n^{(1)}U^{(1)}]_\xi - k_0[n^{(1)}\theta^{(1)}]_\xi,\\ A_2 &= U_\tau^{(1)} + \frac{1}{2}U^{(1)}U_\xi^{(1)} - \Omega\,\theta^{(1)}\theta_\xi^{(1)}\\ &\quad + \frac{1}{2}n_{\xi\xi\xi}^{(1)} + k_0 v_\xi^{(2)},\\ A_3 &= \theta_\tau^{(1)} + \frac{1}{2}U^{(1)}\theta_\xi^{(1)} - \frac{1}{2}v_\xi^{(2)}\\ &\quad + k_0\theta^{(1)}\theta_\xi^{(1)} + k_0 n^{(1)}n_\xi^{(1)},\\ A_4 &= -g_2[n^{(1)}\theta^{(1)}]_\xi + \frac{1}{2}\,\theta_{\xi\xi\xi}^{(1)} - c\,v_\xi^{(2)}.\end{aligned} \tag{98}$$

Performing the substitution (94), we can express the system (97) in the following way:

$$\mathbb{K}\begin{pmatrix}n_\xi^{(2)}\\U_\xi^{(2)}\\\theta_\xi^{(2)}\\\varphi^{(3)}\end{pmatrix} = C_\tau\,n_\tau^{(1)} + C_3\,n_{\xi\xi\xi}^{(1)} + C_{\rm nl}\,n^{(1)}n_\xi^{(1)}, \tag{99}$$

with

$$C_\tau = \begin{pmatrix}1\\ \frac{g_1}{c}\\ \frac{1}{c}\frac{k_0 g_1}{\Omega+g_2}\\ 0\end{pmatrix}, \tag{100}$$

$$C_{\rm nl} = \begin{pmatrix} \frac{\sqrt{2}\sqrt{g_1(-2k_0^2+\Omega+g_2)}}{\sqrt{\Omega+g_2}}\\ g_1\left(\frac{\Omega}{\Omega+g_2} + \frac{g_2}{-2k_0^2+\Omega+g_2}\right)\\ k_0\left(\frac{g_1(2k_0^2+\Omega+g_2)}{(\Omega+g_2)(-2k_0^2+\Omega+g_2)} + 1\right)\\ -\frac{2\sqrt{2}\sqrt{g_1}g_2k_0}{\sqrt{\Omega+g_2}\sqrt{-2k_0^2+\Omega+g_2}}\end{pmatrix}, \tag{101}$$

and

$$C_3 = \begin{pmatrix}0\\ \frac{(\Omega+g_1+g_2)k_0^2}{\Omega(\Omega+g_2)} - \frac{1}{2}\\ -\frac{(\Omega+g_1+g_2)k_0}{2\Omega(\Omega+g_2)}\\ \frac{k_0(2(\Omega+g_1+g_2)k_0^2-(\Omega+g_2)(g_1+g_2))}{\sqrt{2}\Omega(\Omega+g_2)^{3/2}\sqrt{\frac{-2k_0^2+\Omega+g_2}{g_1}}}\end{pmatrix}. \tag{102}$$

Left multiplication of Eq. (99) by L^t gives

$$0 = (L^t \cdot C_\tau)\, n_\tau^{(1)} + (L^t \cdot C_{\rm nl})\, n^{(1)} n_\xi^{(1)} + (L^t \cdot C_3)\, n_{\xi\xi\xi}^{(1)}. \quad (103)$$

Equation (103) is a consistency condition: Eq. (97) admits a solution only if the column vector A is in the image space of $\mathbb{K}$, which is implied by (103) [we used the same technique in Sec. III, see Eqs. (40), (53), and (59)]. Explicitly, Eq. (103) reads

$$n_\tau^{(1)} + \frac{3g_1}{4c}\left(1 - \frac{2\,\Omega\, k_0^2}{(\Omega+g_2)^2}\right) n^{(1)} n_\xi^{(1)} - c_3\, n_{\xi\xi\xi}^{(1)} = 0, \quad (104)$$

where c_3 is the third-order coefficient of the dispersion relation (17). Going back to the original coordinates x and t and denoting $n'(x,t) = n(x,t) - 1$, we obtain the KdV equation

$$n'_t + c\, n'_x + \gamma_1\, n'\, n'_x - c_3\, n'_{xxx} = 0, \quad (105)$$

where

$$\gamma_1 = \frac{3g_1}{4c}\left(1 - \frac{2\,\Omega\, k_0^2}{(\Omega+g_2)^2}\right). \quad (106)$$

Once the solution of Eq. (105) is found, the other field variables can be obtained using relations (94), which we rewrite here for completeness in the final notation:

$$\begin{aligned} U(x,t) &= \frac{g_1}{c}\, n'(x,t), \\ \theta(x,t) &= -\frac{\pi}{2} + \frac{k_0 g_1}{c\,(\Omega+g_2)}\, n'(x,t), \\ \varphi(x,t) &= \frac{k_0}{\Omega}\,\frac{\Omega+g_1+g_2}{\Omega+g_2}\, n'_x(x,t). \end{aligned} \quad (107)$$

Note that when n' becomes of order of the nonlinear coefficient in the KdV Eq. (105), that is when $\gamma_1 \sim |n'| \ll 1$, the level of accuracy accepted here is not sufficient: the cubic nonlinear terms ($\sim n'^2 n'_x$) neglected in the present treatment have the same order of magnitude as the quadratic term in Eq. (105). In this limit we have to consider the next order of approximation.

B. Cubic nonlinearity: Gardner regime

As advocated in Sec. IV A, cubic nonlinearities become important when $\gamma_1 \sim n'$ is small: their contributions can therefore be calculated from the system (7) choosing parameters such that $\gamma_1 = 0$. This is achieved when

$$k_0 = \frac{\Omega + g_2}{\sqrt{2\Omega}}. \quad (108)$$

For this choice of parameters, the sound velocity (18) reads

$$c = c^\star \equiv \sqrt{\frac{-g_1 g_2}{2\,\Omega}}. \quad (109)$$

In this case the system can sustain long wavelength perturbations only if $g_2 < 0$, that is if $\alpha_2 > \alpha_1$, which we assume henceforth (see however the discussion at the end of Sec. IV C and Appendix B).

In this regime the coordinates defined in (87) are no longer appropriate for the description of nonlinear excitations. One should instead perform the computations with the new coordinates:

$$\xi = \epsilon(x - ct), \quad \text{and} \quad \tau = \epsilon^3 t. \quad (110)$$

Then the system (88) rewrites:

$$\begin{aligned} \epsilon^3 n_\tau &= \epsilon c\, n_\xi + \epsilon k_0 [n\cos\theta]_\xi \\ &\quad - \frac{1}{2}\epsilon [n(U - v\cos\theta)]_\xi, \quad (111\text{a}) \\ \epsilon^3 U_\tau &= \epsilon c\, U_\xi - \epsilon k_0\, v_\xi \\ &\quad - \epsilon\left[\frac{\epsilon^2}{4}\frac{n_\xi^2}{n^2} - \frac{\epsilon^2}{2}\frac{n_{\xi\xi}}{n} - \epsilon^2 \frac{\cot\theta}{2}\frac{[n\,\theta_\xi]_\xi}{n}\right]_\xi \\ &\quad - \epsilon\left[\frac{\epsilon^2\theta_\xi^2 + U^2 + v^2}{4} + g_1 n\right]_\xi \\ &\quad + \epsilon\theta_\xi \frac{\Omega\cos\theta}{\sin^2\theta}\cos\varphi + \frac{\Omega}{\sin\theta}\, v\, \sin\varphi, \quad (111\text{b}) \\ \epsilon^3 \theta_\tau &= \epsilon c\, \theta_\xi - \Omega\, \sin\varphi \\ &\quad - \frac{1}{2}\epsilon\left(U\theta_\xi + \frac{[n\,(v + 2k_0)\sin\theta]_\xi}{n}\right), \quad (111\text{c}) \\ \epsilon^3 v_\tau &= \epsilon c\, v_\xi - \epsilon k_0\, U_\xi \\ &\quad + \epsilon\left[\frac{\epsilon^2}{2\sin\theta}\frac{[n\,\theta_\xi]_\xi}{n} - \frac{U\, v}{2} + g_2 n\cos\theta\right]_\xi \\ &\quad + \epsilon\theta_\xi \frac{\Omega\cos\varphi}{\sin^2\theta} + \Omega\, v\, \cot\theta\, \sin\varphi. \quad (111\text{d}) \end{aligned}$$

We perform a multiscale analysis using the *Ansätze* (89); U and v now read:

$$U = \epsilon\, \Phi_\xi, \quad \text{and} \quad v = \epsilon\, \varphi_\xi. \quad (112)$$

The leading term in (111) is now $O(\epsilon)$ and reads, as previously, $\varphi^{(1)} = 0$. The next order $O(\epsilon^2)$ is described by the same system as in Eq. (92). Substituting $U_\xi^{(1)}$, $\theta_\xi^{(1)}$, and $\varphi^{(2)}$ by their expression in $n^{(1)}$ defined in (94), the order $O(\epsilon^3)$ then reads:

$$\mathbb{K}\begin{pmatrix} n_\xi^{(2)} \\ U_\xi^{(2)} \\ \theta_\xi^{(2)} \\ \varphi^{(3)} \end{pmatrix} = C_{\rm nl}\, n^{(1)} n_\xi^{(1)}, \quad (113)$$

where $C_{\rm nl}$ is defined in Eq. (101). Since in this subsection k_0 is fixed such that the non-linearity ($\gamma_1 \propto L^t C_{\rm nl}$) in Eq. (105) cancels, the choice of parameter (108) automatically implies $L^t \cdot C_{\rm nl} = 0$ and from Eq. (113) one can only deduces that

$$\begin{pmatrix} n_\xi^{(2)} \\ U_\xi^{(2)} \\ \theta_\xi^{(2)} \\ \varphi^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2\sqrt{g_1(-g_2)}}{\Omega+g_2} \\ \frac{g_1(3g_2\Omega - g_2^2 + 2\Omega^2) - g_2(g_2+\Omega)^2}{\sqrt{2}g_2\Omega(g_2+\Omega)} \end{pmatrix} n^{(1)} n_\xi^{(1)}. \quad (114)$$

Equation (113) is thus not conclusive and the expansion at order $O(\epsilon^3)$ is not sufficient to describe the dynamic of nonlinear excitations. At next order $[O(\epsilon^4)]$, taking into account the formula (114), we obtain:

$$\mathbb{K}\begin{pmatrix} n_\xi^{(3)} \\ U_\xi^{(3)} \\ \theta_\xi^{(3)} \\ \varphi^{(4)} \end{pmatrix} = C_\tau n_\tau^{(1)} + D_{\rm nl}\, {n^{(1)}}^2 n_\xi^{(1)} + C_3\, n_{\xi\xi\xi}^{(1)}, \tag{115}$$

where C_τ and C_3 are defined in Eqs. (100), (102) and

$$D_{\rm nl} = \begin{pmatrix} \frac{\Omega\sqrt{-g_1^3 g_2}+\sqrt{-g_1 g_2}\,g_2(g_1-6g_2)}{2\sqrt{2}\sqrt{\Omega}g_2^2} \\ -\frac{3\Omega g_1}{\Omega+g_2} \\ -\frac{2g_2(\Omega+g_2)^2+g_1\left(\Omega^2-8g_2\Omega-5g_2^2\right)}{2\sqrt{2}\sqrt{\Omega}g_2(\Omega+g_2)} \\ \frac{\sqrt{-g_1 g_2}\left(-6g_2^3-g_1(2\Omega-g_2)(\Omega+g_2)\right)}{2g_2^2(\Omega+g_2)} \end{pmatrix}. \tag{116}$$

Left multiplication of Eq. (115) by L^t gives

$$0 = (L^t\cdot C_\tau)\, n_\tau^{(1)} + (L^t\cdot D_{\rm nl})\, {n^{(1)}}^2 n_\xi^{(1)} + (L^t\cdot C_3)\, n_{\xi\xi\xi}^{(1)}, \tag{117}$$

which reads

$$0 = n_\tau^{(1)} + \frac{3\,g_1}{8\,c^*\,|g_2|}\left(g_1 - \frac{4g_2^2}{\Omega+g_2}\right){n^{(1)}}^2 n_\xi^{(1)} - c_3\, n_{\xi\xi\xi}^{(1)}. \tag{118}$$

Going back to the original coordinates x and t we obtain the modified KdV (mKdV) equation

$$n'_t + c^*\, n'_x + \gamma_2\, n'^2 n'_x - c_3\, n'_{xxx} = 0, \tag{119}$$

where

$$\gamma_2 = \frac{3\,g_1}{8\,c^*\,|g_2|}\left(g_1 - \frac{4g_2^2}{\Omega+g_2}\right) = \frac{3}{4\sqrt{2}}\sqrt{\frac{\Omega g_1}{-g_2^3}}\left(g_1 - \frac{4g_2^2}{\Omega+g_2}\right). \tag{120}$$

In the regime where γ_1 is not exactly zero, but of order n', we also have to take into account the quadratic nonlinearity of Eq. (105), which finally yields

$$n'_t + c\, n'_x + \gamma_1 n' n'_x + \gamma_2\, n'^2 n'_x - c_3\, n'_{xxx} = 0. \tag{121}$$

This is the Gardner equation describing the evolution of nonlinear polarization pulses in a coherently coupled two-component condensate in the limit where the parameters of the system are close to satisfy the condition (108). This can be considered as an intermediate region where the quadratic and cubic nonlinearities make contributions of the same order of magnitude in the wave dynamics. In the limit of very small γ_1, the quadratic nonlinearity effects can be neglected, the nonlinear polarization waves are correctly described by the modified KdV equation (119). If instead γ_1 is large, then the cubic nonlinearity effects are negligible and the evolution of nonlinear polarization pulses is described by the KdV equation (105). Note that for consistency reasons, the value of the sound velocity in (121) has to evaluated as not being exactly equal to c^*, but has to include corrections $\propto \gamma_1^2$.

Once the solution of the Gardner equation (121) has been found, the other field variables can be expressed in terms of n' by the formulas (107) with account of (108).

C. Quartic nonlinearity: generalized KdV equation

The parameters of the system can be chosen in such a way that not only γ_1, but also γ_2 cancels. This is achieved for the choice (108) with the additional constrain

$$g_1 = \frac{4g_2^2}{\Omega+g_2}, \tag{122}$$

which ensures that $\gamma_2 = 0$. Note that for having a positive value of g_1, one must have $\Omega + g_2 > 0$, but this condition is automatically fulfilled in phase III [cf. the definition (18) of the sound velocity]. Computations very similar to the ones exposed in Secs. IV A and IV B now lead to a higher-order mKdV equation

$$0 = n_\tau^{(1)} + \gamma_3\, {n^{(1)}}^3 n_\xi^{(1)} - c_3\, n_{\xi\xi\xi}^{(1)}, \tag{123}$$

where

$$\gamma_3 = \frac{5\sqrt{2}\,g_2^2\,\Omega(\Omega-2g_2)(\Omega+g_2)^{9/2}}{\sqrt{-g_2\Omega}}. \tag{124}$$

Finally, we can write the general form, which is able to account for choices of parameters such that $\gamma_1 \simeq 0$ and $\gamma_2 \simeq 0$:

$$n'_t + C(n')n'_x - c_3\, n'_{xxx} = 0, \tag{125}$$

with

$$C(n') = c + \gamma_1\, n' + \gamma_2\, n'^2 + \gamma_3\, n'^3. \tag{126}$$

Equation (125) is known as a generalized KdV equation [42].

At this point, it might be helpful to remind the strategy followed in the present section: we study excitations of the lower branch of the spectrum, which, in the linear regime and in the long wave limit, are described by Eq. (86). In order to analyze how nonlinearity affects these excitations, we consider the modifications of the pulse propagation velocity induced by the nonlinear effects: $c \to C(n')$ with an expansion of the form $C(n') = c + \sum_{\ell\geqslant 1}\gamma_\ell\, n'^\ell$. The multiscale analysis consists in rescaling the variable (x,t) in the following way:

$$\xi = \epsilon^a(x - c\,t) \quad \text{and} \quad \tau = \epsilon^b t, \tag{127}$$

transforming Eq. (86) into

$$\epsilon^b n'_\tau = \sum_{\ell\geqslant 1}\epsilon^{a+\ell}\gamma_\ell\, n'^\ell\, n'_\xi + \epsilon^{3a} c_3\, n'_{\xi\xi\xi}. \tag{128}$$

The analysis amounts to determine the coefficients γ_ℓ; this has been done in Eqs. (106), (120) and Eq. (124). The first correction is $\ell = 1$:

$$\epsilon^b n'_\tau = \epsilon^{a+1}\gamma_1\, n'\, n'_\xi + \epsilon^{3a} c_3\, n'_{\xi\xi\xi}. \tag{129}$$

The approach is coherent if all orders in ϵ in Eq. (129) are identical, i.e., if the stretched variables in (127) are chosen with $b = a + 1 = 3\,a \Rightarrow (a = 1/2, b = 3/2)$.

The parameter $\gamma_1(g_1,g_2,k_0,\Omega)$ can vanish or become small for a particular value of k_0; the first-order correction $\ell = 1$ is then no longer sufficient, and we must consider the correction $\ell = 2$, which corresponds, by the same argument as the one used after Eq. (129), to $(a = 1, b = 3)$; these are the exponents used in Sec. IV B. If $\gamma_2(g_1,g_2,k_0,\Omega)$ is also small, one has to consider the next order, as done in the beginning of the present section. The different orders considered and their regime of relevance are recalled in Table I.

TABLE I. List of the different nonlinear equations describing the weakly nonlinear and weakly dispersive dynamics of excitations which, in the linear regime, pertain to the lower dispersion branch. The right column shows their successive regime of relevance of the equations. The conditions $\gamma_1 = 0$ and $\gamma_2 = 0$ are precisely defined in Eqs. (108) and (122). Note that each row assumes that the regime of relevance of the upper rows is fulfilled.

Nonlinear equation	Regime of relevance
$\ell = 1$: KdV	phase III
$\ell = 2$: Gardner	$\gamma_1 \simeq 0$ and $g_2 < 0$
$\ell = 3$: generalized KdV	$\gamma_2 \simeq 0$ and $\lvert g_2\rvert < \Omega$

It is appropriate to discuss if the different regimes identified in Table I can be reached with present-day experimental realization of spin-orbit coupled BECs. The references [8] consider the two states $|m_F = 0\rangle = |\uparrow\rangle$ and $|m_F = -1\rangle = |\downarrow\rangle$ of a ^{87}Rb Bose-Einstein condensate in the $F = 1$ hyperfine structure. The s-wave scattering lengths are (in units of the Bohr radius) $a_{\uparrow\uparrow} = 101.41$ and $a_{\uparrow\downarrow} = a_{\downarrow\downarrow} = 100.94$. Since $a_{\uparrow\uparrow} - a_{\downarrow\downarrow} \ll \frac{1}{2}(a_{\uparrow\uparrow} + a_{\downarrow\downarrow})$ the simplifying assumption of a common value of the nonlinear coupling $\alpha_1 = \hbar\omega_\perp(a_{\uparrow\uparrow} + a_{\downarrow\downarrow})$ in (1) is legitimate (in this expression $\omega_\perp$ is the angular frequency corresponding to a tight harmonic radial trapping which ensures a quasi-1D behavior of the condensate [43]). Besides, if necessary, the present formalism can be extended to take into account the fact that $a_{\uparrow\uparrow}$ and $a_{\downarrow\downarrow}$ are not equal, see Ref. [44]. Note that the recoil energy is typically $\frac{1}{2}k_0^2 \sim 2$ kHz (it is monitored by the wavelength and the relative angle of the Raman lasers), whereas the interaction energy $\frac{1}{2}g_1 = \frac{1}{2}(\alpha_1 + \alpha_2)\rho_0$ is of order (1/5) kHz (it depends on the value of the radial trapping frequency and on the linear atomic density).

One has $\alpha_2 = 2\hbar\omega_\perp a_{\uparrow\downarrow} < \alpha_1$, hence $g_2 > 0$, and it seems from the discussion in the beginning of Sec. IV B that one can never reach the interesting regime where $\gamma_1 = 0$ and where the nonlinear modulations of an excitation formed in the lower branch is described by Gardner equation. However, g_2 is small (since α_1 and α_2 are so close) and, as explained in Appendix B, generalizing the present approach by taking into account the small detuning δ from the Raman resonance—not considered in the main text—one can show that, even with a positive g_2, it is possible to reach a regime where the nonlinearity coefficient γ_1 cancels by correctly fixing the value of δ. However, for keeping the discussion simple we will only consider here the case of a small negative g_2. The more relevant case of a small positive g_2 in the presence of a small detuning is presented in Appendix B. The main conclusions are similar in both cases. Then, for negative g_2, the condition (108) leads to $\Omega = 2k_0^2 - 2g_2 + O(g_2^2)$, which corresponds to a value of the Raman coupling frequency Ω typical in present-day experiments. We recall however that for the system to remain in the good side of the boundary between phase III and phase II one needs to impose $\Omega > 2k_0^2 - g_2$, which is verified by the above choice of Ω, but not by a large extent. Hence, one can reach a regime where the lower excitation branch is described by Gardner dynamics, but this is obtained at the expense of getting close to the phase III–phase II boundary. Away from this boundary, the lower branch has a KdV dynamics.

V. CONCLUSION

In the present paper we have described how nonlinearity affects the dynamics of elementary excitations of a coherently coupled Bose-Einstein condensate. Excitations in the upper branch of the spectrum display a modulational instability. As discussed in the text, this instability is stabilized by a long-wave–short-wave resonance: the momentum imparted by the wave train formed in the upper branch to excitations in the lower branch has a stabilizing effect when it has the same sign that the velocity of the wave train. We also showed that the system can experience second harmonic generation, and that this mechanism may be inhibited by symmetry effects (namely by the complete separation between density and polarization modes, which occurs when $k_0 = 0$).

Excitations in the lower branch are stable. In the long wavelength limit they are affected by nonlinear effects in a manner, which can generically be described by KdV dynamics. For some specific configuration of the system's parameters (close to the phase II–phase III boundary) one has to use a Gardner equation instead. It is interesting to note that the Gardner regime is realized in the lower branch, which is a mode mainly corresponding to density waves: hence, the wide range of nonlinear excitations of Gardner's equation (see, e.g., Ref. [45]) can be generated by means of a simple scalar external potential, whereas for noncoherently coupled two component condensates, where the Gardner regime is obtained for a polarization mode [44], this can be achieved only thanks to a polarization potential [46].

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APPENDIX A: SOLUTION OF EQ. (54)

In this Appendix we briefly explain how the solution of Eq. (54) is obtained. Let us assume that it is of the form

$$\overline{\Phi}^{(1)} = W(k) \int^X dX |\widetilde{\theta}^{(1)}|^2, \tag{A1}$$

where $W(k)$ is a constant [i.e., it depends on k, but not on (X,T_1,T_2)]. One first remarks that

$$\begin{aligned} \partial_{T_1} |\widetilde{\theta}^{(1)}|^2 &= \widetilde{\theta}^{(1)*} \partial_{T_1} \widetilde{\theta}^{(1)} + \text{c. c.} \\ &\overset{\text{Eq.(41)}}{=} -\omega'_+(k)\, \widetilde{\theta}^{(1)*} \partial_X \widetilde{\theta}^{(1)} + \text{c. c.} \\ &= -\omega'_+(k)\, \partial_X |\widetilde{\theta}^{(1)}|^2. \end{aligned} \tag{A2}$$

It follows from this result and from the *Ansatz* (A1) that $\partial_{T_1}^2 \overline{\Phi}^{(1)} = W(k)\, [\omega'_+(k)]^2 \partial_X |\widetilde{\theta}^{(1)}|^2$, and of course

$\partial_X^2 \overline{\Phi}^{(1)} = W(k)\,\partial_X|\widetilde{\theta}^{(1)}|^2$ [this is a direct consequence of (A1)]. Hence

$$\partial_{T_1}^2 \overline{\Phi}^{(1)} - c^2\,\partial_X^2 \overline{\Phi}^{(1)} = W(k)\,([\omega_+'(k)]^2 - c^2)\,\partial_X|\widetilde{\theta}^{(1)}|^2. \tag{A3}$$

Equating the right-hand side of this expression to the right-hand side of (54) determines the value of $W(k)$ as given in Eq. (57).

APPENDIX B: TAKING INTO ACCOUNT A SMALL DETUNING FROM THE RAMAN RESONANCE

In this Appendix we rapidly present the treatment of the lower excitation branch for a spin-orbit coupled condensate ($k_0 \neq 0$) in the case where the system experiences a finite detuning $\hbar\,\delta$ from the Raman resonance. The single-particle Hamiltonian H_0 (2) has now an additional contribution: $\frac{1}{2}\hbar\,\delta\,\sigma_z$. The system (7) is not modified, except for Eq. (7d), which now reads

$$\varphi_t = \frac{1}{2\sin\theta}\frac{(\rho\,\theta_x)_x}{\rho} - \frac{1}{2}\Phi_x(\varphi_x + 2k_0) + (\alpha_1 - \alpha_2)\rho\cos\theta - \Omega\,\cos\varphi\cot\theta + \delta. \tag{B1}$$

The ground-state value of the fields is no longer given by Eq. (9). One has now

$$\Xi^{(0)}(x,t) = \begin{pmatrix} 1 \\ 2k_1 x - 2\mu t \\ \theta_0 \\ 0 \end{pmatrix}, \tag{B2}$$

where k_1 is a variational parameter. Minimizing the energy per particle one obtains [22]

$$k_1 = k_0\cos\theta_0. \tag{B3}$$

The same result can be obtained in a different manner, by using dynamical arguments: one keeps k_1 as a free parameter, and one studies the linear excitations of the system. By demanding that the system is dynamically stable, i.e., that the frequency of elementary excitations remains real, one obtains the result (B3).

In the presence of a finite δ, the ground-state value θ_0 is no longer exactly equal to $-\pi/2$ as in (9) and μ is not given by Eq. (8). Instead one has [from (B1) and (7b)]

$$\left(2k_0^2 - g_2\right)\cos\theta_0 + \Omega\cot\theta_0 = \delta, \tag{B4}$$

and

$$\mu = \frac{k_0^2}{2}(1 + \cos^2\theta_0) + \frac{g_1}{2} + \frac{\Omega}{2\,\sin\theta_0}. \tag{B5}$$

Equation (B4) determines the value of θ_0. Depending on the system's parameters it has either four or two solutions. In the first case, only one corresponds to the minimum of the energy per particle (the other is the maximum) and the system can be considered to be in the single minimum phase III. In the second case there are two nonequivalent minima and the system is in phase II. In the regime where k_0^2 is larger than $g_2/2$ and where $\Omega > 0$, one can show that the boundary between these two regimes corresponds to

$$\left(2\,k_0^2 - g_2\right)^{2/3} = \Omega^{2/3} + |\delta|^{2/3}. \tag{B6}$$

In the case $\delta = 0$ the solution of (B4) is $\theta_0 = -\pi/2$ if $\Omega > 2k_0^2 - g_2$ and (B6) corresponds to the standard transition line between phases II and III, which is reproduced in Fig. 1. It is important to stress that in the presence of a finite detuning δ the second-order phase transition from phase III to phase II strictly speaking disappears because the system does not cross any phase transition line when Ω varies [47]. For instance, the velocity of sound vanishes at the transition region when $\delta = 0$ [cf. Eq. (18)], whereas it remains finite when $\delta \neq 0$ (cf. Fig. 8). Also, when $\delta \neq 0$, even in what has been identified above as the single minimum phase, the system has a small spin polarization and condensates into a state with a small but finite momentum.

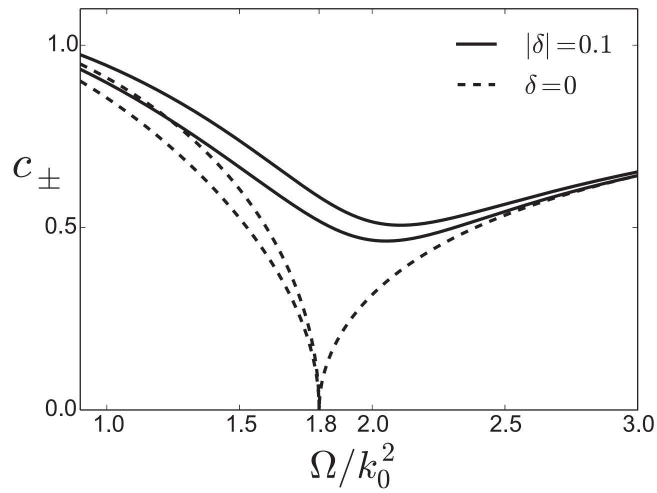

FIG. 8. Sound velocities $c_\pm$ as a function of Ω. The systems' parameters are $k_0 = 1$, $g_1 = 2.2$, and $g_2 = 0.2$. When $\delta = 0$, the transition from phase III to phase II occurs at $\Omega = 2k_0^2 - g_2 = 1.8$, and above $\Omega = 1.8$ one is in phase III with a single sound velocity. When $\delta \neq 0$ one can show that $c_+(-\delta) = c_-(\delta)$ and $c_-(-\delta) = c_+(\delta)$. This is the reason why the sign of δ is not specified in the figure.

The matrix $\mathbb{K}$ of Eq. (93) now reads

$$\mathbb{K} = \begin{pmatrix} c & -\frac{1}{2} & -k_0\sin_0 & 0 \\ -g_1 & c - k_0\cos_0 & \Omega\cos_0\sin_0^{-2} & 0 \\ -k_0\sin_0 & 0 & c - 2k_0\cos_0 & -\Omega \\ g_2\cos_0 & -k_0 & \Omega\sin_0^{-2} - g_2\sin_0 & 0 \end{pmatrix}, \tag{B7}$$

where, for gaining space, we have written $\sin_0$ and $\cos_0$ instead of $\sin\theta_0$ and $\cos\theta_0$. In formula (B7) the sound velocity c is not given by (18): it now depends on δ. It can be determined through the computation of the dispersion relation in the system, or more simply just by imposing the cancelation of $\det\mathbb{K}$. For nonzero δ the ground state breaks Galilean invariance and in our 1D configuration one obtains two velocities of sound, one for each direction of propagation. The first one, denoted as c_+, corresponds to waves propagating in the same direction as the ground state (for which $U_x^{(0)} = 2k_1$) and the other (c_-) propagates in the opposite direction. A typical case is displayed in Fig. 8. Note that this figure is interrupted at low values of Ω in order to prevent the system to get into phase I.

Following the procedure exposed in Sec. IV A one can determine the form of the KdV equation, which describes how long wavelength excitations propagating along the lower branch of the spectrum are affected by nonlinearity when $\delta \neq 0$. In this case the nonlinear parameter γ_1 is different from the value given by expression (106) (which corresponds to the $\delta = 0$ case). We do not write the explicit form of γ_1 when $\delta \neq 0$ because it is too cumbersome. Instead, we rather plot γ_1 as a function of Ω for different values of δ in Fig. 9.

As for the sound velocity, the value of the nonlinear coefficient depends of the direction of propagation of the wave. We denote as γ_1^+ (γ_1^-) the value of γ_1 corresponding to wave trains propagating in the same (the opposite) direction than the momentum of the ground state. One has the symmetry relation $\gamma_1^+(-\delta) = \gamma_1^-(\delta)$. When $\Omega < 2k_0^2 - g_2$ the nonlinear coefficient is discontinuous at $\delta = 0$, as θ_0 and k_1 are, and this corresponds to the crossing of the first-order transition line in the plane (Ω,δ) [47].

One sees in the figure that there exist values of δ for which the nonlinear coefficient cancels even for $\Omega > 2k_0^2 - g_2$, provided Ω is not too large. It is important to notice that this cancellation of γ_1 is obtained for a positive g_2, contrarily to what occurs when $\delta = 0$. Note however, that for $\Omega = 2.5$ the nonlinear coefficient cannot be canceled by imposing a finite value of the detuning δ. When γ_1 cancels, the effective nonlinear dynamics of the system is no longer described by a KdV equation, but rather by a Gardner equation.

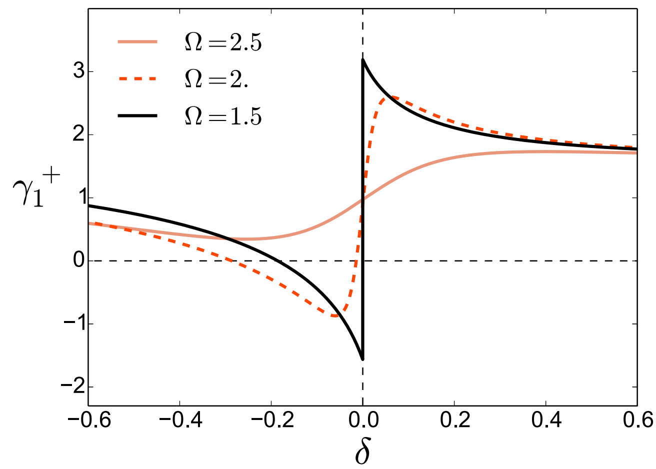

FIG. 9. γ_1^+ as a function of δ for several values of Ω. The system's parameters are $k_0 = 1$, $g_1 = 2.2$, and $g_2 = 0.2$. The curve is discontinuous when $\Omega < 2k_0^2 - g_2 = 1.8$ because in this case the system meets a first-order phase transition at $\delta = 0$. For the present choice of parameters γ_1^+ can be canceled by changing the value of δ when $\Omega = 1.5$ or 2.5, but not when $\Omega = 2.5$.

[1] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).

[2] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, Nature (London) **396**, 345 (1998); H.-J. Miesner, D. M. Stamper-Kurn, J. Stenger, S. Inouye, A. P. Chikkatur, and W. Ketterle, Phys. Rev. Lett. **82**, 2228 (1999); M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, *ibid.* **83**, 2498 (1999); Z. Dutton, M. Budde, C. Slowe, L. Vestergaard Hau, Science **293**, 663 (2001); N. S. Ginsberg, J. Brand, and L. V. Hau, Phys. Rev. Lett. **94**, 040403 (2005); L. E. Sadler, J. M. Higbie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Nature (London) **443**, 312 (2006); K. C. Wright, L. S. Leslie, A. Hansen, and N. P. Bigelow, Phys. Rev. Lett. **102**, 030405 (2009); D. M. Weld, P. Medley, H. Miyake, D. Hucul, D. E. Pritchard, and W. Ketterle, *ibid.* **103**, 245301 (2009); L. S. Leslie, A. Hansen, K. C. Wright, B. M. Deutsch, and N. P. Bigelow, *ibid.* **103**, 250401 (2009).

[3] M.-S. Chang, Q. Qin, W. Zhang, L. You, and M. S. Chapman, Nature Phys. **1**, 111 (2005); T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, Phys. Rev. Lett. **105**, 204101 (2010).

[4] P. Böhi, M. F. Riedel, J. Hoffrogge, J. Reichel, T. W. Hänsch, and P. Treutlein, Nature Phys. **5**, 592 (2009); C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature (London) **464**, 1165 (2010); M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, *ibid.* **464**, 1170 (2010); C. Gross, H. Strobel, E. Nicklas, T. Zibold, N. Bar-Gill, G. Kurizki, and M. K. Oberthaler, *ibid.* **480**, 219 (2011).

[5] S. Palzer, C. Zipkes, C. Sias, and M. Köhl, Phys. Rev. Lett. **103**, 150601 (2009); T. Fukuhara, A. Kantian, M. Endres, M. Cheneau, P. Schauß, S. Hild, D. Bellem, U. Schollwöck, T. Giamarchi, C. Gross, I. Bloch, and S. Kuhr, Nature Phys. **9**, 235 (2013).

[6] S. Beattie, S. Moulder, R. J. Fletcher, and Z. Hadzibabic, Phys. Rev. Lett. **110**, 025301 (2013).

[7] Y.-J. Lin, R. L. Compton, A. R. Perry, W. D. Phillips, J. V. Porto, and I. B. Spielman, Phys. Rev. Lett. **102**, 130401 (2009); Z. Fu, P. Wang, S. Chai, L. Huang, and J. Zhang, Phys. Rev. A **84**, 043609 (2011); M. W. Ray, E. Ruokokoski, S. Kandel, M. Möttönen, and D. S. Hall, Nature (London) **505**, 657 (2014).

[8] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature (London) **471**, 83 (2011); J.-Y. Zhang, S.-C. Ji, Z. Chen, L. Zhang, Z.-D. Du, B. Yan, G.-S. Pan, B. Zhao, Y.-J. Deng, H. Zhai, S. Chen, and J.-W. Pan, Phys. Rev. Lett. **109**, 115301 (2012); S.-C. Ji, J.-Y. Zhang, L. Zhang, Z.-D. Du, W. Zheng, Y.-J. Deng, H. Zhai, S. Chen, and J.-W. Pan, Nature Phys. **10**, 314 (2014).

[9] L. J. LeBlanc, K. Jiménez-García, R. A. Williams, M. C. Beeler, A. R. Perry, W. D. Phillips, and I. B. Spielman, Proc. Natl. Acad. Sci. USA **109**, 10811 (2012).

[10] C. Qu, C. Hamner, M. Gong, C. Zhang, and P. Engels, Phys. Rev. A **88**, 021604(R) (2013); L. J. LeBlanc, M. C. Beeler, K. Jiménez-García, A. R. Perry, S. Sugawa, R. A. Williams, and I. B. Spielman, New J. Phys. **15**, 073011 (2013).

[11] M. C. Beeler, R. A. Williams, K. Jiménez-García, L. J. LeBlanc, A. R. Perry, and I. B. Spielman, Nature (London) **498**, 201 (2013).

[12] A. J. Olson, S.-J. Wang, R. J. Niffenegger, C.-H. Li, C. H. Greene, and Y. P. Chen, Phys. Rev. A **90**, 013616 (2014).

[13] C. Hamner, C. Qu, Y. Zhang, J. Chang, M. Gong, C. Zhang, and P. Engels, Nature Comm. **5**, 4023 (2014).

[14] M. A. Khamehchi, Y. Zhang, C. Hamner, T. Busch, and P. Engels, Phys. Rev. A **90**, 063624 (2014); S.-C. Ji, L. Zhang, X.-T. Xu, Z. Wu, Y. Deng, S. Chen, and J.-W. Pan, Phys. Rev. Lett. **114**, 105301 (2015).

[15] M. R. Matthews, D. S. Hall, D. S. Jin, J. R. Ensher, C. E. Wieman, E. A. Cornell, F. Dalfovo, C. Minniti, and S. Stringari, Phys. Rev. Lett. **81**, 243 (1998); D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *ibid.* **81**, 1543 (1998).

[16] K. C. Wright, L. S. Leslie, and N. P. Bigelow, Phys. Rev. A **77**, 041601(R) (2008); **78**, 053412 (2008); J. Higbie and D. M. Stamper-Kurn, Phys. Rev. Lett. **88**, 090401 (2002).

[17] P. B. Blakie, R. J. Ballagh, and C. W. Gardiner, J. Opt. B **1**, 378 (1999); C. P. Search and P. R. Berman, Phys. Rev. A **63**, 043612 (2001); P. Tommasini, E. J. V. de Passos, A. F. R. de Toledo Piza, M. S. Hussein, and E. Timmermans, *ibid.* **67**, 023606 (2003); C. Lee, W. Hai, L. Shi, and K. Gao, *ibid.* **69**, 033611 (2004); M. Abad and A. Recati, Eur. Phys. J. D **67**, 148 (2013).

[18] C. Wang, C. Gao, C.-M. Jian, and H. Zhai, Phys. Rev. Lett. **105**, 160403 (2010); T.-L. Ho and S. Zhang, *ibid.* **107**, 150403 (2011); Y. Zhang, L. Mao, and C. Zhang, *ibid.* **108**, 035302 (2012); W. Zheng and Z. Li, Phys. Rev. A **85**, 053607 (2012); Y. Li, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **108**, 225301 (2012); G. I. Martone, Y. Li, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **86**, 063621 (2012); Y. Zhang, G. Chen, and C. Zhang, Sci. Rep. **3**, 1937 (2013); W. Zheng, Z.-Q. Yu, X. Cui, and H. Zhai, J. Phys. B **46**, 134007 (2013); Y. Li, G. I. Martone, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **110**, 235302 (2013). S. Maronov, M. Modugno, and E. Y. Sherman, J. Phys. B: At. Mol. Opt. Phys. **48**, 115302 (2015).

[19] D. T. Son and M. A. Stephanov, Phys. Rev. A **65**, 063621 (2002); J. J. García-Ripoll, V. M. Pérez-García, and F. Sols, *ibid.* **66**, 021602(R) (2002); K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. Lett. **93**, 250406 (2004); B. Deconinck, P. G. Kevrekidis, H. E. Nistazakis, and D. J. Frantzeskakis, Phys. Rev. A **70**, 063605 (2004); V. A. Brazhnyi and V. V. Konotop, Phys. Rev. E **72**, 026616 (2005); I. M. Merhasin, B. A. Malomed, and R. Driben, J. Phys. B **38**, 877 (2005); K. Nakamura, A. Kohi, H. Yamasaki, V. M. Pérez-García, and V. V. Konotop, Europhys. Lett. **80**, 50005 (2007).

[20] M. Merkl, A. Jacob, F. E. Zimmer, P. Öhberg, and L. Santos, Phys. Rev. Lett. **104**, 073603 (2010); X.-Q. Xu and J. H. Han, *ibid.* **107**, 200401 (2011); T. Kawakami, T. Mizushima, M. Nitta, and K. Machida, *ibid.* **109**, 015301 (2012); O. Fialko, J. Brand, and U. Zülicke, Phys. Rev. A **85**, 051605(R) (2012); V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, and D. E. Pelinovsky, Phys. Rev. Lett. **110**, 264101 (2013); V. Achilleos, J. Stockhofe, P. G. Kevrekidis, D. J. Frantzeskakis, and P. Schmelcher, Europhys. Lett. **103**, 20002 (2013); S. Maronov, E. Y. Sherman, J. G. Muga, H.-W. Wang, Y. Ban, and X. Chen, Phys. Rev. A **91**, 043604 (2015).

[21] C. Polymilis, K. Hizanidis, and D. J. Frantzeskakis, Phys. Rev. E **58**, 1112 (1998); P. L. Christiansen, J. C. Eilbeck, V. Z. Enolskii, and N. A. Kostov, Proc. R. Soc. London A **456**, 2263 (2000); C. Eilbeck, V. Z. Enolskii, and N. A. Kostov, J. Math. Phys. **41**, 8236 (2000); A. M. Kamchatnov and V. V. Sokolov, Phys. Rev. A **91**, 043621 (2015).

[22] Y. Li, G. I. Martone, and S. Stringari, *Annual Review of Cold Atoms and Molecules*, Vol. III, edited by K. W. Madison, K. Bongs, L. D. Carr, A. M. Ray, and H. Zhai (World Scientific, Singapore, 2015), p. 201.

[23] K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A **71**, 043611 (2005).

[24] The value $\theta_0 = +\pi/2$ also gives equal densities of the two components, but is unstable when $\Omega > 0$.

[25] A. Jeffrey and T. Kawahara, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982).

[26] T. Taniuti and K. Nishihara, *Nonlinear Waves* (Pitman, Boston, 1983).

[27] A. C. Newell, *Solitons in Mathematics and Physics* (Society for Industrial and Applied Mathematics, Philadelphia, 1985).

[28] A. M. Kamchatnov, *Nonlinear Periodic Waves and Their Modulations - An Introductory Course* (World Scientific, Singapore, 2000).

[29] M. J. Ablowitz, *Nonlinear Dispersive Waves, Asymptotic Analysis and Solitons* (Cambridge University Press, Cambridge, 2011).

[30] The bar is not a complex conjugate. The complex conjugate is denoted in this work with a superscript *.

[31] This is not always true: in some instances the determinant of $\mathbb{M}_2$ cancels. This occurs in the presence of second harmonic generation, as discussed in section III D [after Eqs. (80) and (81)].

[32] We use the simple property that the image space of a given matrix $\mathbb{M}$ is orthogonal—in the sense of the usual scalar product—to the kernel of $\mathbb{M}^t$, to which L belongs by (39). This property is easily demonstrated in $\mathbb{R}^n$: Let C be a column vector in the image space of a $n \times n$ matrix $\mathbb{M}$; this means that there exists a column vector V such that $C = \mathbb{M}V$. Let L be a column vector in the kernel of $\mathbb{M}^t$. This means that $\mathbb{M}^t L = 0 \Leftrightarrow L^t\mathbb{M} = 0$. Then it is clear that the scalar product $L^t C = 0$ (since it reads $L^t\mathbb{M}C$).

[33] From Eqs. (24) and (41) one gets $\partial_{T_2}\widetilde{\theta}^{(1)} = \epsilon^{-2}\partial_t\widetilde{\theta}^{(1)} - \epsilon^{-1}\partial_{T_1}\widetilde{\theta}^{(1)} = \epsilon^{-2}\partial_t\widetilde{\theta}^{(1)} + \epsilon^{-1}\omega'_+(k)\partial_X\widetilde{\theta}^{(1)}$.

[34] V. E. Zakharov and L. A. Ostrovsky, Physica D **238**, 540 (2009).

[35] V. V. Konotop and M. Salerno, Phys. Rev. A **65**, 021602(R) (2002); A. Smerzi, A. Trombettoni, P. G. Kevrekidis, and A. R. Bishop, Phys. Rev. Lett. **89**, 170402 (2002); B. Wu and Q. Niu, New J. Phys. **5**, 104 (2003); M. Machholm, C. J. Pethick, and H. Smith, Phys. Rev. A **67**, 053613 (2003); M. Cristiani, O. Morsch, N. Malossi, M. Jona-Lasinio, M. Anderlini, E. Courtade, and E. Arimondo, Opt. Express **12**, 4 (2004); L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. Lett. **93**, 140406 (2004); L. De Sarlo, L. Fallani, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. A **72**, 013603 (2005).

[36] C. K. Law, C. M. Chan, P. T. Leung, and M.-C. Chu, Phys. Rev. A **63**, 063612 (2001); M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, *ibid.* **84**, 041605(R) (2011); C. Hamner, J. J. Chang, P. Engels, and M. A. Hoefer, Phys. Rev. Lett. **106**, 065302 (2011).

[37] T. Brooke Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).

[38] D. J. Benney, Stud. Appl. Math. **56**, 81 (1976); R. H. J. Grimshaw, *ibid.* **56**, 241 (1977).

[39] Y. S. Kivshar and B. Luther-Davies, Phys. Rep. **298**, 81 (1998).

[40] C. J. Pethick and H. Smith, *Bose-Einstein condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2002).

[41] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, 2003).

[42] M. Tsutsumi, T. Mukasa, and R. Iino, Proc. Japan Acad. **46**, 921 (1970).

[43] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998); A. D. Jackson, G. M. Kavoulakis, and C. J. Pethick, Phys. Rev. A **58**, 2417 (1998); P. Leboeuf and N. Pavloff, *ibid.* **64**, 033602 (2001).

[44] A. M. Kamchatnov, Y. V. Kartashov, P.-É. Larré and N. Pavloff, Phys. Rev. A **89**, 033618 (2014).

[45] A. M. Kamchatnov, Y.-H. Kuo, T.-C. Lin, T.-L. Horng, S.-C. Gou, R. Clift, G. A. El, and R. H. J. Grimshaw, Phys. Rev. E **86**, 036605 (2012).

[46] A. M. Kamchatnov and Y. V. Kartashov, Phys. Rev. Lett. **111**, 140402 (2013).

[47] The situation is similar to the phase transition in a pure ferromagnet: in the absence of external field the system has a line of first-order transition, which terminates with a critical point. For fixed finite external magnetic field, one does not cross this line by varying temperature. It can be crossed at fixed temperature by varying the external field. In our case, the equivalent line corresponds in the (Ω,δ) plane to the segment $\delta = 0$, $|\Omega| \leqslant 2k_0^2 - g_2$, which is a line of first-order phase transition ending with a critical point at $\delta = 0$ and $|\Omega| = 2k_0^2 - g_2$. This line cannot be crossed at fixed nonzero δ by changing Ω, but can be at fixed Ω by changing δ.