
ADVANCED NONLINEAR PHYSICS

Duration: 3 hours. Exercice A corresponds to the first part of the course, exercices B and C to the second part. They are all independent of each other. Please use different sheets for writing up the solution of each exercice.

Dictionaries, handwritten notes on the courses and tutorials are allowed. Books as well as computers, telephones and other electronic devices are forbidden.

A Traffic flow

One considers a traffic flow problem for which the flux-density relation $Q(\rho)$ is of the form depicted in Fig. 1. In the following, one denotes $c(\rho) = \mathrm{d}Q/\mathrm{d}\rho$, $c_0 = c(0)$ and $c_j = -c(\rho_j)$. This exercice aims at studying the time evolution of the initial profile represented in the right panel of Fig. 1.

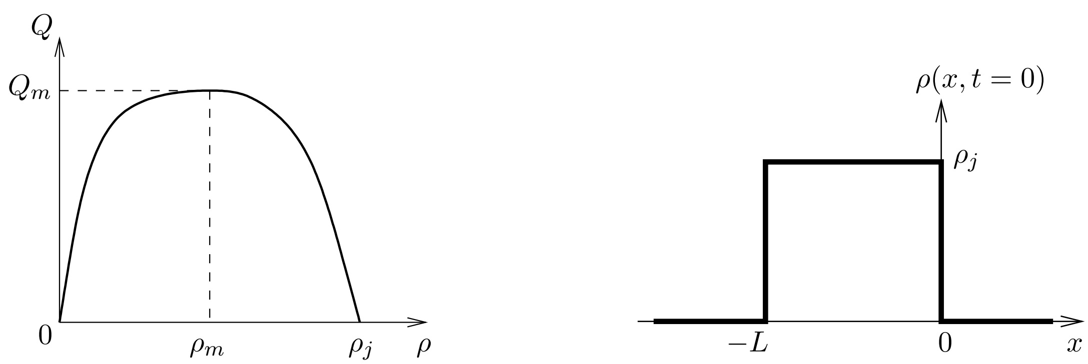

Figure 1: *Left panel: Sketch of the flux-density relation. Right panel: initial vehicle distribution. ρ_j is the traffic jam density.*

1/ Describe (without using any mathematics) the time evolution of this profile, for times "not too large".

2/ Write the partial differential equation which governs the time evolution of $\rho(x, t)$. Solve it for $t < t_j = L/c_j$, i.e., give the explicit expression of $\rho(x, t)$ (you will need to introduce the inverse function of $c = c(\rho)$, use the notation $\rho = f(c)$ for this function). Compute the number of vehicle in the rarefaction wave region as a function of t, c_j and ρ_j.

3/ What happens at $t = t_j$? For solving the problem at later time, it is convenient to use an explicit form of the flux-density relation: one considers the case where

$$Q(\rho) = V_m(\rho - \rho^2/\rho_j) \ , \quad \text{with} \quad \rho \in [0, \rho_j] \ . \tag{A1}$$

Express Q_m as a function of V_m. What is the physical significance of V_m ?

4/ One denotes as $X(t)$ the position of the trailing shock[1]. Write the differential equation which governs its time evolution for $t \geq t_j$. Integrate this equation[2].

Check that the values you obtain for $X(t)$ and for the whole density profile $\rho(x, t > t_j)$ conserve the number of vehicles, i.e., that $\int_{\mathbb{R}} \mathrm{d}x \, \rho(x, t) = L\rho_j$.

[1] En français: "Choc à l'arrière de la distribution".

[2] Hint: you may use the change of variable $y(t) = X(t)/(V_m t)$.