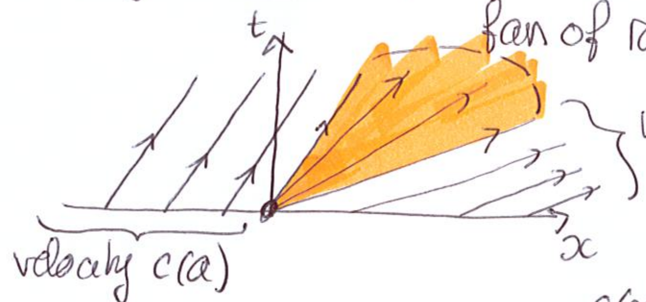
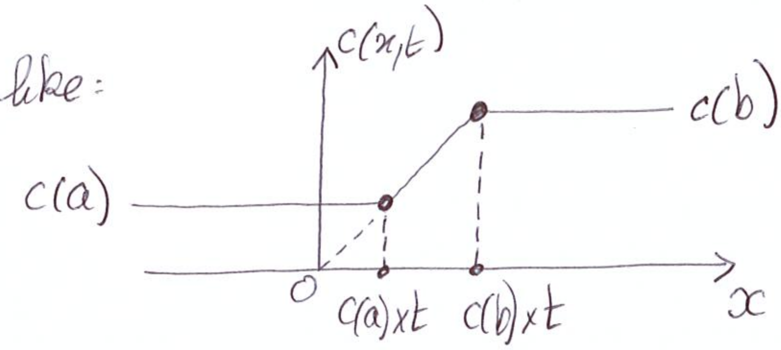
Shock - Rarefaction

1) <u>case $b>a$</u> (>0) clearly $c(b)>c(a)$ and the characteristics in the (x,t) plane look like:



the curve $c(x,t)$ at $t>0$ looks like:

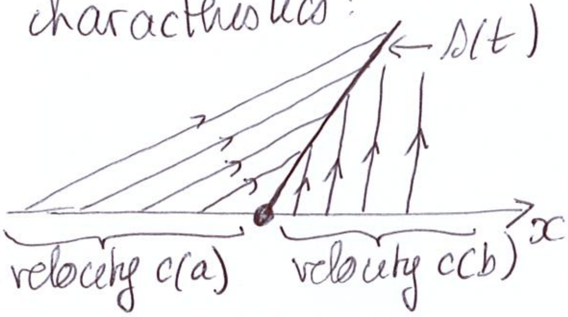
(one can obtain $u(x,t)$ from $u(x,t)=\sqrt{\frac{1}{3}c(x,t)}$)



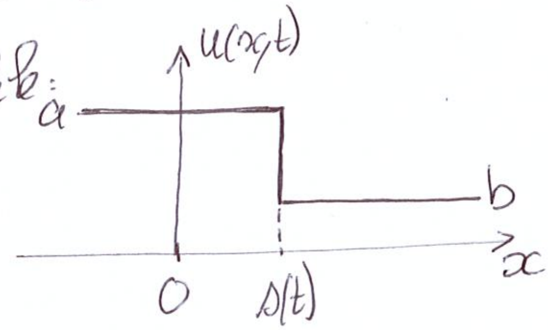
2) <u>case $0<b<a$</u> clearly $0<c(b)<c(a)$ = a shock is formed at $t=0$.

Its velocity is = $\dot{s}(t)=\frac{a^3-b^3}{a-b}=a^2+ab+b^2$ (it's a constant)

characteristics:



profile:



3/ <u>case $b<-a/2<0$</u> one should fulfill the Lax entropy condition =

$$c(b)=3b^2\leq\dot{s}=a^2+ab+b^2\leq 3a^2=c(a)$$

this imposes $2b^2-ab-a^2\leq 0$

roots $=\frac{a\pm\sqrt{9a^2}}{4}=-a/2$ and a

hence one must have

$$-\frac{a}{2}\leq b\leq a$$

this imposes $b^2+ab-2a^2\leq 0$

roots $=\frac{-a\pm\sqrt{9a^2}}{2}=-2a$ and a

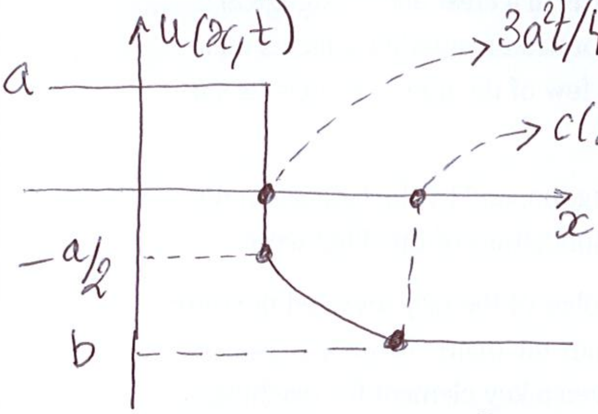
hence one must have

$$-2a\leq b\leq a$$

the left condition is the more stringent.
(ie $c(b)\leq\dot{s}$)

if $b < -a/2$ $\dot{s} < c(b)$ and one violates the Lax entropy condition. SR2

the text proposes the following solution:

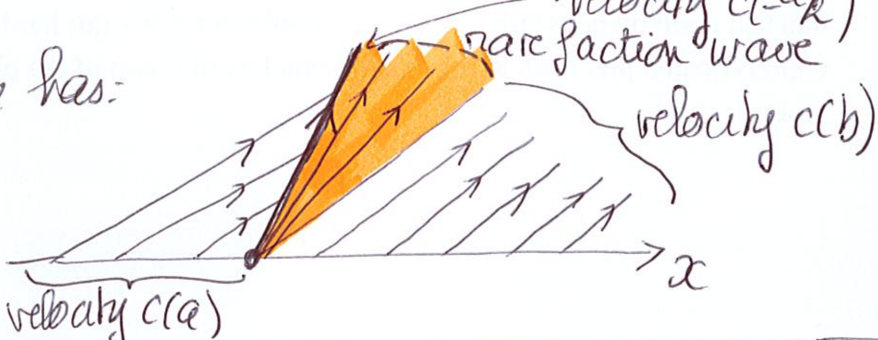


$\to 3a^2t/4$ easily determined since for this shock $\dot{s} = c(-a/2)$

$\to c(b) \times t = 3b^2 t$

between these two times one has $c(x,t) = x/t$ from which one can determine $u(x,t)$ = here $u = -\sqrt{c/3}$

in the (x,t) plane one has:

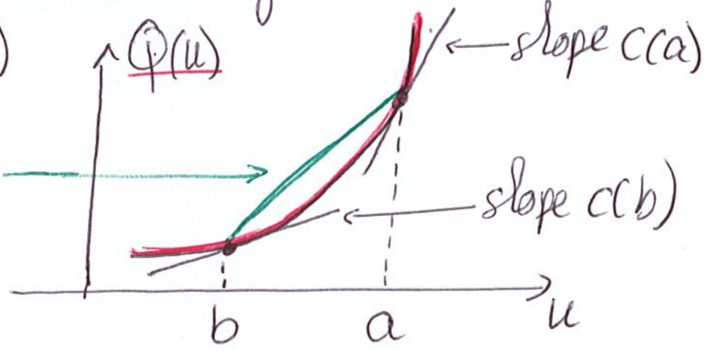


COMMENT ON THE LAX ENTROPY CONDITION

denote $\Phi(u) = \int^u c(u')du'$ so that the nonlinear advection eq. reads $u_t + \Phi_x = 0$ (in this problem $\Phi(u) = u^3$). The shock velocity (if a shock is formed) is:

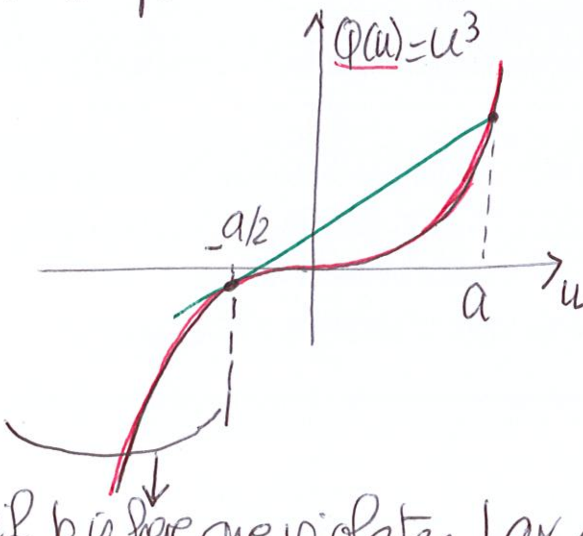
$\dot{s} = \dfrac{\Phi(a) - \Phi(b)}{a-b}$.

Graphical representation of the condition $c(b) \leqslant \dot{s} \leqslant c(a)$



one sees that Lax condition is a condition of convexity of $\Phi(u)$

in the particular case studied in this problem, one has:



The Lax condition is the only one which enables to draw meaningful characteristics in the (x,t) plane. As an illustration, try to make this usual construction here if $b = -a$

if b is here one violates Lax condition

Nagumo model

N1

$u_t = u_{xx} + u(u-a)(1-u)$ traveling wave = $u = U(x-ct)$

then = $-cU' = U'' + U(U-a)(1-U)$

defining a phase-space (U,V) with $V = U'$ one has a flow =

$$\begin{cases} U' = V \\ V' = -cV - U(U-a)(1-U) \end{cases}$$

the stationary points are: $(0,0)$ $(a,0)$ $(1,0)$

the Jacobian matrix is $\begin{pmatrix} 0 & 1 \\ 3U^2 - 2(a+1)U + a & -c \end{pmatrix} = J(U,V)$

⚠ remember: $c > 0$, $a \in [0,1]$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ a & -c \end{pmatrix}$ $\quad J(1,0) = \begin{pmatrix} 0 & 1 \\ 1-a & -c \end{pmatrix}$

eigenvalues $\frac{-c \pm \sqrt{c^2 + 4a}}{2}$ = one is >0, the other is <0 = $(0,0)$ is a saddle

For treating the case of $(1,0)$ it suffices to replace, in the discussion of $(0,0)$ a by $1-a$ = hence $(1,0)$ is also a saddle

$J(a,0) = \begin{pmatrix} 0 & 0 \\ a^2 - a & -c \end{pmatrix}$. again replacing a by $a^2 - a = -a(1-a)$ in the above one sees that the eigenvalues are $\frac{-c \pm \sqrt{c^2 - 4a(1-a)}}{2}$

- if $c^2 > 4a(1-a)$ both eigenvalues are real and < 0 = node
- if $c^2 < 4a(1-a)$ 2 complex conjugate eigen-values with negative real part = attractive spiral

this is the situation in the example given in the text: $a = 0.3$ $c = 0.5$

$4a(1-a) = 0.84 > c^2 = 0.25$

one looks for a solution leaving the saddle $(1,0)$ (hence along the unstable manifold) and reaching the saddle $(0,0)$ (hence along the stable manifold).

educated guess = $V = b\,U(1-U)$ with b = real constant one should have along this heteroclinic orbit:

$$\frac{dV}{dU} = b(1-2U) \quad \text{and also} \quad \frac{dV}{dU} = \frac{V'}{U'} = \frac{-cV - U(U-a)(1-U)}{V}$$

$$\overset{\text{here}}{=} -c - \frac{U-a}{b}$$

equating the two expressions one gets:

$$\begin{cases} b^2 = 1/2, \text{ hence } b = \pm 1/\sqrt{2} \\ c = \frac{a}{b} - b = \frac{1}{b}(a - b^2) = \frac{1}{b}(a - 1/2) \end{cases}$$

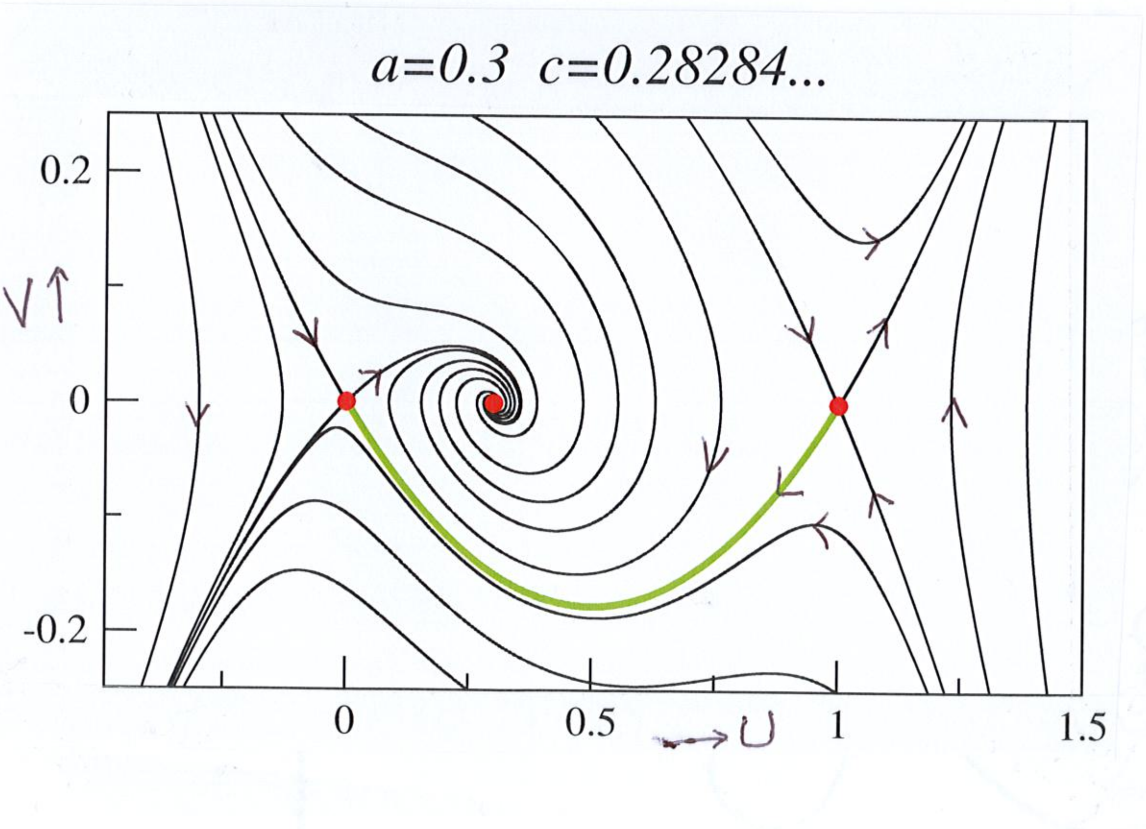
For having $c > 0$ one should have → either $b = 1/\sqrt{2}$ and $a > 1/2$
→ or $b = -1/\sqrt{2}$ and $a < 1/2$

remark: at $(0,0)$ the curve $V = bU(1-U)$ should match the stable manifold. For an eigenvalue λ of $J(0,0)$ the corresponding manifold is defined by $V = \lambda U$: the stable one is for $\lambda < 0$ and should match $V \simeq bU$: hence b should be < 0. and one should check that $\underbrace{\frac{-c - \sqrt{c^2 + 4a}}{2}}_{\text{negative eigenvalue of } J(0,0)} = b$ indeed $-c = \sqrt{2}(a - 1/2)$

$$c^2 + 4a = 2(a - 1/2)^2 + 4a = 2(a + 1/2)^2$$

$$\frac{-c - \sqrt{c^2+4a}}{2} = -\frac{\sqrt{2}}{2}$$

below: (green) thick line = heteroclinic orbit



equation of the front = one should have $\frac{dU}{dz} = bU(1-U)$ with $b = \pm 1/\sqrt{2}$ - We remarked earlier that b should be <0 = let's not make this assumption here and check what happens.

one has $\int \frac{dU}{U(1-U)} = \int b\, dz$ writing $\frac{1}{U(1-U)} = \frac{1}{U} + \frac{1}{1-U}$

this leads to $\ln \frac{U}{1-U} = bz$ the constant of integration has been chosen so that $z=0$ when $U = 1/2$

and one gets $\boxed{U(z) = \frac{1}{1+e^{-bz}}}$ where $b = \pm 1/\sqrt{2}$

for having $U(-\infty) = 1$ and $U(+\infty) = 0$ one should have $b<0$ = this corroborates the analysis of the previous page -