Nonlinear Schrödinger equation for "waves of envelopes"

These notes explain how the nonlinear Schrödinger equation (NLS) can describe the propagation of a quasi-monochromatic wave in a nonlinear medium.

1 Generic treatment of the modulation of a plane wave

One[1] considers a field $\eta(\vec{r},t)$ which admits plane wave solutions of the type

$$\exp\{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)\}\ , \tag{1}$$

where the angular frequency ω relates to the wave vector $\vec{k}$ through the isotropic dispersion relation $\omega=\omega(k)$, with $k=|\vec{k}\,|$.

An initial wave packet $\eta(\vec{r},0)$ can always be decomposed into a sum of Fourier components:

$$\eta(\vec{r},0)=\int_{\mathbb{R}^d}\frac{\mathrm{d}^d k}{(2\pi)^d}F(\vec{k}\,)\exp\{\mathrm{i}\vec{k}\cdot\vec{r}\}\ ,\quad \text{where}\quad F(\vec{k}\,)=\int_{\mathbb{R}^d}\mathrm{d}^d r\,\eta(\vec{r},0)\exp\{-\mathrm{i}\vec{k}\cdot\vec{r}\} \tag{2}$$

is the spatial Fourier transform of $\eta(\vec{r},0)$ (d is the dimension of space).

If the wave obeys a linear equation (which we assume for a while), each of its Fourier components has a dynamics of the type (1) and the time evolved $\eta(\vec{r},t)$ reads

$$\eta(\vec{r},t)=\int_{\mathbb{R}^d}\frac{\mathrm{d}^d k}{(2\pi)^d}F(\vec{k}\,)\exp\{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega(k)\,t)\}\ , \tag{3}$$

Let's assume that the initial condition is almost monochromatic. In this case $F(\vec{k}\,)$ will be peaked around a certain wave vector $\vec{k}_0$. It is then appropriate to make a change of variable $\vec{k}=\vec{k}_0+\vec{q}$ in (3), and to Taylor expand $\omega(|\vec{k}_0+\vec{q}\,|)$ around $\vec{k}_0$: one first writes

$$|\vec{k}_0+\vec{q}\,|^2=k_0^2+2\vec{k}_0\cdot\vec{q}+\vec{q}^{\,2}\quad\text{and}\quad|\vec{k}_0+\vec{q}\,|=k_0+\hat{k}_0\cdot\vec{q}+\frac{q_\perp^2}{2\,k_0}+\cdots \tag{4}$$

where $k_0=|\vec{k}_0|$, $\hat{k}_0=\vec{k}_0/k_0$ and $q_\perp^2=\vec{q}^{\,2}-(\hat{k}_0\cdot\vec{q}\,)^2$. Then, introducing the notations

$$\omega(k_0)=\omega_0\ ,\qquad \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0}=\omega_0'\ ,\qquad \left.\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\right|_{k=k_0}=\omega_0''\ , \tag{5}$$

one obtains

$$\omega(|\vec{k}_0+\vec{q}\,|)=\omega_0+\omega_0'\,\hat{k}_0\cdot\vec{q}+q_\perp^2\,\frac{\omega_0'}{2k_0}+\tfrac{1}{2}(\hat{k}_0\cdot\vec{q}\,)^2\omega_0''+\cdots \tag{6}$$

In the following, for simplicity, of considers a single spatial dimension z (i.e., $d=1$). In this case $\vec{k}_0=k_0\,\hat{z}$, $\vec{q}=q\,\hat{z}$, $q_\perp^2=0$ (where $\hat{z}$ is a unit vector of the z-axis). Then Eq. (6) simplifies to

$$\omega(k_0+q)=\omega_0+q\,\omega_0'+\tfrac{1}{2}q^2\omega_0''+\cdots \tag{7}$$

[1] This section is inspired by §24 of the book by Karpmann "Nonlinear waves in dispersive media" and by chapter 7.8 of the book by Debnath "Nonlinear water waves".

and Eq. (3) reads

$$\eta(z,t) \simeq A(z,t)\,\mathrm{e}^{\mathrm{i}[k_0 z - \omega_0 t]}\ , \quad \text{where} \quad A(z,t) = \int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} F(k_0+q) \mathrm{e}^{\mathrm{i}\left[qz - \omega_0' q\, t - \frac{1}{2}\omega_0'' q^2 t\right]}\ . \tag{8}$$

The term $\exp[\mathrm{i}(k_0 z - \omega_0 t)]$ in the expression of $\eta(z,t)$ is denoted the "carrier wave", and $A(z,t)$ is the "amplitude" or the "envelope" (it is *a priori* complex). Note that if the initial wave is exactly monochromatic (with wave-vector $k_0\,\hat{z}$, and, say, amplitude unity), then, from (2), $F(k) = 2\pi\delta(k-k_0)$ and (8) yields $A(z,t) = 1$: $\eta(z,t)$ remains exactly a plane wave for all time. In the generic case where the initial wave $\eta(z,0)$ is not perfectly monochromatic, $A(z,t)$ truly depends on z and t. From (8) one can write $A(z,t) = \int \mathrm{d}q\, R(q,z,t)$ with

$$A_t = -\mathrm{i} \int_{\mathbb{R}} \mathrm{d}q\, (\omega_0' q + \tfrac{1}{2}\omega_0'' q^2)\, R\ , \quad A_z = \mathrm{i} \int_{\mathbb{R}} \mathrm{d}q\, q\, R\ , \quad A_{zz} = -\int_{\mathbb{R}} \mathrm{d}q\, q^2 R\ .$$

One thus sees that $A(z,t)$ is solution of [2]

$$\mathrm{i}(A_t + \omega_0' A_z) = -\tfrac{1}{2}\omega_0''\, A_{zz} \quad \Longleftrightarrow \quad \mathrm{i}A_\tau = -\tfrac{1}{2}\omega_0''\, A_{xx} \quad \text{where} \quad \begin{cases} x = z - \omega_0'\, t\ , \\ \tau = t\ . \end{cases} \tag{9}$$

The equation at the right in (9) describes the dynamics of the amplitude in a frame moving at the group velocity ω_0' of the carrier wave. If $A(x,0)$ is initially smooth enough, the right-hand side of this equation is negligible ($A_{xx} \sim 0$) and the wave is stationary in the moving frame, or equivalently: it propagates in the original (z,t) frame at constant velocity ω_0' without deformation. One can be more quantitative: let's denote as L the typical width over which $A(x,0)$ varies significantly ($L \sim |A/A_x|$). From an analysis of the orders of magnitude in the equation at the right of (9), one sees that the shape of A starts changing at a time of order $t_{\mathrm{disp}} = L^2/\omega_0''$. This is called the dispersive time[3]. In the absence of dispersion $\omega_0'' = 0$ and $t_{\mathrm{disp}} = \infty$ as it should.

Note also that, from the form of η in the left of Eq. (8), we interpreted A as a position and time-dependent amplitude. This is meaningful only if A depends more slowly on x and t than the carrier wave does, that is if $L \gg k_0^{-1}$ and $t_{\mathrm{disp}} \gg \omega_0^{-1}$.

The equation of the left-hand side of (9) can be simply obtained from the following recipe:

(a) Expand $\omega(k)$ around k_0 according to (7): $\omega - \omega_0 \simeq (k-k_0)\omega_0' + \frac{1}{2}(k-k_0)^2\omega_0'' + \cdots$

(b) Make the identification: $\omega - \omega_0 \equiv \mathrm{i}\partial_t$ and $k - k_0 \equiv -\mathrm{i}\partial_z$.

(c) Apply the resulting identical operators $\mathrm{i}\partial_t$ and $-\mathrm{i}\omega_0'\partial_z - \frac{1}{2}\omega_0''\partial_{zz}$ to A.

This suggests a phenomenological way to account for nonlinear effects the envelope of the carrier wave: if nonlinearity is present, the group and phase velocities of a quasi-monochromatic wave will depend

[2] In dimension higher than 1, the term proportional to $q_\perp^2$ in (6) results in an additional transverse Laplacian $-\frac{\omega_0'}{2\,k_0}\vec{\nabla}_\perp^2$ in the right-hand side of each of Eqs. (9).

[3] Its expression can also be obtained by considering the cases where one can make the approximation $\exp(-\frac{\mathrm{i}}{2}\omega_0'' q^2 t) \simeq 1$ in the integrant of (8).

on its amplitude. One may phenomenologically account for such an effect by adding an amplitude-dependent term to the dispersion relation which should now read $\omega(k\,;|A|^2)$. The simplest modification of Eq. (7) will be:

$$\omega-\omega_0 \simeq (k-k_0)\omega_0' + \tfrac{1}{2}(k-k_0)^2\omega_0'' + \alpha|A|^2 + \cdots \quad \text{where} \quad \alpha = \left.\frac{\partial\omega}{\partial|A|^2}\right|_{|A|^2=0} . \tag{10}$$

If one modifies the above (a,b,c) recipe in order to account for the modification (10) of the dispersion relation (7), one obtains straightforwardly:

$$\mathrm{i}A_t = -\mathrm{i}\omega_0' A_z - \tfrac{1}{2}\omega_0'' A_{zz} + \alpha|A|^2 A \quad \Longleftrightarrow \quad \mathrm{i}A_t = -\tfrac{1}{2}\omega_0'' A_{xx} + \alpha|A|^2 A\ , \tag{11}$$

where $x = z - \omega_0'\,t$. For simplicity we did not make here the change of notation $t \to \tau$ which was done in Eq. (9). Eq. (11) is called the "nonlinear Schrödinger equation" (NLS equation). The coefficient α is real[4], positive or negative; its precise value depends on the nonlinear wave equation satisfied by the physical field η. Eq. (10) indicates that α should typically depend on k_0. As an illustration, the next section provides a multi-scale derivation of equation (11) where such a k_0-dependence is observed. Then section 3 provides a different derivation in the context of a beam propagating in a nonlinear optical medium.

Comparing the NLS equation (11) with the linear equation (9), one sees that the nonlinear term is an additional source of distortion of the wave packet. It becomes effective after a time $t_{\rm NL} = (|\alpha|\cdot|A_{\rm typ}|^2)^{-1}$. The relative values of $t_{\rm disp}$ and $t_{\rm NL}$ determine which phenomenon, between dispersion or nonlinearity, affects the wave packet first. Note that, however small is $|\alpha|$, nonlinear effects cannot be neglected for $t > t_{\rm NL}$.

2 NLS from a multi-scale analysis

In this section we illustrate how a multi-scale analysis of the dynamics of a nonlinear wave train leads to a NLS equation for its slowly varying amplitude. We consider the (rather simple) case of a wave train of the Korteweg de Vries (KdV) equation[5]. Namely, one looks for a solution of the KdV equation

$$u_t + u\,u_z + u_{zzz} = 0\ , \tag{12}$$

under the form

$$u = u_0 + \epsilon\,u_1(z,Z,t,T_1,T_2) + \epsilon^2 u_2(z,Z,t,T_1,T_2) + \epsilon^3 u_3(z,Z,t,T_1,T_2) + \cdots \tag{13}$$

with $|\epsilon| \ll 1$, $Z = \epsilon z$ and $T_2 = \epsilon T_1 = \epsilon^2 t$. We choose here $u_0 \equiv 0$ for simplicity (no background). We consider a wave train which is a slowly modulated plane wave, hence we chose u_1 of the form

$$u_1 = \mathcal{A}(Z,T_1,T_2)\exp[\mathrm{i}\theta(z,t)] + \text{c.c.}\ , \quad \text{where} \quad \theta(z,t) = k_0 z - \omega_0 t\ . \tag{14}$$

The trick is to consider the different scales as independent, and to perform the replacements

$$\partial_t \to \partial_t + \epsilon\partial_{T_1} + \epsilon^2\partial_{T_2}\ , \quad \partial_z \to \partial_z + \epsilon\partial_Z\ , \quad \text{and thus} \quad \partial_z^3 \to \partial_z^3 + 3\epsilon\partial_z^2\partial_Z + 3\epsilon^2\partial_z\partial_Z^2 + \epsilon^3\partial_Z^3\ . \tag{15}$$

[4] A complex α would correspond to a damped or to an unstable wave, depending on the sign of the imaginary part. Can you see that ?

[5] For a more general treatment, cf. the book "Solitons in Mathematics and Physics" by A. C. Newell, 1985.

Inserting the ansatz (13) into KdV equation (12) one gets at leading order in ϵ: $\partial_t u_1 + \partial_z^3 u_1 = 0$. This imposes

$$\omega_0 = -k_0^3 \, , \tag{16}$$

which is the dispersion relation of linear waves obeying KdV equation (12) in the absence of background. At next order one gets

$$(\partial_t + \partial_z^3) u_2 = -(\partial_{T_1} + 3\partial_z^2 \partial_Z) u_1 + u_1 \partial_z u_1 = -(\partial_{T_1} - 3k_0^2 \partial_Z) u_1 + \partial_z(\tfrac{1}{2} u_1^2) \, . \tag{17}$$

This is a linear equation for u_2 with a source which contains terms behaving as $\exp(\pm \mathrm{i}\theta)$ (the term in u_1) and as $\exp(\pm 2\mathrm{i}\theta)$ (the term in u_1^2). The terms behaving as $\exp(\pm\mathrm{i}\theta)$ being resonant with the right-hand side of Eq. (17) lead to a secular behavior in u_2 and should be discarded[6]. This imposes

$$(\partial_{T_1} - 3k_0^2 \partial_Z) u_1 = 0 \, , \to \mathcal{A}(Z, T_1, T_2) = \mathcal{A}(Z - v_g T_1, T_2) \, , \quad \text{where} \quad v_g = -3k_0^2 = \frac{\mathrm{d}\omega_0}{\mathrm{d}k_0} \tag{18}$$

is the group velocity of the carrier wave. The remaining right-hand side of (17) can be written as $\partial_z(\frac{1}{2}u_1^2) = \mathrm{i}\, k_0 \mathcal{A}^2 \exp[2\mathrm{i}\theta] + \text{c.c.}$ and the corresponding solution u_2 of Eq. (17) should be sought under the form

$$u_2 = \mathcal{B}_0(Z, T_1, T_2) + \Big[\mathcal{B}_1(Z, T_1, T_2) e^{\mathrm{i}\theta} + \text{c.c.}\Big] + \Big[\mathcal{B}_2(Z, T_1, T_2) e^{2\mathrm{i}\theta} + \text{c.c.}\Big] \, . \tag{19}$$

When one inserts the form (19) into (17), only terms behaving as $\exp(\pm 2\mathrm{i}\theta)$ remain. Equating their coefficients yields

$$\mathcal{B}_2 = -\mathcal{A}^2 k_0 / (2\omega_0 + 8k_0^3) = -\mathcal{A}^2/(6k_0^2) \, , \tag{20}$$

but $\mathcal{B}_0$ and $\mathcal{B}_1$ are still undetermined.

The final step consists in writing the KdV equation at order ϵ^3:

$$(\partial_t + \partial_z^3) u_3 = -(\partial_{T_2} + 3\partial_z \partial_Z^2) u_1 - (\partial_{T_1} + 3\partial_z^2 \partial_Z) u_2 - \partial_z(u_1 u_2) - \tfrac{1}{2} \partial_Z (u_1^2) \, . \tag{21}$$

In this equation one should remove the resonant sources: the ones which are independent of θ and the ones behaving as $\exp(\pm\mathrm{i}\theta)$. This leads respectively to

$$\partial_{T_1} \mathcal{B}_0 + \partial_Z |\mathcal{A}|^2 = 0 \, , \quad \text{and} \quad (\partial_{T_2} + 3\mathrm{i}k_0 \partial_Z^2)\mathcal{A} + (\partial_{T_1} - 3k_0^2 \partial_Z)\mathcal{B}_1 + \mathrm{i}k_0(\mathcal{A}\mathcal{B}_0 + \mathcal{A}^* \mathcal{B}_2) = 0 \, . \tag{22}$$

Using the form (18) for $\mathcal{A}$ yields $\partial_{T_1}(\mathcal{B}_0 - |\mathcal{A}|^2/v_g) = 0$, and one takes

$$\mathcal{B}_0 = |\mathcal{A}|^2 / v_g = -|\mathcal{A}|^2/(3k_0^2) \, . \tag{23}$$

One thus sees that the terms $\mathcal{A}$, $\mathcal{B}_0$ and $\mathcal{B}_2$ depend on the slow variables Z and T_1 only through the combination $Z - v_g T_1$. This corresponds to the important physical result that the envelope of the wave packet propagates at the group velocity. We want to enforce this property and we impose that $\mathcal{B}_1$ should also depend only on $Z - v_g T_1$. This cancels the term $(\partial_{T_1} - 3k_0^2 \partial_Z)\mathcal{B}_1$ in the right equation of (22), which then takes a NLS form if one expresses $\mathcal{B}_2$ and $\mathcal{B}_0$ in term of $\mathcal{A}$ through (20) and (23):

$$\partial_{T_2}\mathcal{A} = -3\mathrm{i}k_0 \partial_Z^2 \mathcal{A} - \mathrm{i}k_0(-|\mathcal{A}|^2\mathcal{A}/3k_0^2 - \mathcal{A}^*\mathcal{A}^2/6k_0^2) \iff \mathrm{i}\partial_{T_2}\mathcal{A} = 3k_0 \partial_Z^2 \mathcal{A} - \frac{1}{2k_0}|\mathcal{A}|^2 \mathcal{A} \, . \tag{24}$$

[6] Secular behavior refers to a situation in which the iterates u_2, u_3, ... grow algebraically in the fast time or space variables. If this were allowed to happen, the asymptotic series (13) would not be uniformly valid over long times and distances.

Finally one goes back to the original variables z and t. One defines $x = z - v_g t$ and $A(x,t) = \epsilon\mathcal{A}(Z - v_g T_1, T_2)$. Then a term ϵ^2 factorizes out in (24) and one obtains

$$\mathrm{i}A_t = 3k_0 A_{xx} - \frac{1}{2k_0}|A|^2 A \; . \tag{25}$$

Note that the coefficient of the dispersive operator ∂_x^2 in this equation is $-\frac{1}{2}\mathrm{d}^2\omega_0/\mathrm{d}k_0^2$, as expected from Eq. (11) [see Eq. (16)]. In the present case the coefficient α of Eq. (11) is $\alpha = -1/2k_0$. It is indeed k_0-dependent.

3 Modulation of a monochromatic wave in a nonlinear dielectric medium

In this section we study the propagation of a monochromatic optical beam in a nonlinear dielectric medium. The physical problem is again the nonlinear modulation of a plane wave, but the context is slightly different from the one of Secs. 1 and 2. As a result, the form of the resulting NLS differs from the one of Eqs. (11) and (25).

One[7] considers a non magnetic dielectric material. Maxwell equations read

$$\vec{\nabla}\wedge\vec{E} = -\partial_t\vec{B}\;, \quad \vec{\nabla}\wedge\vec{B} = \frac{1}{\varepsilon_0 c^2}\,\partial_t\vec{D}\;, \quad \vec{\nabla}\cdot\vec{D} = 0\;, \quad \vec{\nabla}\cdot\vec{B} = 0\;, \tag{26}$$

where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$, $\vec{P}(\vec{r},t)$ being the electric polarization density. We consider a medium in which $\vec{P}$ depends nonlinearly on $\vec{E}$ (see below). Assuming that $\vec{\nabla}\cdot\vec{E} = 0$ (this will be justified *a posteriori*), one gets $\vec{\nabla}\wedge\vec{\nabla}\wedge\vec{E} = -\vec{\nabla}^2\vec{E}$ and the two first of Eqs. (26) yield

$$\vec{\nabla}^2\vec{E} = \frac{1}{\varepsilon_0 c^2}\,\partial_t^2\vec{D} = \frac{1}{c^2}\,\partial_t^2\vec{E} + \frac{1}{\varepsilon_0 c^2}\,\partial_t^2\vec{P}\;. \tag{27}$$

One considers a linearly polarized beam, of the form

$$\vec{E}(\vec{r},t) = \left(\tfrac{1}{2}\mathcal{E}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t} + \mathrm{c.c.}\right)\hat{x}\;, \tag{28}$$

where $\mathcal{E}(\vec{r})$ is a complex function. This will be denotes as a stationary beam (since the beam profile does not depend on time).

One has two types of dielectric response of the medium: the linear polarization density $\vec{P}_{\mathrm{L}}(\vec{r},t)$, and the nonlinear one, $\vec{P}_{\mathrm{NL}}(\vec{r},t)$. One has

$$\vec{P}_{\mathrm{L}}(\vec{r},t) = \varepsilon_0\,\chi^{(1)}\vec{E}(\vec{r},t) = \left\{\tfrac{1}{2}\varepsilon_0\,\chi^{(1)}\mathcal{E}(\vec{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t} + \mathrm{c.c.}\right\}\hat{x}\;, \tag{29}$$

where $\chi^{(1)}$ is the linear electric susceptibility. In a non-isotropic medium, the susceptibility is a second rank tensor: $P_{\mathrm{L},i} = \varepsilon_0\,\underset{\sim}{\chi}{}^{(1)}_{ij}E_j$; it may also depend on position, and be nonlocal in time. In the following we consider none of these effects.

[7]This section is largely inspired by §109 of the 8$^{\text{th}}$ volume of the course of Landau and Lifshitz: "Electrodynamics of continuous media".

In standard materials, with spatial inversion symmetry[8], the dominant nonlinear dielectric response reads (assuming it is local in time)

$$P_{\mathrm{NL},i}(\vec{r},t) = \varepsilon_0\, \underset{\sim}{\chi}^{(3)}_{ijkl}\, E_j(\vec{r},t) E_k(\vec{r},t) E_l(\vec{r},t)\ , \tag{30}$$

where $\underset{\sim}{\chi}^{(3)}$ is a fourth rank tensor. In an isotropic medium, it is customary to simply write $\underset{\sim}{\chi}^{(3)}_{xxxx} = \chi^{(3)}$, and counting the different equal contributions to (30) one gets[9]

$$\vec{P}_{\mathrm{NL}}(\vec{r},t) = \left\{ \tfrac{3}{8}\varepsilon_0 \chi^{(3)} |\mathcal{E}(\vec{r})|^2 \mathcal{E}(\vec{r})\, \mathrm{e}^{-\mathrm{i}\omega_0 t} + \mathrm{c.c.} \right\} \hat{x} = \tfrac{3}{4}\varepsilon_0 \chi^{(3)} |\mathcal{E}(\vec{r})|^2 \vec{E}(\vec{r},t)\ . \tag{31}$$

In the following one considers an almost monochromatic wave propagating mostly along the z axis and one writes

$$\mathcal{E}(\vec{r}) = A(\vec{r}_\perp, z)\, \mathrm{e}^{\mathrm{i}\beta_0 z}\ , \quad \text{where} \quad |\partial_z A| \ll \beta_0 |A|\ , \tag{32}$$

meaning that one considers a weak modulation of a plane wave. Then

$$\vec{\nabla}^2 \mathcal{E} = \mathrm{e}^{\mathrm{i}\beta_0 z} \left(\vec{\nabla}_\perp^2 A - \beta_0^2 A + 2\mathrm{i}\beta_0 A_z + \cancel{A_{zz}} \right)\ . \tag{33}$$

Inserting this expression (without the A_{zz} term[10]) in (27) one sees that the amplitude A verifies

$$2\mathrm{i}\beta_0 A_z + \vec{\nabla}_\perp^2 A - \beta_0^2 A = -\frac{\omega_0^2}{c^2} \left(1 + \chi^{(1)} + \tfrac{3}{4}\chi^{(3)} |A|^2 \right) A\ . \tag{34}$$

Discarding nonlinear effects, for a perfect plane wave (constant A), Eq. (34) yields $\beta_0^2 = (1+\chi^{(1)})\omega_0^2/c^2$, which is customarily written as $\beta_0 = n_0 k_0$, where $k_0 = \omega_0/c$ and $n_0 = (1+\chi^{(1)})^{1/2}$ is the linear index of refraction. Eq. (34) is thus written as

$$\mathrm{i}\, A_z = -\frac{1}{2\beta_0} \vec{\nabla}_\perp^2 A - k_0 n_2 |A|^2 A\ , \quad \text{where} \quad n_2 = \frac{3}{8\, n_0} \chi^{(3)}\ . \tag{35}$$

This is a nonlinear Schrödinger equation in which z (the propagation distance within the medium) plays the role of time [compare with (11)]. Typically $n_0 \simeq 1$ and $k_0 = 8.05 \times 10^6$ m^{-1} ($\lambda_0 = 780$ nm for a typical infrared laser). In present time experiments, choosing units such that A^2 is an intensity, one has typically $A^2 \sim 10^5$ W.m^{-2} and in highly nonlinear media $n_2 \sim 10^{-10}$ W^{-1}. m^2.

It remains to check that, indeed, the approximation $\vec{\nabla} \cdot \vec{E} = 0$ – made before Eq. (27) – is sound. From (26) $\vec{\nabla} \cdot \vec{D} = 0$, where $\vec{D} = \varepsilon_0 (1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)} |\mathcal{E}|^2) \vec{E}$. Taking the divergence of both sides of this equality yields[11]

$$0 = (1 + \chi^{(1)} + \tfrac{3}{4}\chi^{(3)} |\mathcal{E}|^2) \vec{\nabla} \cdot \vec{E} + \tfrac{3}{4}\chi^{(3)} (\vec{E} \cdot \vec{\nabla}) |\mathcal{E}|^2\ .$$

From the form (28) one gets $\vec{E} \cdot \vec{\nabla} = \mathcal{E}(\vec{r})\partial_x$, and the above equation reads

$$\vec{\nabla} \cdot \vec{E} = -\frac{2\, n_0\, n_2}{1 + \chi^{(1)} + 2\, n_0 n_2 |\mathcal{E}|^2} \mathcal{E}(\vec{r})\, \partial_x |\mathcal{E}|^2\ .$$

This terms is non zero only through the transverse derivative of the slowly varying envelope. It is additionally small because the nonlinear contribution is small ($n_2 |\mathcal{E}|^2 \ll 1$): it can safely be neglected.

[8] A second order term ($\chi^{(2)}$ contribution) is discarded because it only exists in a medium in which, when $\vec{E} \to -\vec{E}$, the nonlinear response is not affected. This does not occur in materials with spatial inversion symmetry.

[9] Discarding terms $\propto \mathrm{e}^{-3\mathrm{i}\omega_0 t}$ which correspond to third harmonic generation.

[10] Neglecting A_{zz} is legitimate because of the inequality at the left of (32). This is called the "paraxial approximation".

[11] For a scalar field $f(\vec{r})$ and a vector field $\vec{E}(\vec{r})$ one has $\vec{\nabla} \cdot (f\, \vec{E}) = f\, \vec{\nabla} \cdot \vec{E} + \vec{E} \cdot \vec{\nabla} f$.

4 Stability of a plane wave

Using the notations of Sec. 1, we say that a perfect plane wave solution $\eta(z,t)$ has an amplitude $A(x,t)$ of constant modulus. The corresponding solution of Eq. (11) is of the form[12] $A_0 \exp(-\mathrm{i}\alpha A_0^2 t)$, where A_0 can be chosen in $\mathbb{R}^+$. The carrier wave gets an additional phase $\alpha A_0^2 t$ induced by nonlinearity: this phenomenon is denoted as "self phase modulation" (SPM) in optics.

For studying the stability of a plane wave, one looks for the dynamics of a small perturbation to its amplitude and one writes:

$$A(x,t) = A_0 e^{-\mathrm{i}\alpha A_0^2 t}\,[1 + a(x,t)] \;, \quad \text{where} \quad |a| \ll 1 \;. \tag{36}$$

One gets, upon linearizing Eq. (11)

$$\mathrm{i}\, a_t = -\tfrac{1}{2}\omega_0'' a_{xx} + \alpha A_0^2\,(a + a^*) \;, \tag{37}$$

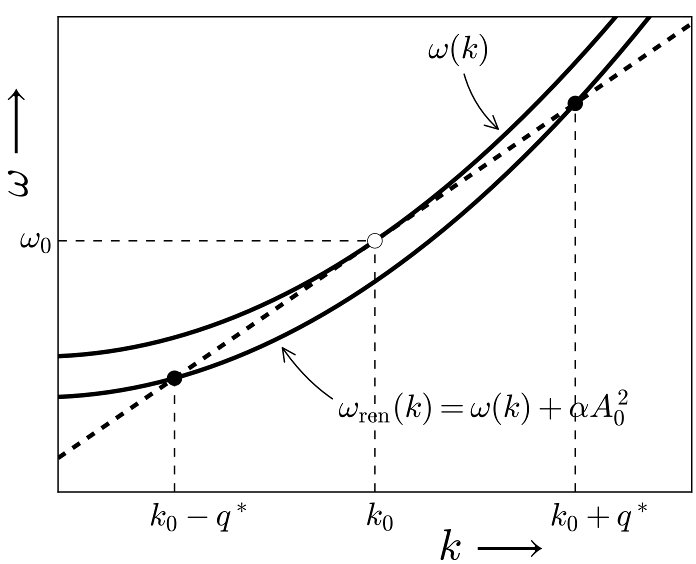
where a^* is the complex conjugate of a. Writing $a = u(x,t) + \mathrm{i}v(x,t)$ where u and v are real, one gets

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\omega_0''\partial_x^2 \\ \frac{1}{2}\omega_0''\partial_x^2 - 2\alpha A_0^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \;. \tag{38}$$

Looking for solutions (u,v) under the form of plane waves with a phase $(qx - \Omega t)$ one gets

$$\Omega^2 = \alpha A_0^2\, \omega_0''\, q^2 + (q^2 \omega_0''/2)^2 \;, \tag{39}$$

which is called the Bogoliubov dispersion relation. It displays an instability if $\alpha\omega_0'' < 0$. This is called the Benjamin-Feir-Lighthill criterion of modulational instability. For instance, this criterion tells us that the plane wave solution of KdV equation studied in Sec. 2 is stable (since in this case $\alpha\omega_0'' = (-2k_0)^{-1}(-6k_0) > 0$). In the unstable case, the most unstable mode is the one for which Ω has the largest imaginary part. Simple computation shows that it has a wave-vector $q^* = (-2\alpha A_0^2/\omega_0'')^{1/2}$.

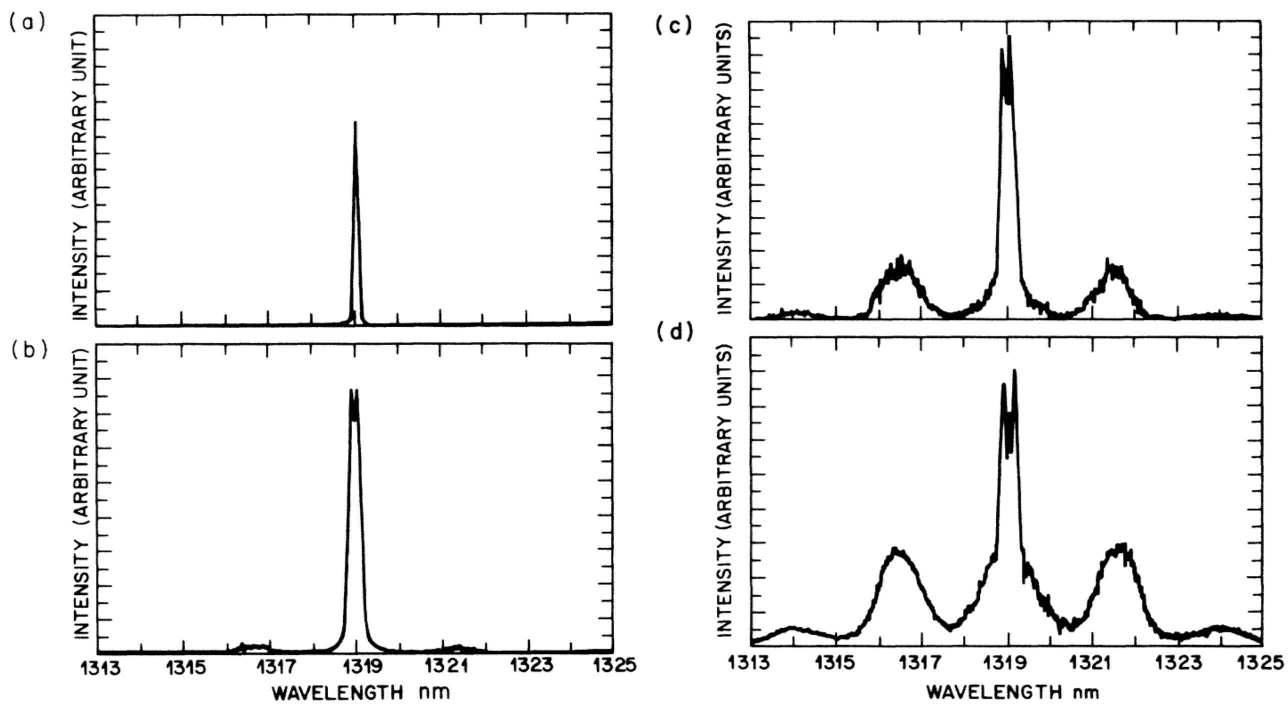


[12]This can be verified by direct insertion in Eq. (11), or by looking for the simplest among the possible constant modulus solutions of (11) of the form $A(x,t) = A_0 \exp[\mathrm{i}\,\vartheta(x,t)]$.

On can devise a geometric construction illustrating the mechanism of modulational instability. This is done in the above figure which corresponds to the case $\omega_0'' > 0$ and $\alpha < 0$. One first plots the approximate parabolic $\omega(k)$ of Eq. (7): $\omega = \omega_0 + (k - k_0)\omega_0' + \frac{1}{2}(k - k_0)^2\omega_0''$. One also plots the "renormalized dispersion relation" $\omega_{\rm ren}(k)$ given by (10) which entails nonlinear effects. It is easy to check that the tangent to $\omega(k)$ at (k_0, ω_0) intercepts the shifted parabola $\omega_{\rm ren}(k)$ at points of abscissa $k_1 = k_0 - q^*$ and $k_2 = k_0 + q^*$, and that, besides, $\omega_{\rm ren}(k_1) + \omega_{\rm ren}(k_2) = 2\,\omega_0$ and $k_1 + k_2 = 2\,k_0$. This suggests that the initial carrier wave at frequency ω_0 decays into two side-bands according to the process:

$$\begin{cases} \omega_0 + \omega_0 \to \omega_{\rm ren}(k_1) + \omega_{\rm ren}(k_2)\ , \\ k_0 + k_0 \to k_1 + k_2\ . \end{cases} \tag{40}$$

This behavior, with the formation of two side-bands at k_1 and k_2, is illustrated by the figure below[13] which represents a power spectrum measured at the output of a nonlinear optic fiber, for increasing input powers [(a) low-power (or input) case, (b) 5.5 W, (c) 6.1 W, and (d) 7.1 W]. The largest the input power, the largest A_0, the more developed is the instability at the output.



Finally some terminology: if $\alpha\omega_0'' < 0$ one speaks of focusing NLS. In the opposite situation one speaks of defocusing NLS. As just seen, constant amplitude solutions of the focusing NLS suffer from a modulational instability, but there exists many other stable solutions of this equation, in particular solitons. The defocusing NLS also admits stable soliton solutions[14], but of a slightly different type (they are called "dark solitons"), cf. tutorial # 2.

[13] From K. Tai, A. Hasegawa, and A. Tomita, Phys. Rev. Lett. **56**, 135 (1986).

[14] They have been observed, for instance, in the envelope of surface water waves, see A. Chabchoub *et al.*, Phys. Rev. E **89**, 011002(R) (2014).