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# Fermat's principle, Huygens' principle, Hamilton's optics and sailing strategy

**J C Kimball and Harold Story**

Physics Department, University at Albany, Albany, NY 12222, USA

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**Abstract.** Fermat's principle asserts that light takes the path of minimum (actually extremal) time. Sailors often wish to find paths of minimum time. Thus by thinking of a sailboat as a light ray, a sailor can use Fermat's principle to describe the optimum sailing strategy. Huygens' principle and Hamiltonian optics follow from Fermat's principle, so a sailor can use Huygens' principle to visualize least-time paths, and Hamilton's optics provides a mathematical description of these paths. In especially simple cases, the optics-based formalism can be used to describe and quantify the basic tactics of sailboat racing.

## 1. Fermat's principle

Perhaps Albert Einstein loved sailing [1–3] because of the tricky geometry associated with sailing strategy. An expert on light like Einstein could have used Fermat's minimum-time principle to navigate most efficiently. Fermat's guess that light takes the path of minimum time was based on an assumption Einstein might have liked; 'nature operates by the simplest and most expeditious ways and means' [4]. The updated version of Fermat's principle only specifies that light's travel time is an 'extremum'. But for applications to sailing strategy, the primitive and historically accurate version of Fermat's minimum-time principle is particularly appropriate. Fermat derived Snell's law of refraction by minimizing the time it takes light to pass from a point in a less dense material to a point in a more dense material. He was one of a handful of people who knew enough about the precursors of calculus to do this. Fermat died in 1665; the famous plague year when Newton 'discovered' calculus and universal gravitation.

Fermat's principle is more than an alternative form of Snell's law [5]. It is the foundation of geometrical optics from which Huygens' principle and Hamilton's optics follow. If a sailor follows a light-like path, Fermat's principle assures us that the travel time will be a minimum, and translating Huygens' principle and Hamiltonian optics into sailing language provides a description of these least-time sailing paths.

## 2. Speed diagrams

Sailboats cannot sail directly against the wind. More generally, a sailboat's speed varies widely with the angle $\theta$ between the wind direction and the direction of a sailboat's motion. A polar plot of the angular dependence of a boat's maximum speed $\boldsymbol{V}(\theta)$ is called a 'speed diagram'. The apple-shaped curve of figure 1 is a typical speed diagram. It has been decorated with top

  

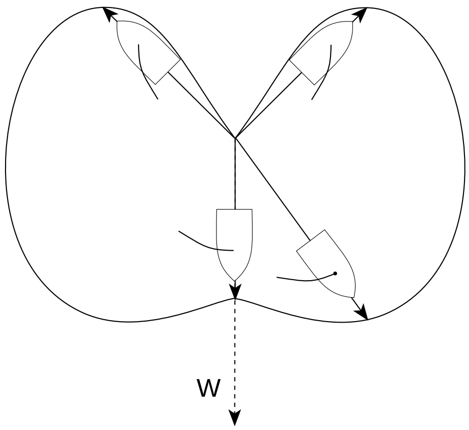


**Figure 1.** A sailboat's speed diagram is a polar plot of $\boldsymbol{V}(\theta)$, which is its speed versus its direction. Also shown are the wind direction $\boldsymbol{W}$ and four possible boat velocities.

views of little sailboats which show the sail positions needed to travel most quickly in four example directions. The 'notch' at the top of the speed diagram (opposite the wind direction $\boldsymbol{W}$) reflects the sailboat's inability to sail toward the wind. If the speed diagram is wrapped with a string, 'fast' parts of the diagram touch the string, but 'slow' parts do not. The boat in figure 1 which is sailing parallel to $\boldsymbol{W}$ (toward the bottom of the figure) is going in a slow direction, and it would never win a sailboat race. Sailing 'with the wind' is slower for some high-tech sailboats because the relative wind is decreased by the sailboat's motion. The slow parts of a speed diagram give rise to the 'zigzag' paths sailboats sometimes follow. In principle, a speed diagram could be derived from the principles of hydrodynamics. In practice, speed diagrams are determined experimentally by observing boat speeds in steady breezes.

The power for sailboats comes from wind moving over water. A stronger breeze (to within limits) means faster sailing and an expanded speed diagram. Variations in the wind's direction change the orientation of the speed diagram, and currents translate the speed diagram. Wind is turbulent, and this unavoidable complication of sailing produces a speed diagram whose shape changes in space and time. Since the maximum speed must therefore be written as $\boldsymbol{V}(\boldsymbol{r}, t, \theta)$, the sailor's task of finding the quickest path is quite complex. Rather than sailing a straight line, the quickest sailing path can meander to keep the boat in favourable winds.

The optical analogue of a speed diagram is light's 'indicatrix', which is polar plot $\boldsymbol{V}(\theta, \phi)$ of the speed of light in an anisotropic material. Light's ellipsoidal indicatrix is simpler than a speed diagram because it has no 'slow' parts. Thus there is no optical analogue for the zigzag paths of sailing. The time-dependent fluctuations of a speed diagram is another complication not commonly encountered in optics. Despite the geometrical and temporal differences between a speed diagram and an indicatrix, Fermat's principle remains as a fundamental connection between sailing strategy and optics.

## 3. Huygens' principle

In optics, Huygens' principle is the geometric construction which describes both the paths of light rays and the evolution of a wavefront. When applied to sailing, the light ray becomes the minimum-time path of a sailboat, and the wavefront becomes the line describing an idealized fleet of racing sailboats, which we call a 'fleet curve'.

In real sailboat races, some are faster than others, so the boats become more or less randomly distributed over the water. In an idealized sailboat race (where the optics analogy applies) boats are identical and sailors are perfect. Thus no boat is ahead of any other boat and all the boats lie on the fleet curve. The time evolution of a fleet curve (or a wavefront) may be described by Hamilton's time function $T(\boldsymbol{r})$, with the possible positions of all the boats at time

$t$ determined by the condition $T(\boldsymbol{r}) = t$. For example, if all the boats started at $\boldsymbol{r} = \mathbf{0}$ when $t = 0$, then $T(\boldsymbol{r}) = |\boldsymbol{r}|/v$ would describe a fleet moving out from the origin at a constant speed $v$ in every direction. Of course the time function is never this simple for sailboats because the velocity depends on space, time and direction. The two curves labelled $T(\boldsymbol{r}) = t_0$ and $T(\boldsymbol{r}) = t_1$ in figure 2 show fleet curves at two fixed times $t_0$ and $t_1$ which could be obtained from a more complicated time function.

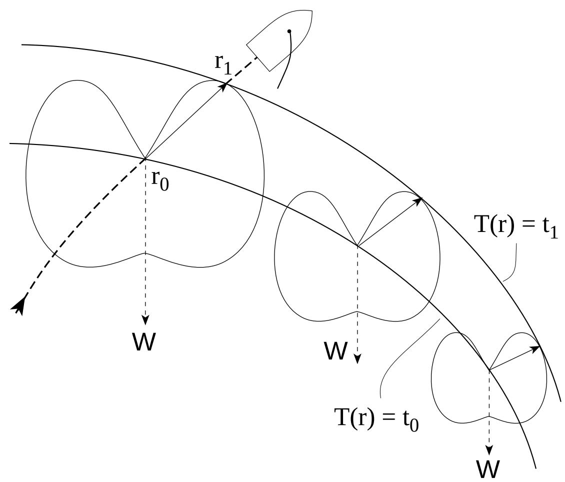


**Figure 2.** Two curves showing the positions of a fleet of boats at times $t_0$ and $t_1$. The evolution of this fleet curve is determined by Huygens' principle and the speed diagrams. The dotted curve shows the time-minimizing path for a sailboat which passes through $\boldsymbol{r}_0$ at time $t_0$.

Huygens' principle and the speed diagrams determine the time evolution of the fleet curves. Figure 2 illustrates how the fleet curve at the later time $t_1$ is obtained from the fleet curve at the earlier time $t_0$. If $t_1 = t_0 + \Delta t$, and $\Delta t$ is sufficiently small, then all the possible boat positions at time $t_1$ are obtained from speed diagrams placed at all points on the $t_0$ curve, multiplied by $\Delta t$ to change the speeds into the appropriate distances. Huygens' principle is the observation that the envelope curve touching the outer edges of this speed diagram array is also the fleet curve $T(\boldsymbol{r}) = t_1$. Each boat must sail as quickly as possible to this curve (Fermat's principle) so the sailing directions are given by the full arrows which are directed from the $T(\boldsymbol{r}) = t_0$ curve to the $T(\boldsymbol{r}) = t_1$ curve.

The example of figure 2 depicts a case where the wind is stronger on the left-hand side of the figure. Thus the speed diagram on the left is larger, and this gradient in the wind twists the fleet curve. The twisting is represented algebraically by the gradient term in equation (9), which will be derived later.

## 4. A Hamiltonian view of sailing

Hamilton's optics provides an algebraic description of the path of minimum time which results from applying Huygens' principle at every point along the path. Since time minimization is the sailor's goal, it should come as no surprise that Hamilton's time function $T(\boldsymbol{r})$ plays a fundamental role in the theory. For the sailboat analogy to optics, $T(\boldsymbol{r})$ is defined (at any position $\boldsymbol{r}$) as the minimum time needed for a boat to reach $\boldsymbol{r}$. Of course, this definition depends on the choice of an initial starting point and starting time. For simplicity, one can assume that the boat starts at the origin when $t = 0$, so $T(\boldsymbol{r})$ represents the minimum time interval needed to sail from the origin to $\boldsymbol{r}$. Of course the choice of an origin is arbitrary and does not affect the physics.

In addition to this time function, two vector fields characterize the minimum-time paths;

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} \tag{1}$$

is the velocity of the minimum-time path which passes through $\boldsymbol{r}$, and

$$\boldsymbol{p}(\boldsymbol{r}) = \boldsymbol{\nabla} T(\boldsymbol{r}) \tag{2}$$

is the 'vector of normal slowness' [6], or simply 'slowness'. Even though $\boldsymbol{p}(\boldsymbol{r})$ has the formal properties of momentum, it has the units of an inverse velocity; if all velocities were halved, the slowness would be doubled. This explains the name 'slowness', which was introduced by Hamilton.

The slowness yields the evolution of the shortest-time path because: (i) $\boldsymbol{p}(\boldsymbol{r})$ determines $\boldsymbol{v}(\boldsymbol{r})$ and (ii) the time dependence of $\boldsymbol{p}(\boldsymbol{r})$ is given by a 'Hamiltonian' differential equation. These complimentary aspects of the shortest-time path determination are described below.

### *4.1. Velocity determination*

For any given $\boldsymbol{r}$ and $\boldsymbol{p}(\boldsymbol{r})$, a sailor knows which way to steer because $\boldsymbol{v}(\boldsymbol{r})$ is a velocity on the speed diagram which satisfies the equation $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1$. To show this, consider the minimum-time path $\boldsymbol{r}(t)$, illustrated by the dotted curve in figure 2. For this path $\boldsymbol{r}(t_0) = \boldsymbol{r}_0$ lies on the curve $T(\boldsymbol{r}) = t_0$, and $\boldsymbol{r}(t_1) = \boldsymbol{r}_1$ lies on the curve $T(\boldsymbol{r}) = t_1$. Since $t_1 - t_0 = \Delta t$,

$$T(\boldsymbol{r}_1) - T(\boldsymbol{r}_0) = \Delta t. \tag{3}$$

In the limit of small $\Delta t$ this time difference is also given by the first term in the Taylor series expansion of $T(\boldsymbol{r})$ about the point $\boldsymbol{r}_0$:

$$T(\boldsymbol{r}_1) - T(\boldsymbol{r}_0) = \boldsymbol{\nabla} T(\boldsymbol{r}_0) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_0). \tag{4}$$

Also, for vanishing $\Delta t$

$$\boldsymbol{r}_1 - \boldsymbol{r}_0 = \boldsymbol{v}(\boldsymbol{r}_0)\, \Delta t. \tag{5}$$

Combining equations (3)–(5), and using the definition $\boldsymbol{p}(\boldsymbol{r}) = \boldsymbol{\nabla} T(\boldsymbol{r})$ from equation (2), gives

$$\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1. \tag{6}$$

The subscript '0' has been discarded from equation (6) because this relation is valid for any point along any time-minimizing path, and thus for all $\boldsymbol{r}$.

A geometrical illustration of this result is summarized in figure 3(*a*), which shows a speed diagram, the line $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v} = 1$, the direction of $\boldsymbol{p}(\boldsymbol{r})$ (labelled '$p$'), and $\boldsymbol{v}(\boldsymbol{r})$ (labelled '$v$'). The line $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v} = 1$ is tangent to the speed diagram; the direction $p$ is an arrow drawn perpendicular to this tangent line, and $v$ is the velocity on the speed diagram which touches this tangent line. Figure 3(*a*) can be obtained from the right-hand speed diagram of figure 2 by a $1/(\Delta t)$ magnification in the $\Delta t \to 0$ limit. In this limit, the infinitesimal segment of the fleet curve at $t_0 + \Delta t$ which touches the right-hand speed diagram in figure 2 becomes the tangent line $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v} = 1$ in figure 3(*a*).

Figure 3(*b*) is similar to figure 3(*a*) except $\boldsymbol{p}(\boldsymbol{r})$ is antiparallel to the wind velocity, $\boldsymbol{W}$. For this case, the sailor is 'tacking to windward' and there are two velocities, labelled '$v_1$' and '$v_2$' in figure 3(*b*), which both satisfy the condition $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1$. A sailor can choose either velocity and make equal progress to windward ($v_1$ corresponds to 'starboard tack' where the wind blows from the right-hand side of the boat, and $v_2$ corresponds to 'port tack').

Sailing to satisfy the condition $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1$ is equivalent to steering to maximize the velocity component in the direction of $\boldsymbol{p}(\boldsymbol{r})$. However, even though the direction of $\boldsymbol{p}(\boldsymbol{r})$ is the most desirable direction to travel, the optimum velocity $\boldsymbol{v}(\boldsymbol{r})$ is generally not parallel to $\boldsymbol{p}(\boldsymbol{r})$ because boats may slow down (or even stop) if they are steered to align $\boldsymbol{v}$ with $\boldsymbol{p}$. This

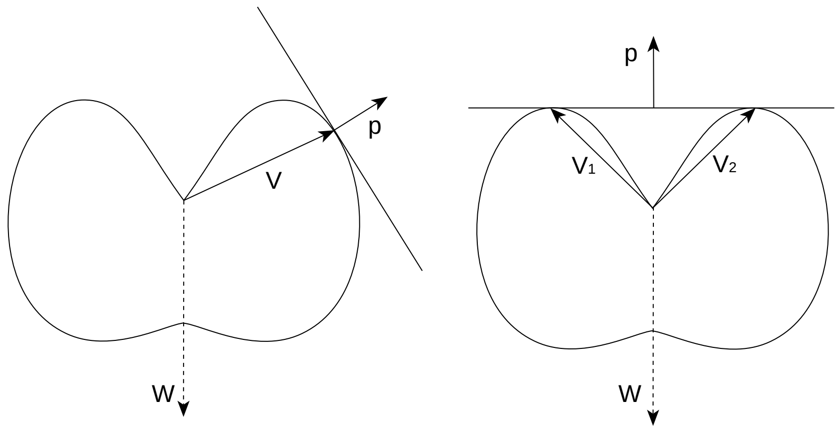


**Figure 3.** Examples of how $\boldsymbol{p}$ and the speed diagram determine the velocity. For both values of $\boldsymbol{p}$, the tangent line to the speed diagram is described by $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1$, and the speed is determined by the point(s) of tangency. The geometry of this velocity determination is equivalent to figure 2 (Huygens' principle) in the limit of a vanishing time interval.

is clearly the case in figure 3(*b*), where $\boldsymbol{p}$ is directed toward the wind. Since a boat cannot sail toward $\boldsymbol{p}$, the best way to make rapid progress in the $\boldsymbol{p}$-direction is to pick $\boldsymbol{v}$ to be along one of the two directions which are about 45° away from windward.

### 4.2. Time dependence of *p(r)*

In order to continue sailing in the best direction, the sailor needs to know the variation of $\boldsymbol{p}(\boldsymbol{r})$ along his minimum-time sailing path. Hamilton's optics gives this variation; the time derivative of $\boldsymbol{p}(\boldsymbol{r})$ is determined by the analogue to one of Hamilton's equations of classical mechanics (see the appendix), which is

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\boldsymbol{\nabla}(\boldsymbol{p} \cdot \boldsymbol{v}(\boldsymbol{r})). \tag{7}$$

Here the time derivative means the time variation as experienced by a sailor moving along a path of minimum time, and the gradient operates on $\boldsymbol{v}(\boldsymbol{r})$, but not on $\boldsymbol{p}$. To express this result in more practical terms, we write $\boldsymbol{v}(\boldsymbol{r})$ in terms of the polar representation of the speed diagram as

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{V}(\boldsymbol{r}, T(\boldsymbol{r}), \theta) \tag{8}$$

where $\theta$ is a direction which maximizes the component of speed in the direction of $\boldsymbol{p}$. Using this to calculate the gradient of $\boldsymbol{v}(\boldsymbol{r})$ yields $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ in terms of the space and time derivatives of the speed diagram.

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\boldsymbol{\nabla}(\boldsymbol{p} \cdot \boldsymbol{V}(\boldsymbol{r}, t, \theta)) - \boldsymbol{p}\frac{\partial}{\partial t}(\boldsymbol{p} \cdot \boldsymbol{V}(\boldsymbol{r}, t, \theta)) \tag{9}$$

where the first $\boldsymbol{p}$ in the second term comes from the gradient of $t = T(\boldsymbol{r})$ (see equation (2)). A possible additional term in equation (9) (associated with $\theta$ differentiation) vanishes because changes of $\boldsymbol{v}$ caused by changes in $\theta$ are perpendicular to $\boldsymbol{p}$.

## 5. Applications

In principle, equation (9) represents a formal solution to the least-time problem. In practice, sailors cannot perform the differentiation of equation (9), and they do not know the initial $\boldsymbol{p}$ needed to sail to the correct destination. However, it is still possible to gain insight into sailing strategy by applying the formal relations to the four particularly simple examples described below.

### *5.1. Example 1: a constant wind*

This drastically oversimplified example serves as a first step in devising sailing strategy. If the wind is constant, and if the direction from starting point A to destination B corresponds to a 'fast' direction on the speed diagram, then the quickest path is a straight line. Choosing $\boldsymbol{v}$ to point from A to B also determines a $\boldsymbol{p}$ which is generally not directed from A to B. This is illustrated in figure 3(*a*). Because $\boldsymbol{p}$ is normal to the surfaces of constant $T$ (constant time), a small displacement along $\boldsymbol{p}$ corresponds to having a 'head start'. Thus if two nearby boats are sailing toward B in a constant wind, the lead boat has made the most progress in the $\boldsymbol{p}$-direction, and this is not necessarily the boat closest to B.

When sailing toward the wind, the direction from A to B is a 'slow' direction on the speed diagram. For this case, one must sail a 'zigzag' course to B, and $\boldsymbol{p}$ must be chosen so that the tangent line touches the speed diagram at the two points shown in figure 3(*b*). To arrive at B, one sails about 45$^\circ$ to the right of the wind for a while, and then changes direction (tacks) and sail to the other side of the wind. Even when sailing this more complicated zigzag path, displacement in the $\boldsymbol{p}$-direction remains the true indicator of which boat is ahead.

### *5.2. Example 2: wind depends only on time*

A purely time-dependent wind is a reasonable approximation for slow sailboats. When one cannot sail to a more favourable wind before the wind moves on, the spatial dependence of the wind is of no consequence. Ignoring spatial derivatives of the speed diagram in equation (9) means the direction (but not the magnitude) of $\boldsymbol{p}$ is conserved. The sailor must first determine the fixed direction for $\boldsymbol{p}$ (this is the hard part), and then maximize speed in the fixed $\boldsymbol{p}$-direction. Maximizing speed along $\boldsymbol{p}$ in a time-dependent wind leads to complicated sailing paths. This is especially noticeable when sailing a zigzag path toward the average wind. As the wind direction varies, the sailor must change from one lobe of the speed diagram to the other (tack). This is illustrated in figure 4 by the broken-line path of a sailboat in a clockwise rotating wind. At each $\boldsymbol{r}$ along this path, the sailor determines his sailing direction using $\boldsymbol{p}(\boldsymbol{r})$ and the speed diagram corresponding to the wind at that point. Speed diagrams, tangent lines described by $\boldsymbol{p}\cdot\boldsymbol{v} = 1$, and the constructions used to find the correct sailing directions are superimposed at three points along this path toward the top of figure 4. The sailor tacks at the middle speed diagram because the wind direction becomes antiparallel to $\boldsymbol{p}$ at this point. Note that the sailor chooses to maximize speed 'to windward' (opposite $\boldsymbol{W}$) only when $\boldsymbol{p}$ points directly toward the wind. At other points of the zigzag path, the sailor steers slightly less toward the wind to minimize his travel time [7].

The geometry of figures 4–6 combine real-space sailing paths with velocity-space speed diagrams. In order to place the speed diagrams in a real-space figure, we have multiplied each speed diagram by a fixed time interval $\tau$. This scaling gives each speed diagram a non-zero real-space size, but $\boldsymbol{v}(\boldsymbol{r})$ and $\boldsymbol{p}(\boldsymbol{r})$ for each speed diagram refer to these quantities evaluated at the origin of the speed diagram. Thus each speed diagram determines the boat's velocity only at the point where the boat's path touches the origin of the speed diagram. At this point, the trajectory should be tangent to the vector $\boldsymbol{v}(\boldsymbol{r})$.

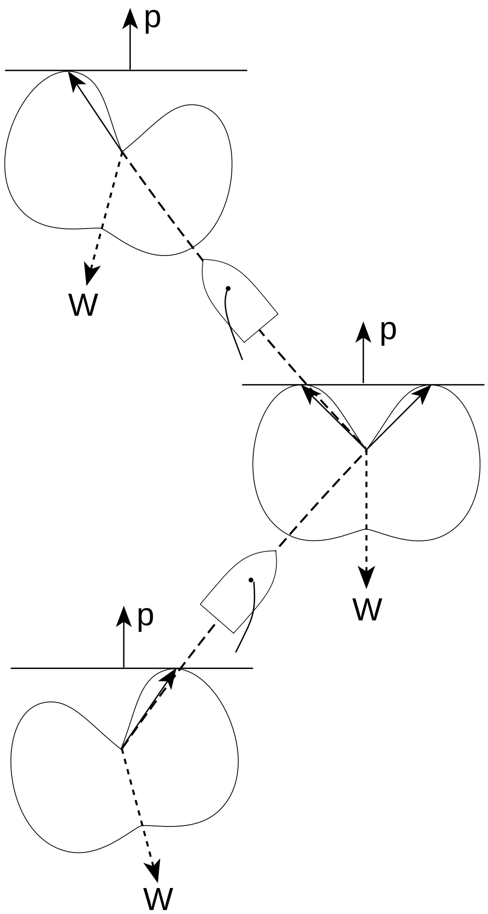


**Figure 4.** If the wind depends on time only, the direction of $\boldsymbol{p}$ is fixed, but the direction of $\boldsymbol{v}$ can change with time. For sailing toward the wind, the correct zigzag path (indicated by the broken line) is determined by the orientation of the wind with respect to $\boldsymbol{p}$. Speed diagram constructions similar to those of figure 3 have been superposed at three points along the path to show how the velocity at each point is determined.

### *5.3. Example 3: time-independent wind varies on only one direction*

Sometimes geography (such as a shoreline) produces a time-independent wind velocity which varies in just one direction. We consider here the example where the wind increases with the coordinate $y$ (the vertical direction in figure 5) and is independent of $x$ (the horizontal direction). For this simplification, translational invariance along $x$ yields a conservation law; the $x$-component of the momentum-like slowness is constant. This follows directly from equation (9). The conserved $x$-component of $\boldsymbol{p}$ has a simple geometric interpretation for speed diagrams. The tangent line $\boldsymbol{p} \cdot \boldsymbol{v} = 1$ intersects the $v_x$-axis at a fixed speed $U = 1/p_x$. An example of how speed diagrams with this fixed point on the tangent line determine the quickest sailboat course is illustrated in figure 5. For each of the three speed diagrams superposed on the boat's path in this figure, the displacement of the 'dot' from the centre of the speed diagram is the same speed, $U$, in the $x$-direction.

In figure 5, the wind speed was chosen to be larger for larger $y$, as is indicated by the larger size for the speed diagram nearer the top of the figure. Conservation of $p_x$ for this case means a sailor follows the curved path indicated by the broken curve from left to right in the figure. Since the least-time path passes through the region of larger $y$ where the wind is stronger, this construction is a quantitative description of the sensible sailing adage 'sail to the wind'. When the direction of $\boldsymbol{p}$ is not fixed, as in this example, the question of which boat is ahead cannot be answered simply.

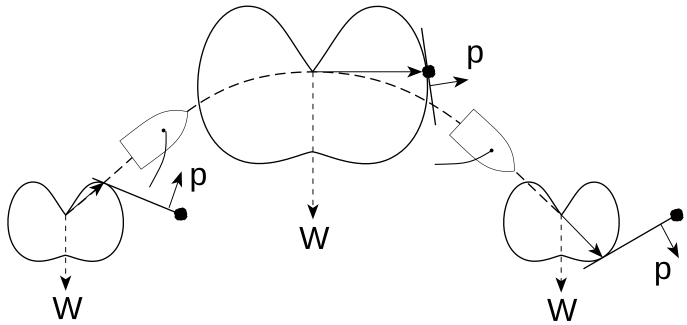


**Figure 5.** When the wind depends only on $y$, the conserved $p_x$ means the tangent line $\boldsymbol{p}(\boldsymbol{r}) \cdot \boldsymbol{v}(\boldsymbol{r}) = 1$ crosses the $v_x$-axis at a fixed point, which is indicated by the 'dot' adjacent to the three speed diagrams placed along the boat's path. The different sized speed diagrams reflect different wind speeds. Since the 'dot' position is fixed relative to the centre of the speed diagram, the sailing direction must change when the wind changes. The resulting trajectory, shown by the broken curve, takes the boat toward the top of the figure ($+y$-direction) where the wind is greater.

### *5.4. Example 4: wind depends only on y + ut*

By watching the surface of a lake or a field of grass, one can see that wind patterns often drift with a velocity, $\boldsymbol{u}$, which is comparable to the average wind velocity. For simplicity, we overstate this observation (known as the Taylor hypothesis [8]) and consider the case where the wind speed depends only on $y + ut$. This assumes the wind is independent of $x$, and fluctuations in the direction and magnitude of the wind are moving at a constant speed $u$ in the $-y$ direction. Again, $y$ is vertical and $x$ is horizontal in figure 6.

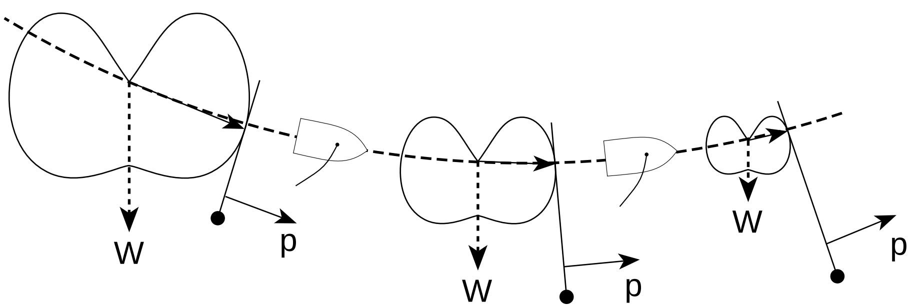


**Figure 6.** If the wind speed depends on $y + ut$, the tangent lines to the speed diagram pass through a fixed point (the dot) which is displaced by $-u$ along the $v_y$-axis. The resulting curved path (broken curve) keeps the boat in larger winds. For this example of a 'dying' wind, the quickest path is toward the bottom of the figure where the wind has decreased less.

As with the other examples, this case has a simple graphical interpretation. Assuming the Taylor hypothesis and ignoring the $x$ dependence of the wind, the time derivatives of the $x$- and $y$-components of the slowness are related. Using equation (9) to compare $\mathrm{d}p_y/\mathrm{d}t$ to $\mathrm{d}p_x/\mathrm{d}t$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{1 + up_y}{up_x}\right) = 0. \qquad (10)$$

Since the argument of the above logarithm must be a constant, the geometry of the lines tangent to the speed diagram (described by $\boldsymbol{p} \cdot \boldsymbol{v} = 1$) are restricted; all the tangent lines go through the fixed point

$$(v_x, v_y) = (U, -u) \tag{11}$$

where $1/U$ is the conserved $x$-component of the slowness in a coordinate system which moves along the $y$-axis with speed $-u$.

The significance of this fixed point on sailing strategy is most clearly illustrated when sailing roughly perpendicular to the wind. As is shown in figure 6, a decreasing wind bends the sailing direction toward the $+y$-direction. Thus a sailor who wishes to travel in the $x$-direction must initially start his trip with a velocity component in the $-y$-direction. This curved path is the quickest path because it allows a sailor to 'stay with the wind', steering to follow the high wind and to cut across the lighter wind. The curvature of this path will be large only when boat speeds are comparable to $u$. Thus when sailing crosswise to the wind, fast boats should sail a more 'wobbly' course than slow boats.

## 6. Conclusion

We have shown how the formalism of classical geometric optics can be applied to sailing strategy. Even though sailors must contend with an incomplete knowledge of the wind, some of the applications cited above are useful and are not completely obvious. Furthermore, seeing how formal optics can be extended to distant realms, like sailing strategy, gives one a renewed appreciation of the work of Fermat, Huygens, Hamilton and others. While considering this recreational application of classical optics, we were repeatedly impressed by the insights and accomplishments of these great scientists from past centuries.

## Appendix

The time derivative of $\boldsymbol{p}(\boldsymbol{r})$ for a boat (or light ray) moving along its minimum-time path is given by the chain rule. For the $x$-component of $\boldsymbol{p}$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_x = v_x \frac{\mathrm{d}p_x}{\mathrm{d}x} + v_y \frac{\mathrm{d}p_x}{\mathrm{d}y}. \tag{A1}$$

Since $\boldsymbol{p} = \boldsymbol{\nabla} T$, and the curl of a gradient is zero, $\hat{z} \cdot (\boldsymbol{\nabla} \times \boldsymbol{p}) = 0$ or

$$\frac{\mathrm{d}p_x}{\mathrm{d}y} = \frac{\mathrm{d}p_y}{\mathrm{d}x}. \tag{A2}$$

Combining equations (A1) and (A2) gives

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = v_x \frac{\mathrm{d}p_x}{\mathrm{d}x} + v_y \frac{\mathrm{d}p_y}{\mathrm{d}x}. \tag{A3}$$

Since $\boldsymbol{p} \cdot \boldsymbol{v} = 1$ (see equation (6)), the derivative of this product must vanish and

$$\frac{\mathrm{d}}{\mathrm{d}x}(\boldsymbol{p} \cdot \boldsymbol{v}) = v_x \frac{\mathrm{d}p_x}{\mathrm{d}x} + p_x \frac{\mathrm{d}v_x}{\mathrm{d}x} + v_y \frac{\mathrm{d}p_y}{\mathrm{d}x} + p_y \frac{\mathrm{d}v_y}{\mathrm{d}x} = 0. \tag{A4}$$

Subtracting equation (A4) from (A3) gives

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -p_x \frac{\mathrm{d}v_x}{\mathrm{d}x} - p_y \frac{\mathrm{d}v_y}{\mathrm{d}x}. \tag{A5}$$

An analogous expression applies for $\mathrm{d}p_y/\mathrm{d}t$, and combining them gives

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\boldsymbol{\nabla}(\boldsymbol{p} \cdot \boldsymbol{v}(\boldsymbol{r})). \tag{A6}$$

Here the gradient operator applies only to the spatial dependence of the velocity field.

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