
ADVANCED NONLINEAR PHYSICS

3 hours. The two exercises A and B are related to the two parts of the course and are independent. A corresponds to the first part of the course, B to the second part. Written notes of the courses and dictionaries are allowed. Books as well as computers, telephone and other electronic devices are forbidden.

A Bogomolny bound

The purpose of this problem is to show that, in non-trivial configurations, the solution with lowest energy of a wide class of non-linear partial differential equations corresponds to a soliton[1].

One considers the one dimensional nonlinear partial differential equation

$$\tfrac{1}{c^2}\phi_{tt} - \phi_{xx} + u(\phi) = 0 \,, \tag{A1}$$

where ϕ is a real field, and $u(\phi)$ is a real function. Up to a redefinition of the variable t one can take $c \equiv 1$. This will be assumed in all the following. One assumes that there exists a primitive function $\mathscr{U}(\phi)$ of $u(\phi)$ such that[2]

$$\mathscr{U}(\phi) \geq 0 \quad \text{for all} \quad \phi \in \mathbb{R} \,. \tag{A2}$$

Equation (A1) is supplemented by the boundary conditions $\phi(x \to \pm\infty, t) = \phi_{(\pm)}$, where $\phi_{(-)}$ and $\phi_{(+)}$ are fixed real constants.

1. Show that the function $\mathscr{U}(\phi)$ reaches an extremum for $\phi = \phi_{(+)}$ and also for $\phi = \phi_{(-)}$.
2. Let's define the fields ε and S by

   $$\varepsilon(x,t) = \tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 + \mathscr{U}(\phi(x,t)) \,, \quad S(x,t) = -\phi_t\,\phi_x \,. \tag{A3}$$

   Verify that $\varepsilon(x,t)$ and $S(x,t)$ obey a conservation equation of the form: $\varepsilon_t + S_x = 0$. Show then that the quantity

   $$E = \int_{-\infty}^{\infty} \mathrm{d}x\, \varepsilon(x,t) \,, \tag{A4}$$

   does not depend on time. We will denote E as the "energy of the system".
3. Show that

   $$\tfrac{1}{2}\phi_x^2 + \mathscr{U}(\phi) \geq \pm\phi_x \sqrt{2\,\mathscr{U}(\phi)} \,. \tag{A5}$$

   Deduce from this result that the energy of the system is bounded from below:

   $$E \geq E_{\mathrm{B}} = \left| \int_{\phi_{(-)}}^{\phi_{(+)}} \mathrm{d}\phi\, \sqrt{2\,\mathscr{U}(\phi)} \right| = \int_{\min\{\phi_{(-)},\phi_{(+)}\}}^{\max\{\phi_{(-)},\phi_{(+)}\}} \mathrm{d}\phi\, \sqrt{2\,\mathscr{U}(\phi)} \,, \tag{A6}$$

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[1] In the generic case, these non-trivial solutions have a non-trivial topological structure. In our simplified unidimensional treatment, the asymptotic boundary conditions at $x \to \pm\infty$ play the role of a topological constrain.

[2] This is always possible since $\mathscr{U}(\phi)$ is defined up to a constant (unless pathological case of diverging $\mathscr{U}(\phi) \to -\infty$ which will not be considered here).

$E_{\rm B}$ is called the "Bogomolny bound". Since $\mathscr{U}(\phi)$ is defined up to an additive constant, there is some latitude in the choice of the bound $E_{\rm B}$. Its lower value compatible with the condition (A2) is obtained when $\mathscr{U}$ is chosen so that its minimum value $\mathscr{U}_{\rm min} \equiv \min\{\mathscr{U}(\phi)\}_{\phi\in\mathbb{R}}$ is 0, which will be assumed henceforth.

4. One looks for a configuration where the Bogomolny bound is reached, i.e., $E = E_{\rm B}$. In this case show that ϕ is time-independent and verifies one of the two following first order differential equations
$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \pm\sqrt{2\,\mathscr{U}(\phi)}\ . \tag{A7}$$
A solution of (A7) with the "+" sign is called a "kink", a solution with the negative sign is an "anti-kink". Show that one can pass from one sign to the other by a simple transformation.

5. Rewrite Eqs. (A7) as an effective conservation equation for the total mechanical energy of a fictitious classical particle of "mass" unity, "position" ϕ, "time" x. What is the potential experienced by the fictitious particle ? What is the total classical mechanical energy of the fictitious particle ?

6. One first considers the case where $\mathscr{U}(\phi)$ reaches its minimum $\mathscr{U}_{\rm min} = 0$ for a single value ϕ_0 of ϕ. For which solution of (A1) and (A7) is the Bogomolny bound reached ?

7. One now considers the case where $\mathscr{U}(\phi)$ reaches its minimum $\mathscr{U}_{\rm min} = 0$ for two values ϕ_0 and ϕ_1 ($\phi_1 > \phi_0$). Draw the corresponding effective potential. Show that for a particular choice of the boundary conditions $\phi_{(+)}$ and $\phi_{(-)}$ one can obtain a nontrivial[3] solution of (A1) and (A7) which reaches the Bogomolny bound. Sketch the corresponding $\phi(x)$.

8. For being specific one considers the case
$$\mathscr{U}(\phi) = (\phi^2 - m^2)^2\ . \tag{A8}$$
where m is a positive constant.

   Give the explicit form of the kink solution $\phi(x)$ [hint: $\int \mathrm{d}x\,(1-x^2)^{-1} = \mathrm{artanh}(x)$]. Compute the corresponding energy.

[3]This solution is "nontrivial" in the sense that it is less boring than the one studied in the previous question 6/.