Bogomolnyi bound

$$\phi_{tt} - \phi_{xx} + u(\phi) = 0$$

$$\mathcal{E} = \frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + \mathcal{U}(\phi) \qquad S = -\phi_t\phi_x$$

$$\mathcal{E}_t + S_x = \phi_t\phi_{tt} + \cancel{\phi_x\phi_{xt}} + u(\phi)\phi_t - \phi_{xx}\phi_t - \cancel{\phi_{tx}\phi_x}$$

$$= \phi_t(\phi_{tt} - \phi_{xx} + u(\phi)) = 0$$

let's consider the time derivative of $E = \int_{\mathbb{R}} dx\, \mathcal{E}$

$$\frac{dE}{dt} = \int_{-\infty}^{+\infty} dx\, \mathcal{E}_t = -\int_{-\infty}^{+\infty} dx\, S_x = S(-\infty,t) - S(+\infty,t)$$

and at $x \to \pm\infty$ both ϕ_x and $\phi_t \to 0$ this $S(\pm\infty,t) = 0$

$\frac{dE}{dt} = 0 \Rightarrow E$ is a conserved quantity

By the way: it is clear that ϕ_{tt} and $\phi_{xx} \xrightarrow[x\to\pm\infty]{} 0$ (since $\phi \to C^{st}$)
this implies that $u(\phi_\pm) = 0 \Rightarrow \phi_\pm$ are extrema of $\mathcal{U}$

l'inégalité (AS) résulte de $\left(\frac{1}{\sqrt{2}}\phi_x \pm \sqrt{U}\right)^2 \geq 0$
then one can write:

$$E = \int_{-\infty}^{\infty} dx\, \mathcal{E}(x,t) \geq \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\phi_x^2 + U\right) \geq \pm \int_{-\infty}^{\infty} dx\, \phi_x \sqrt{2U(\phi)}$$

(since $\phi_t^2 > 0$) (with AS)

in this integration define $\phi = \phi(x)$ as the new variable of integration - this yields:

$$E \geq \pm \int_{\phi_-}^{\phi_+} d\phi \sqrt{2U(\phi)} \quad \text{thus} \quad E \geq \left|\int_{\phi_-}^{\phi_+} d\phi \sqrt{2U(\phi)}\right| = \int_{\min(\phi_+,\phi_-)}^{\max(\phi_+,\phi_-)} d\phi \sqrt{2U(\phi)}$$

since the integrand $\sqrt{2U}$ is > 0 the integral is > 0 when the boundaries of integration are arranged in ascending order

☒ the bound E_B is reached when:

$$E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\phi_x^2 + U \right) = \pm \int_{\mathbb{R}} dx \, \phi_x \sqrt{2U(\phi)}$$

this imposes $\phi_t \equiv 0$

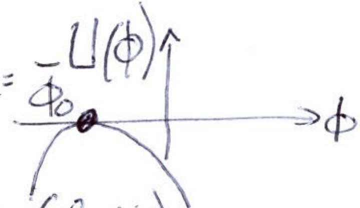
this is realized <u>only</u> in the case of equality in (A5), i.e. when $\left(\frac{1}{\sqrt{2}}\phi_x \pm \sqrt{U} \right)^2 = 0$

this yields $\boxed{\frac{d\phi}{dx} = \pm\sqrt{2U(\phi)}}$

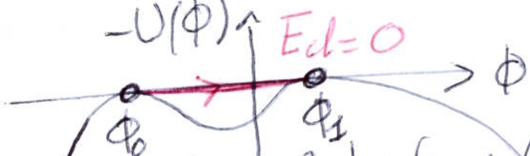
$\pm$ = kink or anti-kink = one can pass from one solution to the other by changing $x \to -x$ and $\phi_+ \leftrightarrow \phi_-$.

eqs. (A7) can be cast under the form $\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - U(\phi) = 0$ which is the conservation of mechanical energy of a fictitious particle experiencing a potential $-U$ and have a total mechanical energy $E_{cl} = 0$

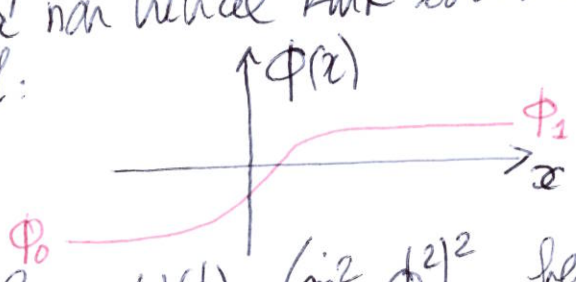
- boring case = U has a single minimum, then:
  the bound is reached
  for $\phi(x,t) = \phi_0$, the energy is zero (cf A4).



- interesting case:



if one takes $\phi_- = \phi_0$ and $\phi_+ = \phi_1$ one has a non trivial kink solution which reaches the Bogomolny bound:



$\phi_1 = +m$
$\phi_0 = -m$
here (cf the above sketch)

- to be more specific: $U(\phi) = (m^2 - \phi^2)^2$
  the kink solution of (A7) is solution of: $\frac{d\phi}{dx} = \sqrt{2}(m^2 - \phi^2)$
  write $\phi = m \, \mathrm{th}\,\theta$, $d\phi = m(1 - \mathrm{th}^2\theta)\, d\theta$ then $\frac{d\theta}{dx} = m\sqrt{2}$
  thus $\phi(x) = m \, \mathrm{th}\left(\sqrt{2}\, m(x - x_0)\right)$ - the corresponding energy is

$$E = E_B = \int_{-m}^{m} d\phi \sqrt{2U(\phi)} = 2\sqrt{2} \int_0^m d\phi (m^2 - \phi^2) = \frac{4}{3}\sqrt{2}\, m^3$$