ADVANCED NONLINEAR PHYSICS

Duration: 3 hours. Problems A and B correspond to the first part of the course, problem C to the second part. They are all independent of each other. Please use different sheets for writing up the solution of each problem.

Dictionaries, handwritten notes on the courses and tutorials are allowed. Books as well as computers, telephones and other electronic devices are forbidden.

A Non convex flux: shock-rarefaction

One considers the nonlinear advection equation:

$$u_t + (u^3)_x = 0 \iff u_t + c(u)u_x = 0 \ , \quad \text{where} \quad c(u) = 3\,u^2 \ . \tag{A1}$$

The initial condition is of the Riemann type:

$$u(x,0) = \begin{cases} a & \text{if } x < 0 \ , \\ b & \text{if } x > 0 \ , \end{cases}$$

where a and b are two constants, with $a > 0$.

1/ Assume that $b > a$. Schematically draw the characteristics in the (x,t) plane and sketch the shape of $u(x,t)$ for $t > 0$.

2/ Same question for $0 < b < a$.

3/ One now considers the situation $b < -a/2$. In this case, although one can write a shock condition fulfilling the conservation equation [left equation in (A1)], one cannot fulfill the so called Lax entropy condition:

$$c(b) \le \dot{s}(t) \le c(a) \ , \tag{A2}$$

where $s(t)$ is the position of the shock and $\dot{s}(t)$ its velocity[1].

(a) Show that when $b < -a/2$ a solution of the type found in question 2/ violates the left inequality in the Lax condition (A2).

Note: this is a simple, but technical question. You may use the property $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$. You may also admit the result, remember that, just at threshold, when $b = -a/2$, one has $c(-a/2) = \dot{s}$ and go to the next question.

(b) In this case, one looks for a solution of the shock-rarefaction type: a shock with a spatial discontinuity of $u(x,t)$ from a to $-a/2$ attached to a rarefaction wave. Draw the associated profile $u(x,t)$ and the corresponding characteristics in the (x,t) plane.

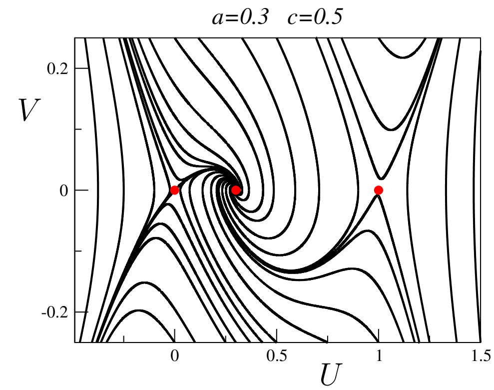
[1] The physical origin of this condition is clear: the shock should be at the meeting of characteristics and cannot be faster than the fast ones and slower than the slow ones.

B Nagumo model

We consider here a reaction-diffusion process governed by the equation[2]

$$u_t = u_{xx} + u(u-a)(1-u) \ , \quad \text{where} \quad a \in [0,1] \ . \qquad \text{(B1)}$$

One looks for a traveling wave solution of the form $u(x,t) = U(x - ct)$, where c is a positive constant. One will denote $z = x - ct$ and $U' = \mathrm{d}U/\mathrm{d}z$.



1/ Write the equation of the corresponding flow in the (U,V) phase space where $V = U'$. A typical phase portrait is displayed in the figure at the right.
Identify the 3 equilibrium points. Compute the corresponding Jacobian matrices, and discuss the stability of these points as a function of the value of the parameters a and c.

2/ One looks for a traveling wave solution of (B1) joining the value $u = 1$ at $x \to -\infty$ to the value $u = 0$ at $x \to +\infty$. This corresponds to a heteroclinic orbit joining two saddles: it is natural to expect that it is only realized for particular values of the parameters.

As an educated guess, one looks for an equation for this heteroclinic orbit of the form $V = bU(1-U)$, where b is a constant. Determine the possible values of b and the corresponding value of c, imposing that c should be positive.

3/ Give the corresponding explicit expression of the traveling wave solution[3] $U(z)$ (choosing the integration constant so that $U(0) = 1/2$). Which is the unique possible value of b fulfilling the expected boundary conditions ?

[2]This is a version of the Fisher model where the logistic map is modified in order to account for the so called Allee effect: negative growth for low population densities (lower than a). The physical idea is that at small densities, animals are less able to find mates.

[3]Indication: $\int \mathrm{d}U/[U(1-U)] = \ln[U/(1-U)]$.