Linear and Nonlinear Waves
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CHAPTER 12

Wave Patterns

Some of the most interesting wave patterns are found in water waves. Some of them, such as the V-shaped ship wave pattern or the pattern of rings spreading out from a stone thrown in a pond, are familiar to everyone, and others are relatively easily observed. We start with these. Here the dispersion relation, the only input required, will be merely quoted. We shall need to look further into the subject of water waves later, since it was the first and most fruitful source of ideas on nonlinear dispersive waves. The derivation of the dispersion relation will be included then.

12.1 The Dispersion Relation for Water Waves

In still water elementary solutions for the perturbation η in the height of the surface take the basic form (11.1):

$$\eta = Ae^{i\mathbf{k}\cdot\mathbf{x} - i\omega t},$$

provided

$$\omega^2 = (gk\tanh kh)\left(1 + \frac{T}{\rho g}k^2\right), \qquad k = |\mathbf{k}|. \tag{12.1}$$

Here h is the undisturbed depth, g is the gravitational acceleration, ρ is the density, T is the surface tension. In still water the waves are isotropic and the dispersion involves only the magnitude k of the wave number vector. There are a number of interesting limits which are conveniently used as approximations in appropriate circumstances.

Gravity Waves.

In c.g.s. units, $g = 981$, $\rho = 1$, and $T = 74$, so that $\lambda_m = 2\pi(T/\rho g)^{1/2} = 1.73$ cm. Thus the surface tension effects become negligible for wavelengths several times greater than this value. Then we have the usual formula for gravity waves:

$$\omega^2 = gk\tanh kh, \qquad \lambda \gg \lambda_m. \tag{12.2}$$

For these, the phase and group velocities are

$$c(k) = \left(\frac{g}{k}\tanh kh\right)^{1/2}, \tag{12.3}$$

$$C(k) = \frac{\partial\omega}{\partial k} = \frac{1}{2}c(k)\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{12.4}$$

Within this approximation, we have the limiting cases

$$\omega \sim (gk)^{1/2}, \qquad c \sim \left(\frac{g}{k}\right)^{1/2}, \qquad C \sim \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}, \qquad kh \to \infty, \tag{12.5}$$

$$\omega \sim (gh)^{1/2}k, \quad c \sim (gh)^{1/2}, \qquad C \sim (gh)^{1/2}, \qquad kh \to 0. \tag{12.6}$$

For fixed h, both c and C are increasing functions of $\lambda = 2\pi/k$, with $C < c$; in the long wave limit (12.6), $C \to c$ and the dispersive effects become small. Of course, the approximation (12.5) is appropriate for short waves when $\lambda_m \ll \lambda \ll h$.

Capillary Waves.

For $\lambda \ll \lambda_m$, the surface tension effect may be dominant and (12.1) is then approximated by

$$\omega^2 = \frac{T}{\rho}k^3\tanh kh. \tag{12.7}$$

In this case

$$c(k) = \left(\frac{T}{\rho}k\tanh kh\right)^{1/2}, \tag{12.8}$$

$$C(k) = \frac{3}{2}c\left(1 + \frac{2kh}{3\sinh 2kh}\right). \tag{12.9}$$

The further limits of these are

$$\omega \sim \left(\frac{T}{\rho}\right)^{1/2} k^{3/2}, \qquad c \sim \left(\frac{T}{\rho}\right)^{1/2} k^{1/2}, \qquad C \sim \frac{3}{2}\left(\frac{T}{\rho}\right)^{1/2} k^{1/2}, \qquad kh \to \infty, \tag{12.10}$$

and

$$\omega \sim \left(\frac{Th}{\rho}\right)^{1/2} k^{2}, \qquad c \sim \left(\frac{Th}{\rho}\right)^{1/2} k, \qquad C \sim 2\left(\frac{Th}{\rho}\right)^{1/2} k, \qquad kh \to 0. \tag{12.11}$$

For capillary waves, c and C are decreasing functions of λ, with $C > c$.

Combined Gravity and Surface Tension Effects.

When both effects are important, it is usually sufficient to consider relatively short waves, $kh \gg 1$, and take

$$\omega^2 = gk + \frac{T}{\rho}k^3. \tag{12.12}$$

The phase and group velocities are

$$c = \left(\frac{g}{k} + \frac{T}{\rho}k\right)^{1/2}, \tag{12.13}$$

$$C = \frac{1}{2}c\,\frac{1 + (3T/\rho g)k^2}{1 + (T/\rho g)k^2}. \tag{12.14}$$

The phase velocity has a minimum at $k = k_m$, where

$$k_m = \left(\frac{\rho g}{T}\right)^{1/2}, \qquad \lambda_m = \frac{2\pi}{k_m} = 1.73\,\text{cm}; \tag{12.15}$$

the corresponding values of c and C are equal with

$$c_m = 23.2\ \text{cm/sec}.$$

For $\lambda > \lambda_m$, often known as the gravity branch, $C < c$; whereas for $\lambda < \lambda_m$, known as the capillary branch, $C > c$. For any given value of $c > c_m$, there are two possible wavelengths. The minimum group velocity is attained at $\lambda = 2.54\lambda_m = 4.39$ cm, $C = 0.77c_m = 17.9$ cm/sec.

Shallow Water with Dispersion.

In the limit $kh \to 0$, (12.1) may be expanded as

$$\omega^2 \sim ghk^2\left\{1 + \left(\frac{T}{\rho g h^2} - \frac{1}{3}\right)k^2h^2 + \cdots\right\}, \tag{12.16}$$

and we have

$$c \sim (gh)^{1/2}\left\{1 + \frac{1}{2}\left(\frac{T}{\rho g h^2} - \frac{1}{3}\right)k^2h^2 + \cdots\right\}. \tag{12.17}$$

When dispersion is neglected altogether, the equations for *nonlinear* shallow water theory are hyperbolic and similar to those of gas dynamics; this so-called hydraulic analogy has been exploited for experiments. The dispersion must be kept to a minimum so h is chosen such that

$$\frac{T}{\rho g h^2} - \frac{1}{3} = 0,$$

that is,

$$h = \left(\frac{3T}{\rho g}\right)^{1/2} = 0.48 \text{ cm}.$$

Magnetohydrodynamic Effects.

In a conducting liquid a third vertical restoring force may be introduced when a horizontal magnetic field is applied and horizontal currents flow through the liquid. This has been investigated by Shercliff (1969), who finds that the dispersion relation is

$$\rho\omega^2 = k \tanh kh(\rho g + k^2T + J_s B_n),$$

where B_n is the magnetic field normal to the wave crests and J_s the current along them. The term $J_s B_n$ is the vertical component of the Lorentz force. It is interesting that the propagation depends on the orientation of the waves to the field and becomes anisotropic. Details of the phase and group velocities, and of the various limiting cases, are to be found in the paper quoted. We shall not pursue this case further here, although the various wave patterns can be studied by the methods developed below.

12.2 Dispersion from an Instantaneous Point Source

The waves from a point source spread out isotropically and the different values of k introduced initially propagate out with the corresponding group velocities $C(k)$. At time t any particular value k will be found at $r = C(k)t$. Hence $k(r,t)$ is the solution of

$$C(k) = \frac{r}{t}. \tag{12.18}$$

For deep water gravity waves (12.5), we have therefore

$$k = \frac{gt^2}{4r^2}, \qquad \omega = \frac{gt}{2r}. \tag{12.19}$$

This is the axisymmetric counterpart of the one dimensional problem noted in Section 11.4. This very simple formula for ω has been checked by Snodgrass et al, (1966) against observational data of the swells produced by storms in the South Pacific. At distances of the order of 2000 miles, the frequency was found to vary linearly with t, and the constant of proportionality gave a very accurate determination of the distance of the storm.

On a smaller scale, the typical rings spreading from a stone or other splash in a pond satisfy (12.18) with $C(k)$ given by (12.14). Since $C(k)$ has a minimum value of about 18 cm/sec, there is a quiescent circle of radius $18t$ cm. Beyond that there are two values of k for each r/t, one on the gravity branch and one on the capillary branch, so there are two superimposed wavetrains. Of course, the energy in the different wave numbers is determined by the initial disturbance. Waves with wavelength of the same order as the size of the object will have the largest amplitudes and will be most accentuated.

12.3 Waves on a Steady Stream

The waves produced by an obstacle on a steady stream U in the x_1 direction may be viewed as the waves produced by an obstacle moving with speed U in the negative x_1 direction. For a two dimensional obstacle, with flow independent of x_2, the only waves that can keep up with the obstacle and appear steady when viewed from the obstacle must satisfy

$$c(k) = U. \tag{12.20}$$

We again take the situation when (12.12)–(12.14) apply. There will be no solutions to (12.20) and hence no steady wavetrain if $U < c_m$. In this case there will be local disturbances dying out away from the obstacle but no contribution to the asymptotic wave pattern. If $U > c_m$ there will be two solutions of (12.20): one of them, k_g say, on the gravity branch, and one, k_T say, on the capillary branch. Now $k_g < k_m$ and $k_T > k_m$; hence from the properties of (12.13)–(12.14), we have

$$C(k_g) < c(k_g) = U, \tag{12.21}$$

$$C(k_T) > c(k_T) = U. \tag{12.22}$$

Therefore, the gravity waves k_g have group velocity less than U and will appear behind the obstacle; the capillary waves k_T have group velocity greater than U and will be ahead of the obstacle. The resulting pattern is shown in Fig. 12.1.

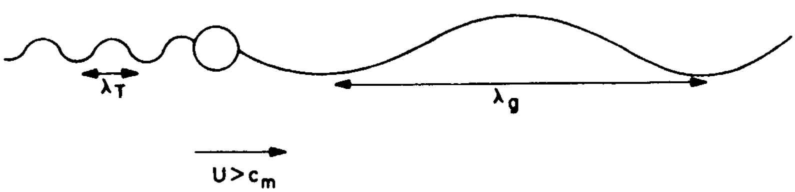

Fig. 12.1. Sketch of capillary waves (upstream) and gravity waves (downstream) produced by an obstacle on the surface of a stream.

This is an interesting use of the group velocity concepts to determine the correct radiation condition in a steady flow problem. For this reason it is also interesting to derive the result in detail from the exact Fourier transform solution and see how the group velocity condition comes out of the usual type of radiation condition used in the techniques of solving boundary value problems. At the same time, the full solution gives the amplitudes of the waves. The asymptotic analysis tells us only that the amplitude remains constant in each wavetrain; the detailed initial conditions have to be analyzed to determine their values. It would interrupt the present discussion of kinematics to give the details here. They will be given in Section 13.9.

12.4 Ship Waves

For an obstacle that is finite in the x_2 direction we have a two dimensional wave pattern on the surface of the water and the analysis is more complicated. We shall study only the gravity wave problem for deep water and use the dispersion relation (12.5). This covers the pattern produced by any object of dimension $l \gg \lambda_m$ moving on water with depth $h \gg l$; this is the usual situation for ship waves.

The most striking result, originally due to Kelvin, is that in deep water the waves are confined to a wedge shaped region of wedge semiangle $\sin^{-1}\frac{1}{3} = 19.5°$. This result is independent of the velocity provided the velocity is constant; it is independent of the shape of the object, and it depends only on the fact that $C/c = \frac{1}{2}$ for deep water.

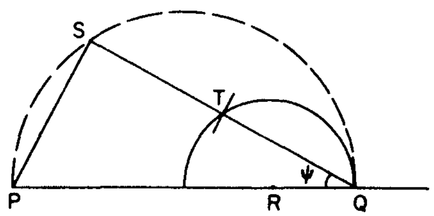

Fig. 12.2. Construction of wave elements in ship wave problem.

A concise form of the argument is given by Lighthill (1957). Consider the "ship" to move from Q to P in Fig. 12.2, in time t, and let its speed be U. For a wave crest to keep a stationary position relative to the ship,

$$U\cos\psi = c(k), \tag{12.23}$$

where ψ is the inclination of the normal (direction of $\mathbf{k}$) to the line of motion QP. This condition is most easily seen by taking the frame of reference in which the stream of velocity U flows past a stationary ship; the stream has component $U\cos\psi$ normal to the wave element and this must be balanced by the phase velocity of the element. The condition tells us the value of k to be found in the direction ψ. It may be represented geometrically in Fig. 12.2 by constructing the semicircle with diameter PQ and noting that $PQ = Ut$, $SQ = Ut\cos\psi = ct$. Therefore wave elements parallel to PS will have $ct = QS$. Now c is the phase speed and is appropriate for condition (12.23), but the group velocity $C = \frac{1}{2}c$ determines the location of these waves. The waves produced at Q will have traveled a

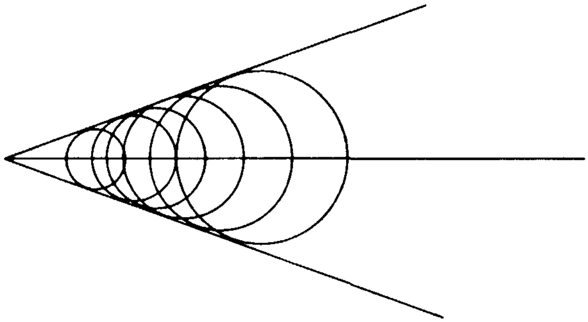

Fig. 12.3. Envelope of the disturbance emitted at successive times.

distance $Ct = \frac{1}{2}ct$. Therefore in the direction ψ they will be found at T, the midpoint of QS. Including all values of ψ, we deduce that those waves produced at Q which can contribute to a stationary pattern lie on a circle of radius $\frac{1}{4}Ut$ centered at R, where $PR = \frac{3}{4}Ut$. Finally, varying t for a fixed point P, we have the pattern of circles of Fig. 12.3. From the construction in Fig. 12.2, each circle has a radius one third the distance of its center from P. Hence they fill a wedge-shaped region with semiangle $\sin^{-1}\frac{1}{3} = 19.5°$. It is amusing to note that the construction in Fig. 12.3 is the same as for supersonic flow with Mach number 3; all swimming objects have effective Mach number 3.

Further Details of the Pattern.

In discussing the pattern in more detail, it is convenient to take the reference frame in which the source is fixed at P and there is a uniform stream U in the x_1 direction (see Fig. 12.4). This raises some general points about handling steady patterns which are also useful in other contexts. The dispersion relations in Section 12.1 apply to waves propagating into still water, but we may transfer to any other reference frame moving with relative velocity $-\mathbf{U}$ by noting that the frequency ω relative to the moving frame is given in terms of the frequency ω_0 in the stationary frame by

$$\omega = \mathbf{U}\cdot\mathbf{k} + \omega_0(\mathbf{k}). \tag{12.24}$$

This is the dispersion relation between ω and $\mathbf{k}$ for waves superimposed on a stream $\mathbf{U}$. Of course the propagation is no longer isotropic since the direction of $\mathbf{U}$ enters. For a steady wave pattern in this frame, $\omega = 0$ and (12.24) becomes a relation between the components k_1 and k_2 of the wave number vector $\mathbf{k}$. With $\omega_0(\mathbf{k}) = \sqrt{gk}$, we have

$$G(k_1,k_2) = Uk_1 + \sqrt{gk} = 0. \qquad (12.25)$$

Since $\cos\psi = -k_1/k$ and $c(k) = \sqrt{g/k}$, this is the same as (12.23). We may also write it using the polar coordinates (k,ψ) for $\mathbf{k}$ as

$$\mathcal{G}(k,\psi) = Uk\cos\psi - \sqrt{gk} = 0. \qquad (12.26)$$

Since the frequency ω is zero and $\mathbf{k}$ is independent of t, the kinematic description (11.43) reduces to the consistency relation

$$\frac{\partial k_2}{\partial x_1} - \frac{\partial k_1}{\partial x_2} = 0. \qquad (12.27)$$

From (12.25), $k_1 = f(k_2)$, say, and (12.27) gives

$$\frac{\partial k_2}{\partial x_1} - f'(k_2)\frac{\partial k_2}{\partial x_2} = 0.$$

Hence k_2 and k_1 are constant on characteristics

$$\frac{dx_2}{dx_1} = -f'(k_2).$$

For a point source P, the characteristics carrying disturbances pass through P and we have a centered wave

$$\frac{x_2}{x_1} = -f'(k_2); \qquad (12.28)$$

this gives k_2 as a function of x_2/x_1, and $k_1 = f(k_2)$ completes the solution for $\mathbf{k}$.

The basic relation (12.28) can be written symmetrically in k_1 and k_2. If $k_1 = f(k_2)$ satisfies (12.25) identically,

$$f'(k_2)G_{k_1} + G_{k_2} = 0,$$

and (12.28) may be written

$$\frac{x_2}{x_1} = \frac{G_{k_2}(k_1,k_2)}{G_{k_1}(k_1,k_2)}, \qquad G(k_1,k_2) = 0. \qquad (12.29)$$

These are to be solved to give k_1 and k_2 as functions of **x**. The distribution of **k** is sufficient to sketch out the pattern, but the phase $\theta(x_1,x_2)$ can also be deduced to give the equations of the crests.

It might be noted that (12.29) is the limit of the nonstationary centered wave solution as $\omega \to 0$. For the centered wave solution in (11.49) we have

$$\frac{x_i}{t} = C_i = -\frac{G_{k_i}}{G_\omega}, \qquad G(\mathbf{k},\omega) = 0.$$

If we take the ratios of the first set to eliminate t and G_ω, (12.29) follows as the limit $\omega \to 0$. We can think of the disturbance propagating out with the group velocity C_i even though its form is unchanging and there is no change in the appearance of the pattern. We may refer to the group velocity in this sense even though it is only its direction $\partial G/\partial k_i$ that appears in the formulas.

A further remark is that polar coordinates are sometimes useful, as in (12.26). In polar coordinates the gradient $\partial G/\partial \mathbf{k}$ has a component $\partial \mathcal{G}/\partial k$ in the direction of **k** and a component $\partial \mathcal{G}/k\,\partial\psi$ perpendicular to **k**. Hence the angle μ in Fig. 12.4 is given by

$$\tan\mu = \frac{1}{k}\frac{\partial \mathcal{G}/\partial\psi}{\partial \mathcal{G}/\partial k}. \qquad (12.30)$$

The content of (12.29) is then equivalent to

$$\xi = \pi - \mu - \psi, \qquad \mathcal{G}(k,\psi) = 0. \qquad (12.31)$$

Equations 12.30 and 12.31 determine k and ψ (and hence **k**) for the direction ξ.

These formulations apply to any steady two dimensional pattern, and

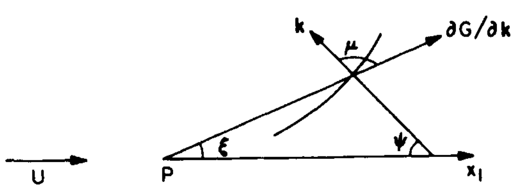

Fig. 12.4. Geometry of wave crests in ship wave problem.

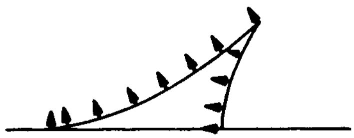

Fig. 12.5. A complete wave crest in ship wave problem.

we now apply them to ship waves. Using (12.25) in (12.29), we have

$$\tan\xi = \frac{x_2}{x_1} = \frac{\dfrac{k_2}{2k}\sqrt{\dfrac{g}{k}}}{U + \dfrac{k_1}{2k}\sqrt{\dfrac{g}{k}}}, \qquad Uk_1 + \sqrt{gk} = 0.$$

It is clearly more convenient to switch to the (k,ψ) description of **k** and reduce these to

$$\tan\xi = \frac{\tan\psi}{1+2\tan^2\psi}, \qquad k = \frac{g}{U^2\cos^2\psi}. \tag{12.32}$$

If the approach via (12.30)–(12.31) is preferred, we have

$$\tan\mu = -2\tan\psi, \tag{12.33}$$

and (12.32) follows.

We may now sketch out a typical wave crest as ψ varies. From (12.25) or (12.26), $k_1<0$ and $\cos\psi>0$, so that only the range $-\pi/2<\psi<\pi/2$ is permissible. The pattern is clearly symmetrical and it is sufficient to take the range $0<\psi<\pi/2$. From (12.32) we see that the values $\psi\to 0$ and

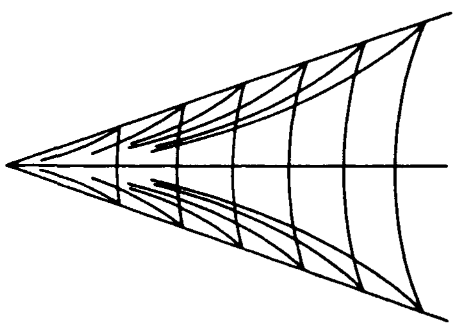

Fig. 12.6. Final ship wave pattern.

$\psi \to \pi/2$ are both to be found on $\xi = 0$, and there must be a maximum value of ξ in the range. It is easily verified that the maximum value is

$$\xi_m = \tan^{-1}\frac{1}{2\sqrt{2}} = 19.5° \qquad \text{at} \quad \psi_m = \tan^{-1}\frac{1}{\sqrt{2}} = 35.3°.$$

This agrees with the wedge angle found earlier and shows that the wave pattern is confined to this wedge. At the maximum $\psi \neq \pi/2$ therefore the wave crest can not turn back smoothly; there must be a cusp at $\psi = \psi_m$ on the boundary of the wedge. We may then complete the shape as given in Fig. 12.5 and the whole pattern must appear as in Fig. 12.6.

The formula for the phase function $\theta(\mathbf{x})$ may be found from

$$\theta = \int_0^{\mathbf{x}} \mathbf{k} \cdot d\mathbf{x}, \tag{12.34}$$

using any convenient path since $\mathbf{k}$ is irrotational. Obviously the rays $\xi = \text{constant}$ are convenient since $\mathbf{k}$ remains constant on them. We have

$$\theta = (k \cos \mu) r, \tag{12.35}$$

where $r = |\mathbf{x}|$ is distance from the origin. Here k and μ are functions of ξ given by (12.32) and (12.33). A phase curve $\theta = \text{constant}$ is given parametrically in terms of ψ by

$$r = \frac{\theta}{k \cos \mu} = -\frac{U^2\theta}{g}\cos^2\psi\{1 + 4\tan^2\psi\}^{1/2},$$

$$\tan\xi = \frac{\tan\psi}{1 + 2\tan^2\psi},$$

and θ is negative. These may also be written

$$\left.\begin{aligned} x_1 &= -\frac{U^2\theta}{g}\cos\psi(1 + \sin^2\psi), \\ x_2 &= -\frac{U^2\theta}{g}\cos^2\psi\sin\psi. \end{aligned}\right\} \tag{12.36}$$

12.5 Capillary Waves on Thin Sheets

One can study steady patterns of capillary waves in similar fashion, and a particularly interesting setting is Taylor's study (1959) of waves on