Optical hydrodynamics and nonlinear diffraction

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Waves Côte d'Azur juin 2019

undular bore in Turnagain Arm, Alaska

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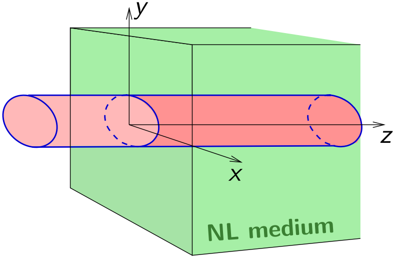
ISAN, Troitsk

$$\vec{\nabla}^2 \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0 c^2}\partial_t^2 \vec{D}(\vec{r},t)$$



$$\begin{cases} \vec{E}(\vec{r},t) = \hat{x}\left\{\tfrac{1}{2}A(\vec{r}_\perp,z)\,\mathrm{e}^{\mathrm{i}(\beta_0 z-\omega_0 t)} + \mathrm{c.c.}\right\} \\ \vec{D}(\vec{r},t) = \varepsilon_0\vec{E} + \vec{P}_L(\vec{r},t) + \vec{P}_{NL}(\vec{r},t) \end{cases}$$

$$\begin{cases} \vec{P}_L(\vec{r},t) = \varepsilon_0\chi^{(1)}_{\omega_0}(\vec{r})\vec{E}(\vec{r},t) \\ \vec{P}_{NL}(\vec{r},t) = \varepsilon_0\underset{\sim}{\chi}^{(3)}\vdots\vec{E}(\vec{r},t)\vec{E}(\vec{r},t)\vec{E}(\vec{r},t) \\ \qquad = \varepsilon_0\tfrac{3}{4}\chi^{(3)}|\vec{E}|^2\vec{E}(\vec{r},t) \end{cases}$$

$$(\chi^{(3)} = \underset{\sim}{\chi}^{(3)}_{xxyy} + \underset{\sim}{\chi}^{(3)}_{xyxy} + \underset{\sim}{\chi}^{(3)}_{xyyx})$$

- linear, homogeneous system: PW with $\beta_0 = \frac{\omega_0}{c}(1+\chi^{(1)}_{\omega_0})^{1/2} = k_0\, n(\omega_0)$

nonlinear, non homogeneous system. paraxial approximation $\partial_z A \ll \beta_0 A$

$$\chi^{(1)}(\vec{r}) = \chi^{(1)}_{\omega_0} + \Delta\chi^{(1)}(\vec{r}_\perp) \qquad \mathrm{i}\partial_z A = -\frac{1}{2\beta_0}\vec{\nabla}_\perp^2 A - k_0\Delta n(\vec{r})A$$

$$\Delta n(\vec{r}) = \Delta n^{(1)}(\vec{r}_\perp) + n_2|A(\vec{r}_\perp,z)|^2 \quad \text{with} \begin{cases} \Delta n^{(1)}(\vec{r}_\perp) = \tfrac{1}{2}\Delta\chi^{(1)}(\vec{r}_\perp)/n(\omega_0) \\ n_2 = \tfrac{3}{8}\chi^{(3)}/n(\omega_0) \quad < 0 \quad \text{in the following} \end{cases}$$

$$\mathrm{i}\partial_z A = -\frac{1}{2\beta_0}\vec{\nabla}_\perp^2 A - k_0 n_2 |A|^2 A \quad \text{where} \quad A = A(\vec{r}_\perp, z)$$

I_0: typical light intensity $\qquad Z_{NL} = -1/(n_2 k_0 I_0)$: nonlinear length

$\xi_\perp = \sqrt{Z_{NL}/\beta_0}$: transverse healing length

$\vec{r}_\perp = \vec{r}_\perp/\xi_\perp \qquad t = z/Z_{NL} \qquad A(\vec{r}_\perp, t) = A(\vec{r}_\perp, z)/\sqrt{I_0}$

$$\mathrm{i}\partial_t A = -\tfrac{1}{2}\vec{\nabla}_\perp^2 A + |A|^2 A$$

dispersionless hydrodynamics

$A(\vec{r}_\perp, t) = \sqrt{\rho}\,\exp\{\mathrm{i}\,S\} \qquad \vec{\nabla}_\perp S = \vec{u}$

$$\begin{cases} \partial_t \rho + \vec{\nabla}_\perp \cdot (\rho \vec{u}) = 0 \\ \partial_t \vec{u} + (\vec{u}\cdot\vec{\nabla}_\perp)\vec{u} + \vec{\nabla}_\perp \rho + \vec{\nabla}_\perp \left(\frac{(\vec{\nabla}_\perp \rho)^2}{8\rho} - \frac{\Delta_\perp \rho}{4\rho} \right) = 0 \end{cases}$$

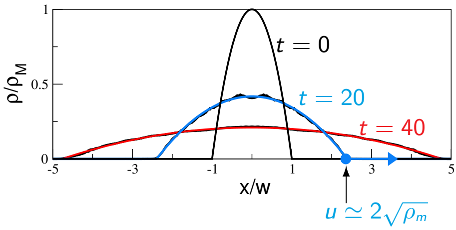
dimensionless units:

$$\rho(\vec{r}_\perp, t=0) = \begin{cases} \rho_m(1-\frac{x^2}{w^2}) \text{ if } |x|<w \\ 0 \text{ otherwise} \end{cases}$$

self-similar profile: Talanov 1965

$$\rho(x,t) = \frac{\rho_m}{f(t)}\left(1 - \frac{x^2}{w^2 \cdot f^2(t)}\right)$$
$$u(x,t) = x \cdot \phi(t)$$

$$\phi = f'/f$$
$$\ln(\sqrt{f}+\sqrt{f-1})+\sqrt{f(f-1)} = 2t\sqrt{\rho_m}/w$$



Spreading vs. wave breaking

dimensionless units:

$$\rho(\vec{r}_\perp, t=0) = \begin{cases} \rho_m(1 - \frac{x^2}{w^2}) \text{ if } |x| < w \\ 0 \text{ otherwise} \end{cases}$$

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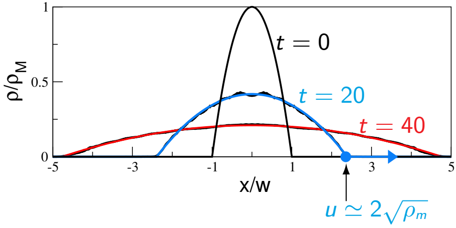
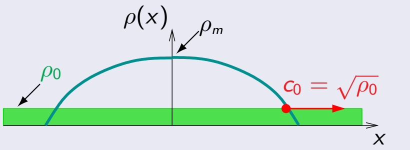
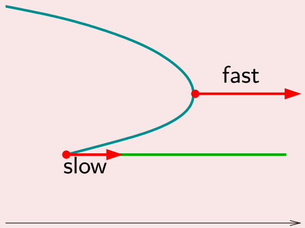


in the presence of ρ_0



dispersive regularization of wave breaking



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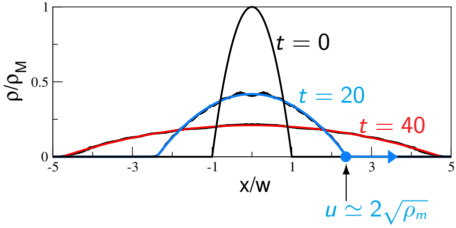

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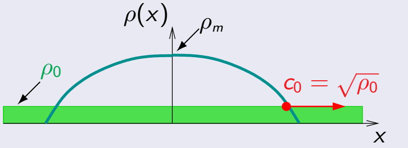

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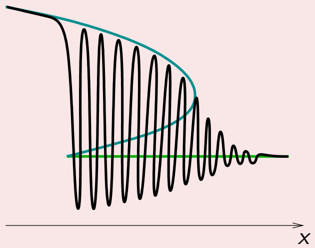
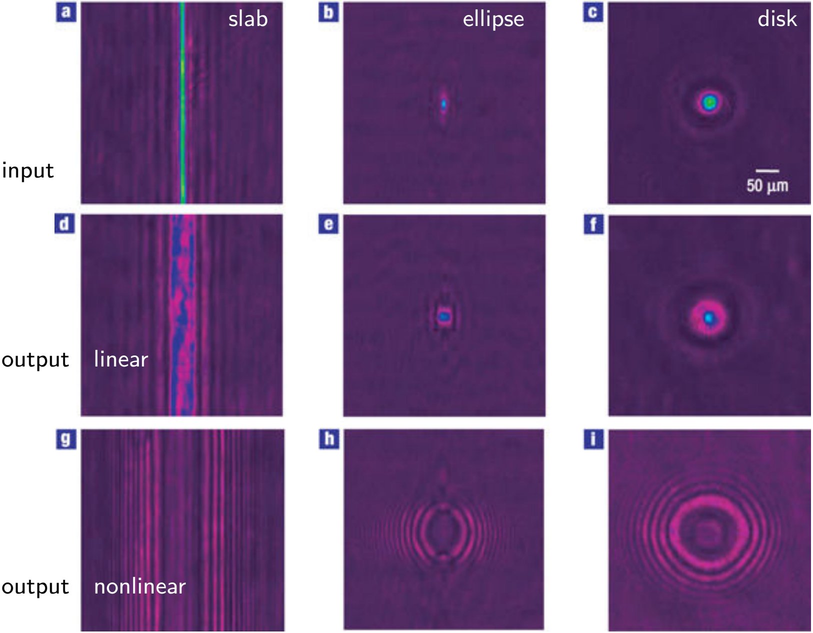
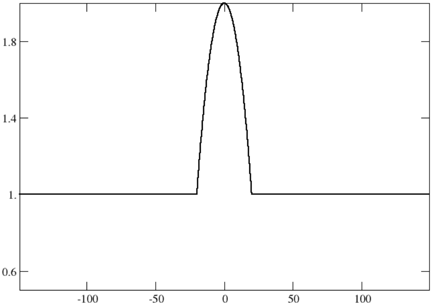
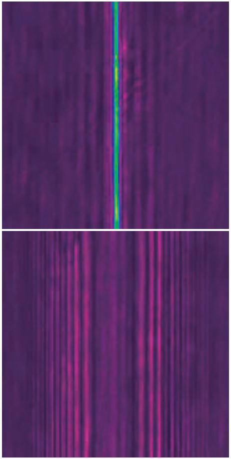


photo-refractive material: NL induced by a voltage bias across the crystal



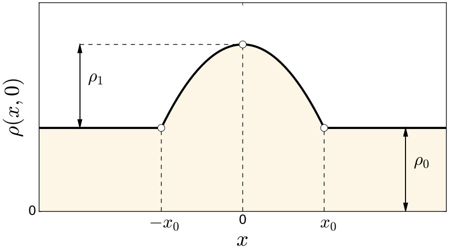


input



output

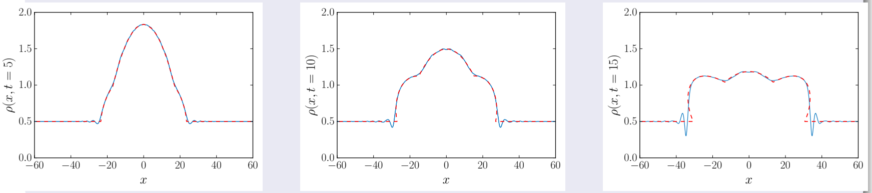
$0 \leq t \leq 60$

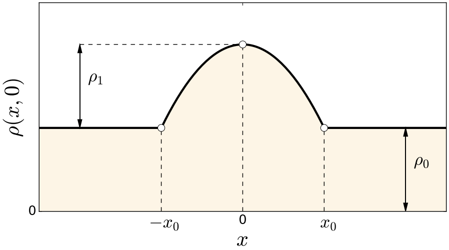


$$\rho(x, t=0) = \rho_0 + \rho_1 \left(1 - \frac{x^2}{x_0^2}\right)$$

$t < t_{WB}$: dispersionless approximation

Riemann method $\simeq$ improved method of characteristics

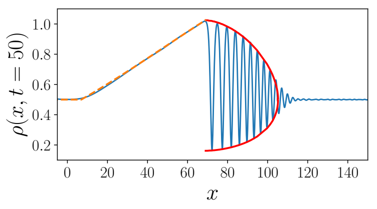


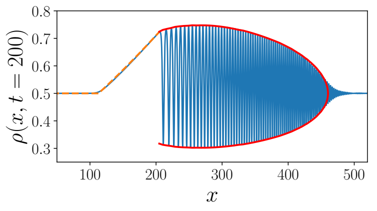


$$\rho(x, t = 0) = \rho_0 + \rho_1 \left(1 - \frac{x^2}{x_0^2}\right)$$

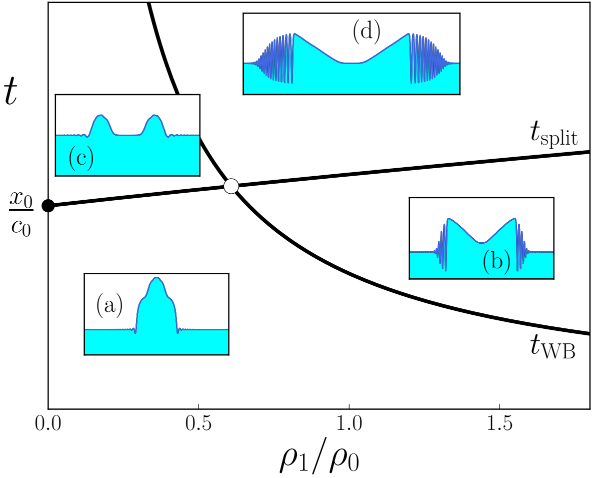
$t > t_{WB}$: dispersive shock wave

Whitham method $\simeq$ adiabatic invariants for modulated nonlinear wave

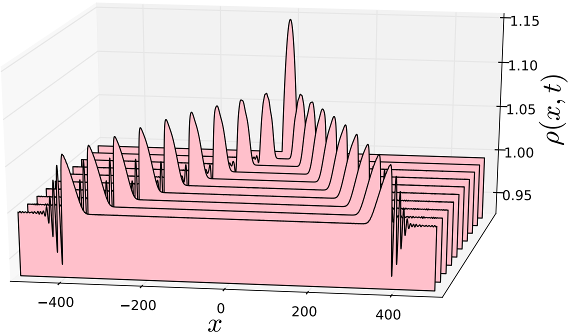


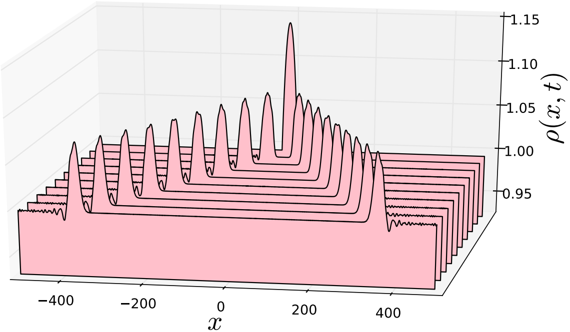


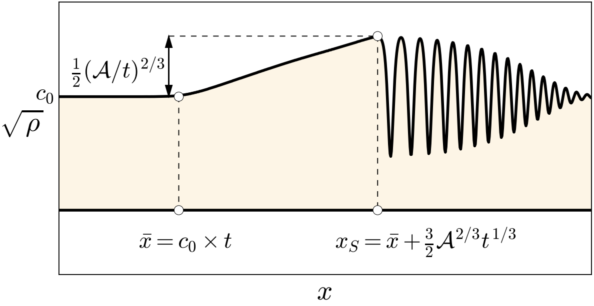
$$t_{WB} = 2\,\frac{x_0}{c_0}\,\frac{\rho_0}{\rho_1} \qquad t_{split} \simeq \frac{x_0}{c_0}\left[1 + \frac{1}{6}\frac{\rho_1}{\rho_0} - \frac{1}{60}\left(\frac{\rho_1}{\rho_0}\right)^2 + \cdots\right]$$



$t \gg t_{WB}$: failure of the perturbative approach ($\rho_1/\rho_0 = 0.15$, $0 \leq t \leq 360$)







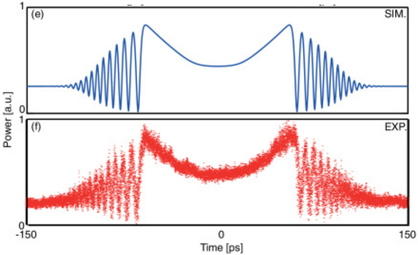
New (asymptotically) conserved quantity

$$\mathcal{A} = \sqrt{2} \int_{\bar{x}}^{x_S} (\sqrt{\rho} - c_0)^{1/2} dx \simeq 2\, x_0 \sqrt{c_0}\, F(\rho_0/\rho_1)$$

where

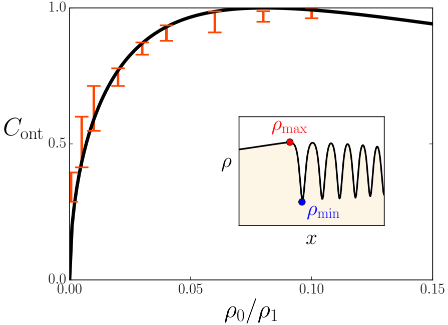
$$F(\alpha) = \int_0^{\pi/2} \cos\theta \left(\sqrt{1 + \frac{\cos^2\theta}{\alpha}} - 1 \right)^{1/2} d\theta$$

Fiber optics : $$-i\partial_z A = -\frac{\beta_2}{2}\partial_t^2 A + \gamma|A|^2 A + \frac{i\alpha}{2}A$$



$t_0 = 18.3$ ps, $L = 3$ km, $P_1 = 5.9$ W
$\gamma = 3W^{-1}.km^{-1}$, $\beta_2 = 2.5 \times 10^{-26}\, s^2/m$
$P_{ref} = 1W$
$\rho_1 = P_1/P_{ref} = 5.9$, $t = \gamma P_{ref} L = 9$,
$x_0 = t_0\sqrt{\gamma P_{ref}/\beta_2} = 6.3$

$$C_{ont} = \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}}$$



C_{ont} is a function of a single scaling parameter : $\xi = \frac{x_0}{c_0 t} F(\rho_0/\rho_1)$

$$C_{ont} = 4(2\xi)^{2/3}/(4 + (2\xi)^{4/3}) \qquad C_{ont} = 1 \text{ for } \xi = \sqrt{2}$$

Conclusion

rich analogy between nonlinear optics and hydrodynamics

- modulational instability
- observation of dispersive shock waves
- analogy with superfluid motion
- in the presence of disorder : competition between SF and Anderson localization
- possible formation of "sonic" horizon

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Mathematical description: Riemann + Whitham

- describes the early, pre-breaking, dispersiveless spreading
- analytic result for the wave-breaking time
- describes the later, post-breaking, dispersive shock
- analytic result for the asymptotic weak shock parameters
  $\rightsquigarrow$ good description of the contrast

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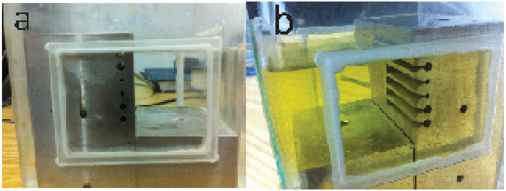
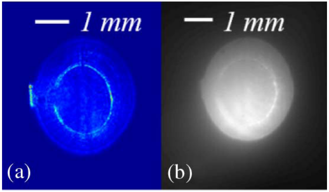
Thank you for your attention



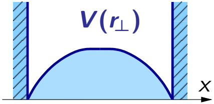


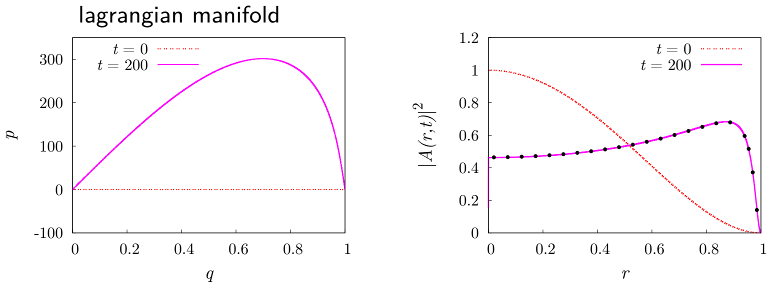
$L_\perp \ll L_{range\ NL} \ll L_z$: highly nonlocal paraxial approximation

Snyder & Mitchell 1997, Folli & Conti 2012

$$\Delta_{NL} n(\vec{r}_\perp, z) = \int \mathrm{d}^2 r'_\perp \chi(\vec{r}'_\perp) A^2(\vec{r}_\perp - \vec{r}'_\perp, z) \simeq \chi(\vec{r}_\perp) \int \mathrm{d}^2 r'_\perp A^2(\vec{r}'_\perp, z)$$
$$= \chi(\vec{r}_\perp) \times C^{st}$$

$$-\mathrm{i}\partial_z \boldsymbol{A} = -\tfrac{1}{2}\vec{\boldsymbol{\nabla}}_\perp^2 \boldsymbol{A} + \boldsymbol{V}(\boldsymbol{r}_\perp)\boldsymbol{A}$$





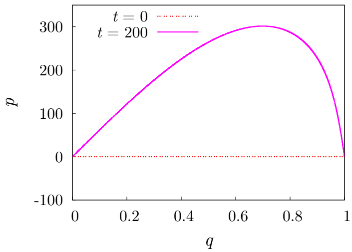
One evolves a swarm of **test points** $(\boldsymbol{r}, \boldsymbol{p})$
in phase space with the Hamilton equations

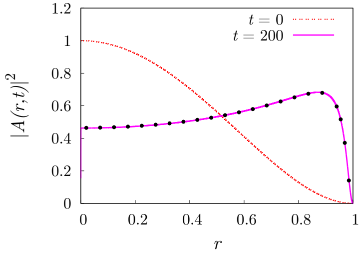
The density conservation eq. gives :

$$\left|A^2\left[\boldsymbol{r}(r_0, t)\right]\right| \mathrm{d}\boldsymbol{r} = \left|A_0^2[\boldsymbol{r}_0]\right| \mathrm{d}\boldsymbol{r}_0$$

$$\Leftrightarrow |A(r,t)| = \left|\frac{\partial \boldsymbol{r}_0}{\partial r}\right|^{1/2}_{\boldsymbol{r}_0(r,t)} |A_0[\boldsymbol{r}_0(r,t)]|$$

lagrangian manifold





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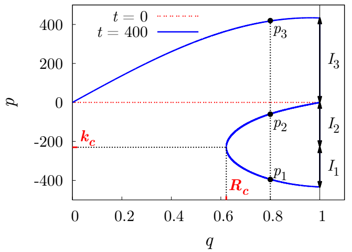
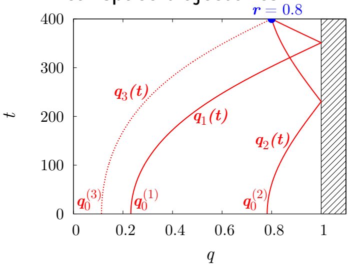
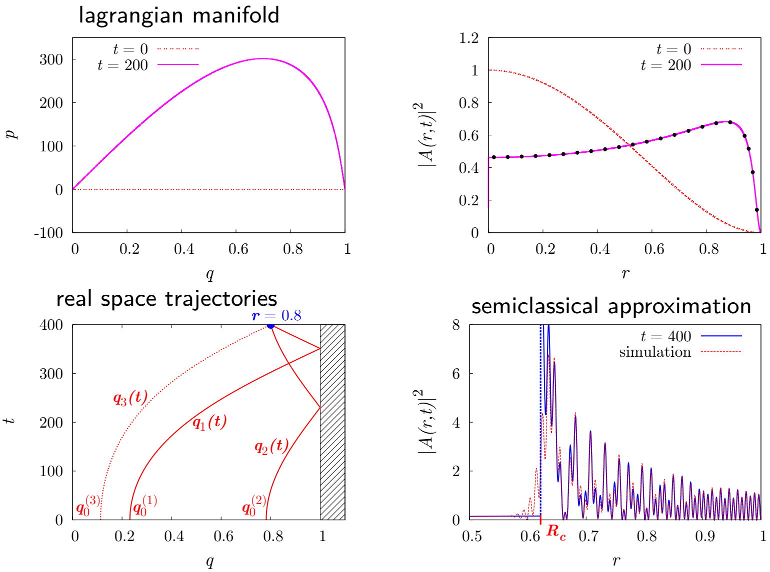


real space trajectories



The fellowship of the ring(s): Bar-Ad's group 2015



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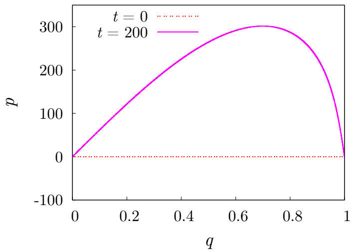

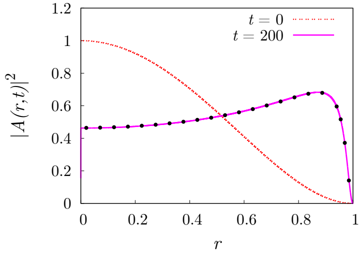

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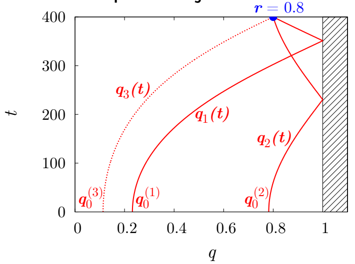
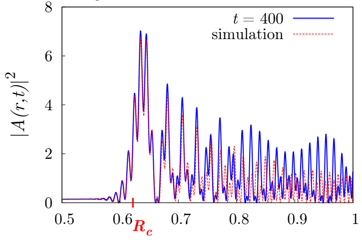
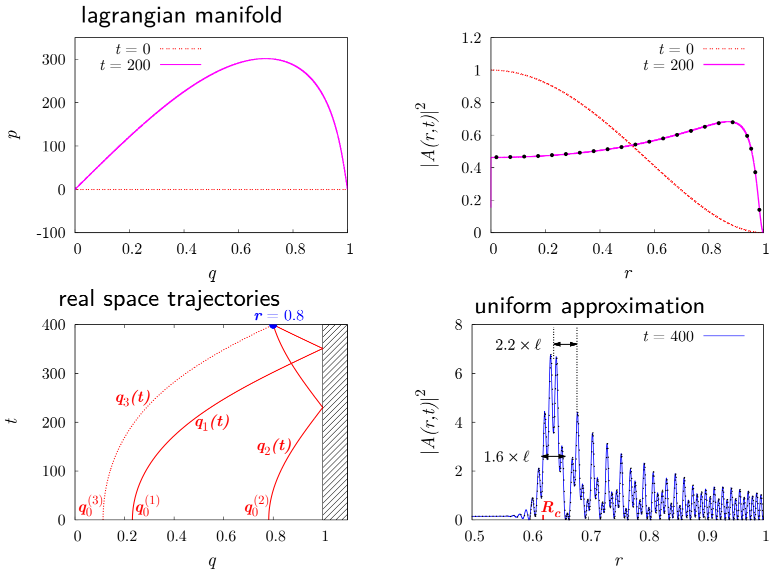


Airy near the caustic




lagrangian manifold
t = 0
t = 200
p
q
|A(r,t)|^2
r
real space trajectories
r = 0.8
q_3(t)
q_1(t)
q_2(t)
q_0^(3)
q_0^(1)
q_0^(2)
t
uniform approximation
t = 400
2.2 × ℓ
1.6 × ℓ
R_c

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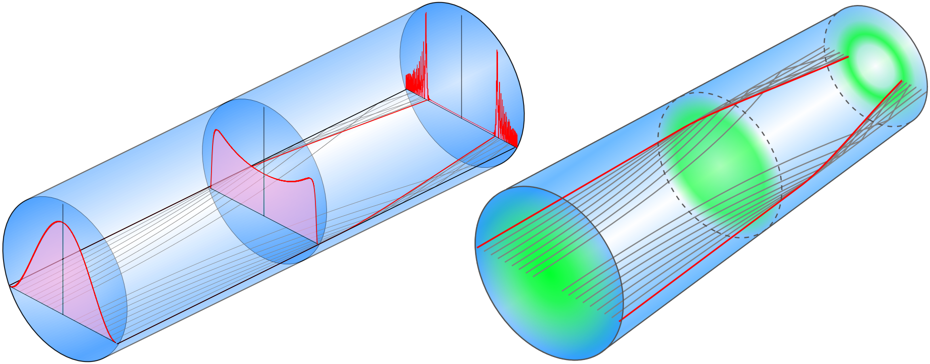
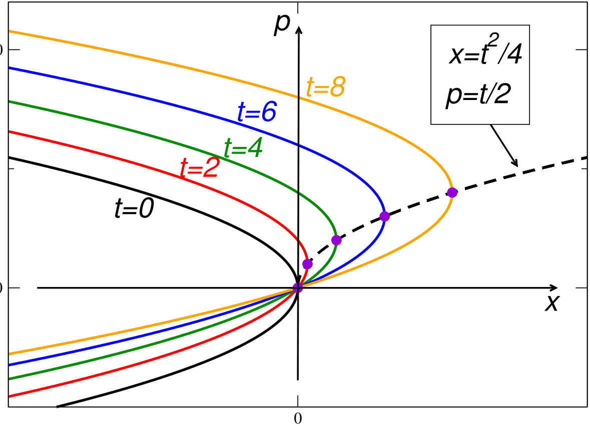


$$i\partial_t\Phi = -\tfrac{1}{2}\partial_x^2\Phi \quad \rightsquigarrow \quad \Phi(x,t) = Ai\left(x - \frac{t^2}{4}\right)\exp\{i(xt/2 - t^3/6)\}$$



- initial swarm of particles $x_0 = -p_0^2$
- free propagation

$$\begin{cases} x(t) = x_0 + p_0 t \\ p(t) = p_0 \end{cases}$$

$$x(t) = -\left(p(t) - \frac{t}{2}\right)^2 + \frac{t^2}{4}$$

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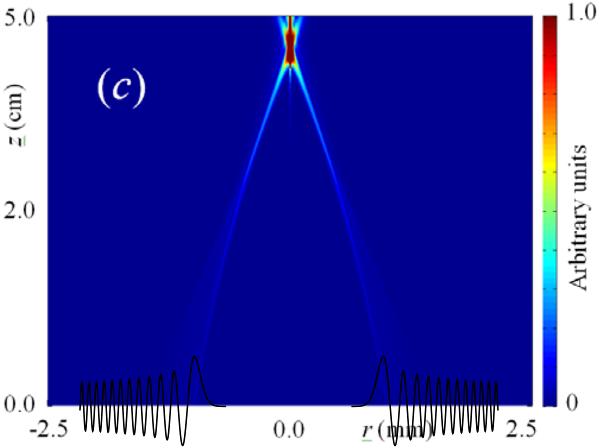

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simple and cheap alternative to a spatial light modulator

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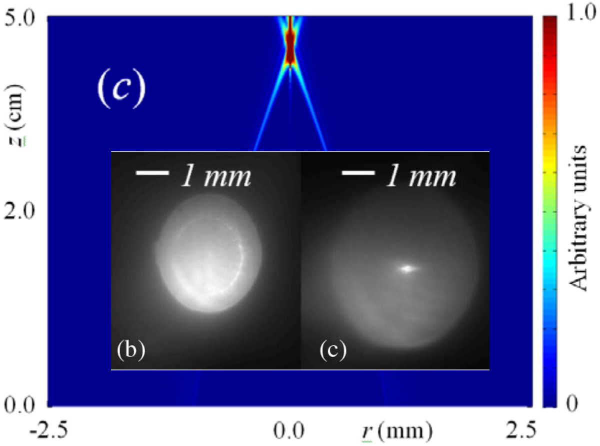

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simple and cheap alternative to a spatial light modulator