Tutorial # 2 : Solitons in the NLS equation

On considers the NLS equation written in dimensionless form:

$$i\phi_t = -\frac{1}{2}\phi_{xx} + \sigma|\phi|^2\phi \,, \tag{1}$$

where $\sigma = \pm 1$. $\sigma = -1$ is the focusing case, $\sigma = +1$ the defocusing one. One looks for traveling wave solutions of the form

$$\phi(x,t) = A(x - Vt)\exp\{\mathrm{i}S(x - Vt) + \mathrm{i}f(t)\} \,, \tag{2}$$

where A, S and f are real functions. The velocity V can be chosen positive. By plugging this ansatz into Eq. (1), show that $A(\zeta = x - Vt)$ verifies[1] ($A' = \mathrm{d}A/\mathrm{d}\zeta$)

$$\frac{1}{2}(A')^2 + W(A) = c_2 \,, \quad \text{where} \quad W(A) = \frac{J^2}{2A^2} + \left(\frac{V^2}{2} - c_1\right)A^2 - \sigma\,\frac{A^4}{2} \,. \tag{3}$$

In this expression c_1, c_2 and J are integration constants. Express the quantity J in terms of A, S' and V and explain its physical content.

2.1 Bright soliton for the focusing NLS ($\sigma = -1$)

Explain why in this case one looks for solutions of (1) for which $A \to 0$ when $\zeta \to \pm\infty$. Give the value of J and S' in this case, and explain why a soliton solution can be obtained only for $c_2 = 0$ and $V^2 < 2c_1$.

Show that in this case the soliton solution reads $A(\zeta) = A_{\mathrm{M}}/\cosh(A_{\mathrm{M}}\zeta)$, where you will express A_{M} in terms of c_1 and V. Express the number of "particles" in this "bright soliton" ($N = \int \mathrm{d}\zeta\, A^2$) as a function of A_{M}.

2.2 Dark soliton for the defocusing NLS ($\sigma = +1$)

One notes here $A_0 = \lim_{|\zeta|\to\infty} A$. Explain why in this case the current J needs not be zero. Show that one can obtain a soliton solution only if

$$V^2 - 2\,c_1 > 3\,J^{2/3} \,. \tag{4}$$

If (4) is verified, one denotes by A_1 the minimum value of A in the soliton solution [$W(A_1) = W(A_0) = c_2$]. Show that in this case the quantity $\rho(\zeta) = A_0^2 - A^2(\zeta)$ obeys a differential equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}\zeta} = \pm 2\rho\sqrt{\rho_{\mathrm{M}} - \rho} \,,$$

where you will express ρ_{M} as a function of A_0 and A_1. Solve this equation and show that the number of the "missing particles" in the soliton is $\Delta N = \int \mathrm{d}\zeta\,\rho = 2\sqrt{A_0^2 - A_1^2}$.

[1] Hints: intermediate steps leading to (3) are: $[A^2(S' - V)]' = 0$ and $A\,(\mathrm{d}f/\mathrm{d}t) = \frac{1}{2}A'' + \frac{1}{2}AV^2 - \frac{1}{2}J^2/A^3 - \sigma A^3$.

2.2.1 Determination of the integration constants

Show that $c_1 = -A_0^2$ and that one must have $W''(A_0) < 0$. Express this condition as a relation between A_0 and V. In the following one will write $c_0 = A_0^2$ and denote this quantity as the "speed of sound". Do you know why ?

Show that a soliton can only exist at subsonic speed and that in this case (4) is automatically verified and $A_1^2 = V^2$. Do you understand why a soliton of zero velocity is called a black soliton ?

2.2.2 Phase of the dark soliton

One writes $V/c = \sin\theta$ where $\theta \in [0, \pi/2]$. Show that the phase obeys the following differential equation

$$\frac{1}{A_0 \sin\theta} \frac{\mathrm{d}S}{\mathrm{d}\zeta} = \frac{-\cos^2\theta}{\cosh^2(\cos\theta A_0\,\zeta) - \cos^2\theta} \;,$$

the solution of which is[2] $\tan(S) = -\tanh(\cos\theta A_0 \zeta)/\tan\theta$. Plot S as a function of ζ for different velocities. Discuss the limit $V \to 0$.

2.3 Conserved quantities

Show that the quantities $\int |\phi|^2 \mathrm{d}x$ and $\frac{1}{2}\int \big(|\phi_x|^2 + \sigma|\phi|^4\big)\,\mathrm{d}x$ are conserved quantities associated with the currents $\frac{\mathrm{i}}{2}(\phi_x^*\phi - \phi^*\phi_x)$ and $-\frac{1}{2}(\phi_t\phi_x^* + \phi_x\phi_t^*)$.

Note: these conserved quantities are the number of particles and the energy of a field theory: consider the Lagrangian density

$$\mathscr{L}(\phi, \phi_x, \phi_t, \phi^*, \phi_x^*, \phi_t^*) = \frac{\mathrm{i}}{2}\left(\phi^*\phi_t - \phi_t^*\phi\right) - \frac{1}{2}|\phi_x|^2 - \frac{\sigma}{2}|\phi|^4 \;. \tag{5}$$

Treating the fields ϕ and ϕ^* as independent variables (why is it allowed ?), show that the Euler-Lagrange equations of motion (you may wish to derive them by yourself)

$$\partial_t \left(\frac{\partial \mathscr{L}}{\partial \phi_t^*}\right) + \partial_x \left(\frac{\partial \mathscr{L}}{\partial \phi_x^*}\right) = \frac{\partial \mathscr{L}}{\partial \phi *}$$

yield the NLS equation (1). Then, one can get the conserved quantities from general considerations[3].

Useful integrals

$$\int \frac{\mathrm{d}x}{x\sqrt{1-x^2}} = -\,\mathrm{arcosh}\left(\frac{1}{x}\right) \;, \quad \int_{\mathbb{R}} \frac{\mathrm{d}x}{\cosh^2(x)} = 2 \;, \quad \int_{\mathbb{R}} \frac{\mathrm{d}x}{\cosh^4(x)} = \frac{4}{3} \;.$$

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[2] You may prefer to verify directly that

$$\phi(x,t) = A_0\,\mathrm{e}^{-\mathrm{i}A_0^2 t}\left\{\sin\theta - \mathrm{i}\cos\theta\tanh\left[A_0\cos\theta(x - Vt)\right]\right\} \;, \quad \text{where} \quad \sin\theta = V/A_0 \;,$$

is a solution of the defocusing version of (1), and that its phase and amplitude are indeed given by S and A.

[3] You may give a look at chapter II.3.1 of the following lecture notes on classical field theory: `http://lptms.u-psud.fr/nicolas_pavloff/enseignement/electrodynamique-classique-et-quantique`. Note however that the NLS equation is "exactly integrable" and has an infinity of conserved quantities, which cannot be obtained so easily from the Lagrangian (5).