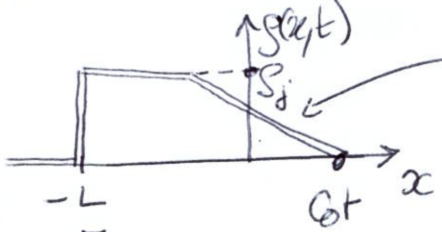
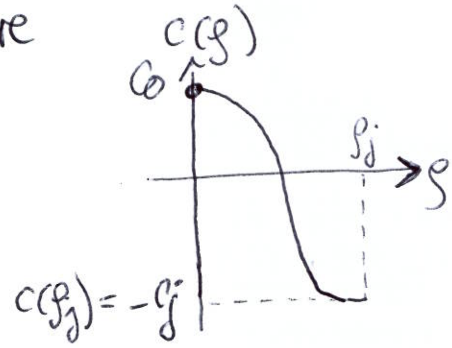
TRAFFIC FLOW

1/ vehicles around $x=0$ move ahead. The first ones have velocity c_0. The vehicles at the rear are not moving. One thus has a density profile of the type:

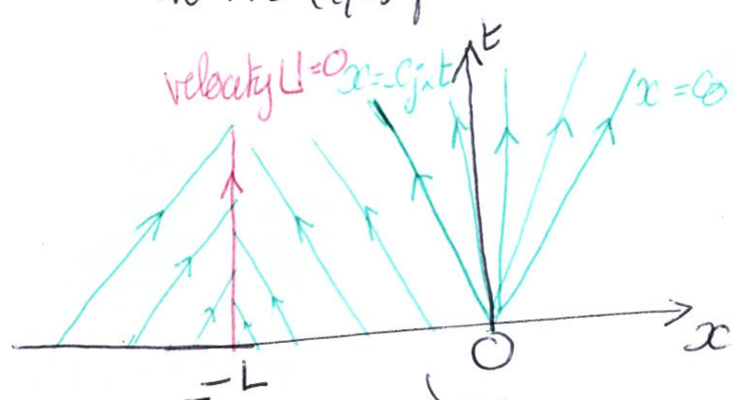


a rarefaction wave is formed ahead, and we will see that the motionless rear end can be conside-red as a zero velocity shock.

2/ the vehicle's conservation law: $\rho_t + Q_x = 0$ yields $\rho_t + c(\rho)\rho_x = 0$ where



let's draw the characteristics in the (x,t) plane:

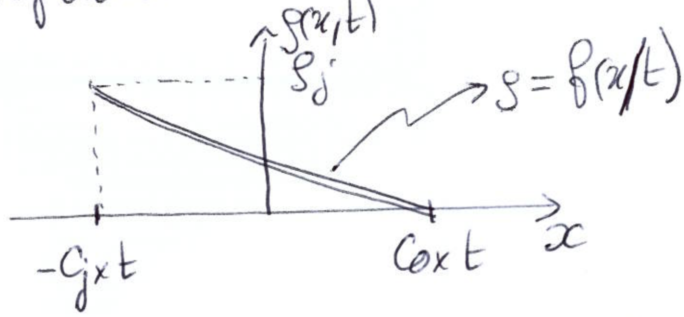


the velocity at the rear is: $U = \dfrac{Q(\rho_j) - Q(0)}{\rho_j - 0} = 0$

for the rarefaction wave:

$$\begin{cases} x/t \geq c_0 : & \rho \equiv 0 \\ -c_j \leq \dfrac{x}{t} \leq c_0 : & \left[\begin{array}{l} c(\rho) = \dfrac{x}{t} \\ \Leftrightarrow \rho = f\left(\dfrac{x}{t}\right) \end{array}\right. \end{cases}$$

the region of the rarefaction wave is:



it contains a number of vehicles $N_{RW} = \int_{-tc_j}^{tc_0} \rho \, dx$. Since these vehicles where initially at density ρ_j between $-c_j t$ and 0 one should have $N_{RW} = t c_j \rho_j$

let's check it directly: one makes a change of variable $\rho = \rho(x)$ since in the RW $c(\rho) = x/t$ one has:

$$dx = t \frac{dc}{d\rho} d\rho$$

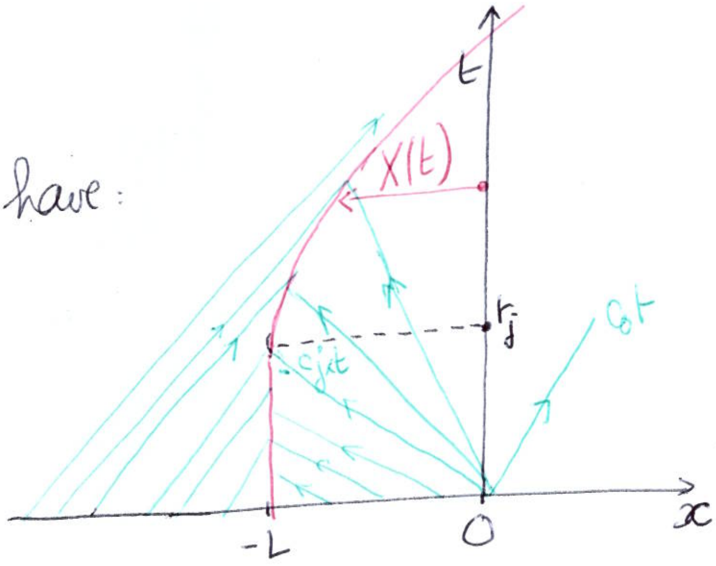
hence $N_{RW} = \int_{-tc_j}^{tc_0} \rho\, dx = t\int_{\rho_j}^{0} \rho \frac{dc}{d\rho} d\rho \underset{\substack{\uparrow \\ \text{integrating} \\ \text{by parts}}}{=} t\left[\rho\, c(\rho)\right]_{\rho_j}^{0} - t \underbrace{\int_{\rho_j}^{0} c(\rho) d\rho}_{\left[\varphi(\rho)\right]_{\rho_j}^{0} = 0}$

thus $N_{RW} = -t \rho_j\, c(\rho_j) = t\, c_j \rho_j$ as expected.

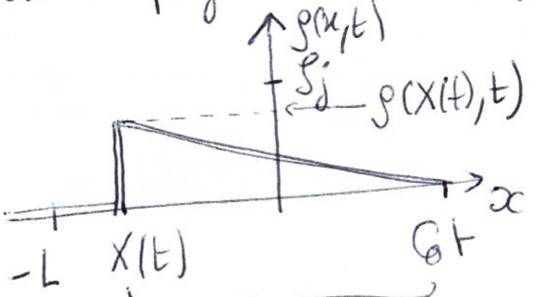
3/ at $t_j = L/c_j$ the rear end of the rarefaction wave reaches $x = -L$. Then, the rear shock begins to move, at velocity $\frac{dX}{dt}$ given by:

$$\frac{dX}{dt} = \frac{Q(\rho(X,t))}{\rho(X,t)}$$

and in the (x,t) plane one will have:



with the profile (at $t > t_j$):



in this region $X(t) \leq x \leq c_0 \times t$

and $\rho = f(x/t) \Leftrightarrow \frac{x}{t} = c(\rho) \overset{\text{here}}{=} V_m(1 - 2\rho/\rho_j)$

this reads

$$\rho = \frac{1}{2}\rho_j\left(1 - \frac{x}{V_m \times t}\right)$$

the corresponding $\varphi(\rho)$ reads:

$$\varphi(\rho) = V_m\, \rho\,(1 - \rho/\rho_j) = V_m \frac{\rho_j}{2}\left(1 - \frac{x}{V_m t}\right)\left[1 - \frac{1}{2}\left(1 - \frac{x}{V_m \times t}\right)\right]$$

$$= \frac{1}{4} V_m \rho_j \left(1 - \left(\frac{x}{V_m t}\right)^2\right)$$

the corresponding equation $\frac{dX}{dt} = \frac{\varphi(\rho(X(t),t))}{\rho(X(t),t)}$ reads:

$\frac{dX}{dt} = \frac{1}{2} V_m \left(1 + \frac{X}{V_m t}\right)$ let's define $y = \frac{X}{V_m t}$

one has $\frac{dy}{dt} = \frac{dX/dt}{V_m t} - \frac{X}{V_m t^2} = \frac{1}{2}\frac{V_m(1+y)}{V_m t} - y/t$

hence $2t\frac{dy}{dt} = 1 - y$ at $t = t_j$ one has $X = -L$ and here $c_j = V_m$ hence $t_j = \frac{L}{V_m}$, $y(t_j) = -1$

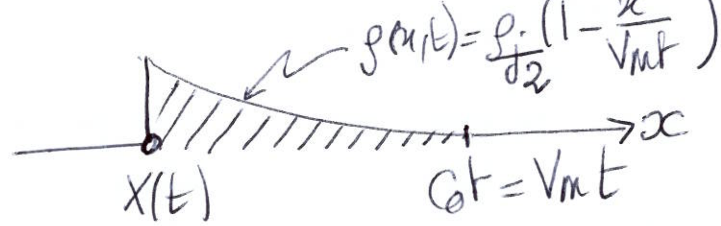
integration: $\frac{dy}{1-y} = \frac{dt}{2t}$ hence $\begin{cases} 1 - y = \frac{K}{\sqrt{t}} \\ \text{with } K = 2\sqrt{\frac{L}{V_m}} \end{cases}$

this yields $\boxed{X = V_m t\left(1 - 2\sqrt{\frac{L}{V_m t}}\right)}$

$X = 0$ when $t = \frac{4L}{V_m} = 4 t_j$

at large t one has $X \simeq V_m t = c_0 t$ (in our model)
this is expected from the figure of the characteristics on the previous page.

one has a profile:



= straight line. hence the total number of vehicles is = (area of the triangle)

$N_{tot} = \frac{1}{2}(V_m t - X(t)) \times \rho(X(t),t)$

this yields: $N_{tot} = \frac{1}{2}\, 2\sqrt{L V_m t} \times \frac{1}{2}\rho_j \left(2\sqrt{\frac{L}{V_m t}}\right) = \rho_j L$ as it should.

<u>Note</u>: using the conservation of the # of particles one could have determined $X(t)$ without solving a differential equation.