Tutorial # 1 : Method of characteristics and traffic flow

1 Solve the following:

(a) $\phi_t + \mathrm{e}^{-t}\phi_x = 0$ for $x \in \mathbb{R}$ and $t > 0$. At $t = 0$, $\phi = (1 + x^2)^{-1}$.

(a') $\phi_t + \mathrm{e}^{t}\phi_x = 0$ for $x \in \mathbb{R}$ and $t > 0$. At $t = 0$, $\phi = (1 + x^2)^{-1}$.

(b) $\phi_t + c_0\phi_x + \alpha\phi = 0$ for $x \in \mathbb{R}$ and $t > 0$. c_0 and α positive constants. At $t = 0$, $\phi = \phi_0(x)$.

(c) $\phi_t + t^2\phi_x + x\phi = 0$ for $x \in \mathbb{R}$ and $t > 0$. At $t = 0$, $\phi = \phi_0(x)$.

(d) Same equation as above, but region $x \geq 0$, $t \geq 0$.
At $t = 0 : \phi = f(x)$ $(x \geq 0)$ and at $x = 0 : \phi = g(t)$ $(t \geq 0)$

(e) $\phi_t + \phi\phi_x + \alpha\phi = 0$ for $x \in \mathbb{R}$ and $t > 0$. α positive constant. At $t = 0$, $\phi = \phi_0(x)$ which looks like $(1 + x^2)^{-1}$ for instance.

Show that breaking needs not always to occur, i.e., the solution is single-valued for all t in some cases.

2 Traffic flow

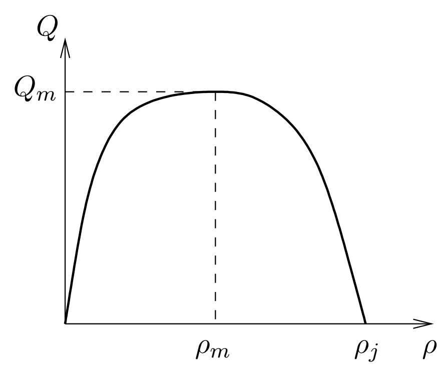

Figure 1: The following exercices are dedicated to the study of model traffic flow situations. Unless specified, one will consider a relation flux-density of vehicles of the generic form:

2.1

Assume an initial car distribution of the form ($a > 0$):

$$\rho(x, t = 0) = \begin{cases} \rho_j & \text{for} \quad x < 0 \ , \\ \rho_m & \text{for} \quad 0 < x < a \ , \\ 0 & \text{for} \quad a < x \ . \end{cases}$$

Describes what happens for $t > 0$. Draw the corresponding characteristics in the (x, t) plane.

2.2

How, and at which velocity does the rear end of a traffic jam (cars stopped at density ρ_j) evolve when it is feeded by a continuous flux of vehicles (with density ρ_0) ?

2.3

What is the effect on a constant flow of vehicles (uniform density ρ_0) of a traffic light suddenly going red ?

1/ Show that, as far as vehicles upstream of the traffic light are concerned, this problem is identical to the preceeding one (2.2), and that in the present case one can draw characteristics issued from the vertical axis.

2/ Describe what happens when the traffic light turns green again. Consider first short times after the change to green, and then qualitatively describe what happens for longer times.

2.4

One aims here at describing quantitatively the problem studied in the previous exercice. To be specific one considers the case where

$$Q(\rho) = V_m(\rho - \rho^2/\rho_j) \; , \quad \text{with} \quad \rho \in [0, \rho_j] \; . \tag{1}$$

Express Q_m as a function of V_m. What is the physical significance of V_m ?

1/ One denotes by ρ_0 the initial uniform car density. Compute the velocities of the two shocks forming when the traffic light (located at $x = 0$) turns red (time $t = 0$). Draw the corresponding characteristics.

2/ At time τ the traffic light turns green again. At what times t_r and t_l are the right and the left shock affected ?

3/ The traffic light turning green creates a rarefaction wave. Show that its equation is $\rho(x,t) = \frac{1}{2}\rho_j[1 - x/(V_m(t-\tau))]$.

4/ Show that after times t_r and t_l the positions of both the left (x_l) and the right (x_r) shock obey the same differential equation:

$$\frac{\mathrm{d}x_{(r/l)}}{\mathrm{d}t} = V_m\left(\frac{1}{2} - \frac{\rho_0}{\rho_j}\right) + \frac{x_{(r/l)}}{2(t-\tau)} \; . \tag{2}$$

5/ Show that the solution of Eq. (2) is

$$x_{(r/l)}(t) = B_{(r/l)}(t-\tau)^{1/2} + V_m\left(1 - \frac{2\rho_0}{\rho_j}\right)(t-\tau) \; , \tag{3}$$

where the integration constants are given by $B_{(r/l)} = (\pm) 2\, V_m \sqrt{\tau\rho_0/\rho_j}\sqrt{1-\rho_0/\rho_j}$.

6/ How does the amplitude of the discontinuity at each shock evolve as a function of time ?

7/ One considers the case of light traffic: $\rho < \rho_m$ [$\rho_m = \rho_j/2$ according to model (1)].

Describe what happens. Show that the left shock reaches the position of the traffic light ($x = 0$) at time $t_{\text{end}} = \tau(1 - 2\rho_0/\rho_j)^{-2}$. Explain why an observer located at the same position than the traffic light will consider t_{end} as the ending time of the jam.