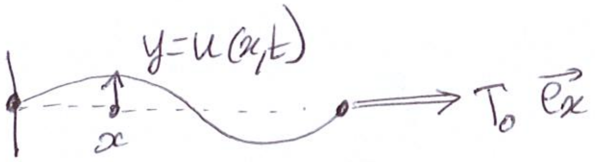
non linear waves and first order equations

Let's start ~~to~~ with smthg familiar (neither non linear nor first order)
D'Alembert equation for a vibration string. Let's consider the situation:

at B one applies a constant force (tension) T_0.

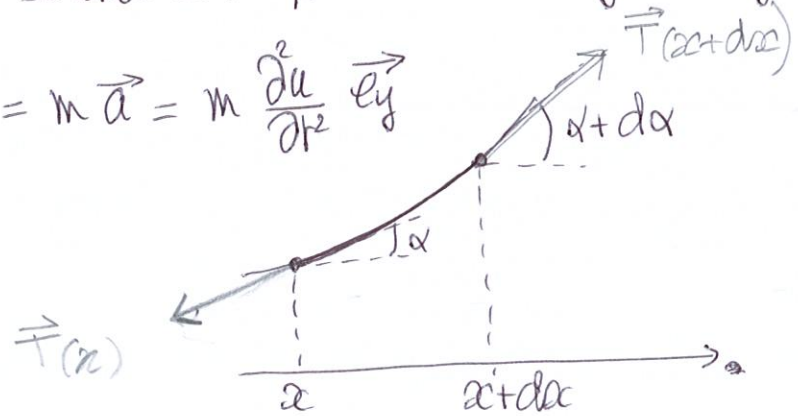
small vibrations:



assumption: $\left|\frac{\partial u}{\partial x}\right| \ll 1$ (notation: $\frac{\partial u}{\partial x} = \partial_x u = u_x$)

Newton's second law for a small segment of the string:

$$\Sigma \vec{F} = m\vec{a} = m\frac{\partial^2 u}{\partial t^2}\vec{e_y}$$



$$\begin{cases} \frac{\partial u}{\partial x} = \tan\alpha \simeq \alpha \ll 1 \\ \text{hence } \cos\alpha = 1 + O(\alpha^2) \end{cases}$$

if the gravity is neglected (a guitar sounds the same when it's horizontal or vertical!) one has:

$$T(x,t)\underbrace{\cos\alpha}_{\simeq 1} = T(x+dx,t)\underbrace{\cos(\alpha + d\alpha)}_{\simeq 1}$$

hence T is x-independent. since the tension at B is constant (time indep.)

$$T(x,t) \equiv T_0$$

projecting on the y-axis : $\underbrace{T_0 \sin(\alpha + d\alpha) - T_0 \sin\alpha}_{T_0\, d\alpha \text{ where } d\alpha = \partial_x\left(\frac{\partial u}{\partial x}\right) dx} = \rho_0\, dl \frac{\partial^2 u}{\partial t^2}$

$dl \cos\alpha = dx$

(ρ_0 is the lineic mass density of the string)

one thus has $\boxed{\begin{array}{l} u_{tt} - c^2 u_{xx} = 0 \\ \text{where } c^2 = T_0/\rho_0 \end{array}}$

c is homogeneous to a speed : $[c^2] = \dfrac{M L T^{-2}}{M L^{-1}} = (L \cdot T^{-1})^2$

indeed it is a speed of wave propagation : $\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$

$$\begin{cases} \dfrac{\partial \cdot}{\partial x} = \dfrac{\partial \cdot}{\partial \eta}\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \cdot}{\partial \xi}\dfrac{\partial \xi}{\partial x} = \partial_\eta + \partial_\xi \\ \dfrac{\partial \cdot}{\partial t} = \dfrac{\partial \cdot}{\partial \eta}\dfrac{\partial \eta}{\partial t} + \dfrac{\partial \cdot}{\partial \xi}\dfrac{\partial \xi}{\partial t} = c(\partial_\eta - \partial_\xi) \end{cases}$$

D'Alembert equation : $\left[(\partial_\eta - \partial_\xi)^2 - (\partial_\eta + \partial_\xi)^2\right] u = 0$

$\boxed{\text{plus efficace : } \partial_t^2 - c^2 \partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)}$

$\Leftrightarrow \quad u_{\eta\xi} = 0$

the obvious solution is $u = f(\eta) + g(\xi) = \underbrace{f(x+ct)}_{\text{propagating to the left at velocity } c} + \underbrace{g(x-ct)}_{\text{propagating to the right}}$

(solution for an infinite string)

parenthesis : how to determine f and g starting from the initial shape of the string?

$$\begin{cases} u(x,0) = f(x) + g(x) \equiv F(x) \\ u_t(x,0) = c f'(x) - c g'(x) \equiv G(x) \end{cases}$$

= given quantities (ie initial shape and "velocity").

differentiating the 1st eq : $F'(x) = f'(x) + g'(x)$ and combining with the second :

$$\begin{cases} f' = \frac{1}{2}(F' + G/c) \\ g' = \frac{1}{2}(F' - G/c) \end{cases}$$

hence $\begin{cases} f(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)d\xi + c_1 \\ g(x) = \frac{1}{2}F(x) \cancel{+} - \frac{1}{2c}\int_0^x G(\xi)d\xi + c_2 \end{cases}$

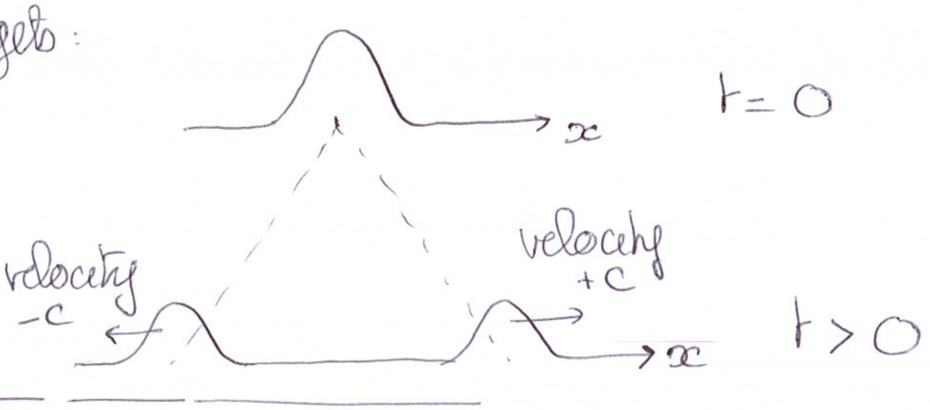
and $u(x,t) = f(x+ct) + g(x-ct) = \dfrac{F(x+ct) + F(x-ct)}{2}$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)d\xi + c_1 + c_2$$

comme $u(x,0) = F(x)$ on trouve que $c_1 + c_2 = 0$.

simple case $G(x) \equiv 0$ (no initial speed, just an initial deformat°)

there one gets:



parenthesis = what about the other case: $F \equiv 0$ & $G \neq 0$?

one has $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)d\xi = \frac{1}{2c}\{\mathcal{G}(x+ct) - \mathcal{G}(x-ct)\}$

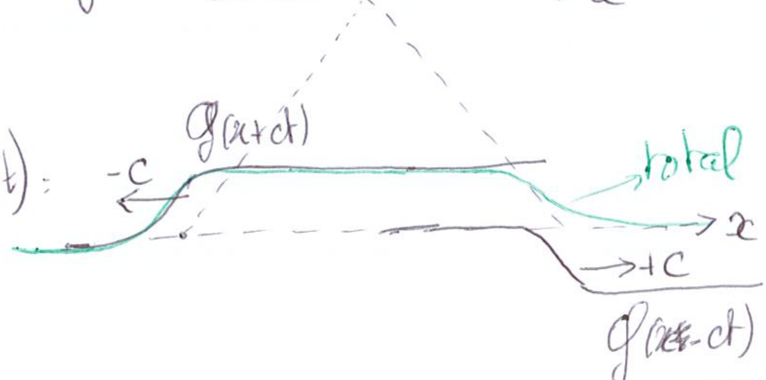
where $\mathcal{G}$ is a primitive of G.

consider for instance the situation where $G(x) = u_t(x,0)$ has the shape:

then $\mathcal{G}$:

and

$u(x,t)$:



instead of bi-directional motion one can simplify and study the equation: $u_t - c\,u_x = 0$. But now the eq. is made non linear: $c = c(u)$: the speed of propagation is a function of the local disturbance $u(x,t)$.

Example of a specific problem of this type: traffic flow. Simplified model: only one lane and no overtaking (dépassement). Denoting as $\rho(x,t)$ the car density (# of cars per unit length) and as $Q(\rho)$ the flux ~~(débit)~~ per unit time (débit). one has:

$$\partial_t \rho + \partial_x Q = 0$$

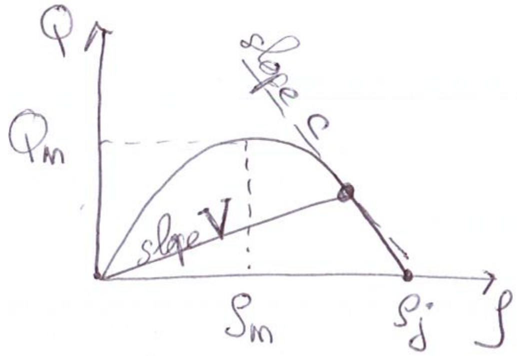
this comes directly from a simple reasoning: the # of cars is conserved and thus the rate of change of the total # of cars in a section $[x_1, x_2]$ must be balanced by the net inflows accross x_1 and x_2:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dt = Q(x_1,t) - Q(x_2,t)$$

and then take the limit $x_1 \to x_2$

there is an assumption here: $Q = Q(\rho)$: the flux depends only of the local density. there have been some ~~experi~~ measurements which seem to ~~go in this direction~~ support this assumption.

Reasonable form for $Q(\rho)$. the flow velocity $V = Q/\rho$ depends here only on ρ. It must be a decreasing fct of ρ which starts from a finite maximum value at $\rho = 0$ and decreases to 0 as $\rho \to \rho_j$, the value for which the cars are bumper to bumper. Thus $Q(\rho)$ is zero both at $\rho = 0$ and ρ_j and looks like



the eq. $\rho_t + Q_x = 0$ reads:

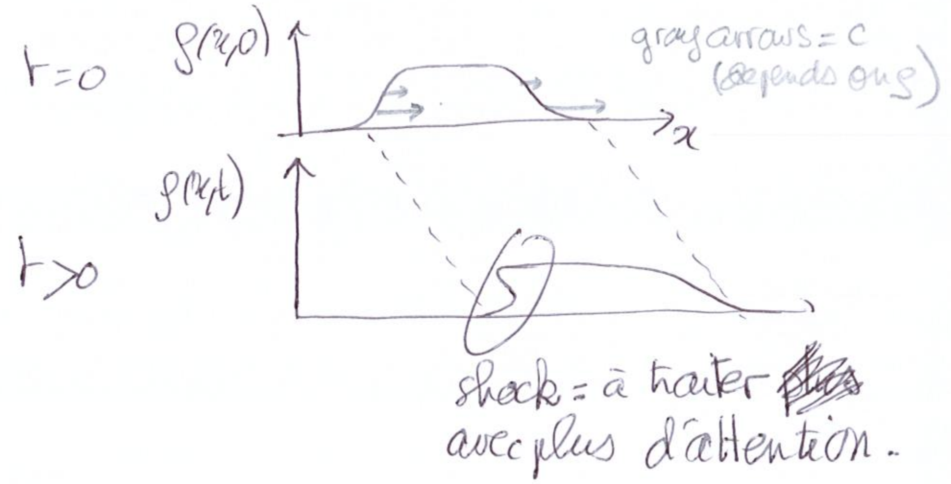
$$\boxed{\rho_t + \underbrace{Q'(\rho)}_{c(\rho)}\, \rho_x = 0}$$

Q has a maximum Q_m for some value ρ_m of the car density.
$\rho_j \simeq 140$ vehicle/km
$\rho_m \simeq 50$ vehicle/km & $Q_m \sim 1500$ vehicle/hour.
note that the maximum flow rate is attained at a velocity
$V \simeq 30$ km/h (quite low).

the propagation velocity for the wave is $c(\rho) = Q'(\rho) = V(\rho) + \rho V'(\rho)$
since $V \downarrow$ as a function of ρ: $c(\rho) < V(\rho)$ = waves propagate backward through the stream of traffic and drivers are warned of disturbances ahead.

- $c = Q'(\rho)$ hence the traffic wave moves forward or backwards relative to the road depending on wether $\rho < \rho_m$ or $\rho > \rho_m$.

- typical situation:



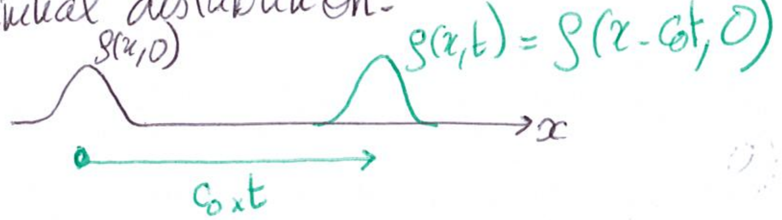
(PS: traffic flow eqs. can be obtained from car-following theory, cf Whitham p 78)

Method of characteristics

* $\boxed{\rho_t + c(\rho)\,\rho_x = 0}$

if $c(\rho) \equiv c_0$ then we know that $\rho(x,t) = f(x - c_0 t)$ [clearly solves the eq. and can be obtained from the change of variable of page 2 = the eq. reads $\rho_\eta = 0 \rightarrow \rho = f(\xi = x - c_0 t)$]

where $f(x) = \rho(x,0)$ initial distribution.



* What happens when $c(\rho)$ is not a constant? Formally one can still write $\rho(x,t) = f(x - c(\rho)t)$ where $f(x) = \rho(x,0)$ is the initial distribution

proof = $\rho_t = -f'\,\partial_t[c(\rho)t] = -f' \times [c(\rho) + t\,c'(\rho)\,\rho_t]$

$$\Rightarrow \rho_t = \frac{-c f'}{1 + t\,c' \times f'}$$

$$\rho_x = f'\left(1 - \partial_x(c(\rho)t)\right) = f' \times [1 - t\,c'(\rho)\,\rho_x]$$

$$\Rightarrow \rho_x = \frac{f'}{1 + t\,c' \times f'}$$

one has indeed $\rho_t + c(\rho)\,\rho_x = 0$

so, for a given x and t one could determine ρ as a solution of $x - c(\rho) \times t = f^{-1}(\rho)$... this is not very helpfull and could miss some important features which are revealed by the method of characteristics (see below) easily seen by taking $t=0$ in the equation $x - c(\rho)t = f^{-1}(\rho)$

- first method: $x - c(\rho)t = f^{-1}(\rho) = x_0(\rho)$

  ↳ inverse of $\rho_0(x) = f(x)$ = initial distribution.

  hence in the plane (x,t) ρ is a constant along a straight line = $\begin{cases} x = \bar{x} + c(\rho_0(\bar{x}))t \\ \text{for given } \bar{x} \quad (\bar{x} = x_0(\rho_0)) \end{cases}$

  the constant value of ρ along the straight line

- second method:

  one notices that $\rho_t + c(\rho)\rho_x$ is the total derivative of ρ along a curve of the (x,t) plane, which slope $\frac{dx}{dt} = c(\rho)$ since $\frac{d\rho}{dt} = \rho_t + \frac{dx}{dt}\rho_x$

  Of course the curve $\mathcal{C}$ which has a slope $\boxed{\frac{dx}{dt} = c(\rho)}$ at every of its points cannot be determined explicitely in advance since its definition involves the unknown values of ρ on the curve.

  However on $\mathcal{C}$, from $\frac{dx}{dt} = c(\rho)$ and from $\rho_t + c(\rho)\rho_x = 0$ we deduce that $\frac{d\rho}{dt} = 0$ = ρ remains constant $\longrightarrow$ $\mathcal{C}$ is a straight line.

  as obvious from the equation $dx/dt = c(\rho)$

  let's then consider the initial value problem:

  $$\rho = f(x) \quad \text{at } t=0 \quad \text{for } x \in \mathbb{R}.$$

  if one of the curves $\mathcal{C}$ intersects the axis of abscissae at $x = \bar{x}$ then $\rho = f(\bar{x})$ on the whole of that curve. The corresponding constant slope is $c(f(\bar{x}))$ which we will denote by $F(\bar{x}) = c(\rho_0(\bar{x}))$.

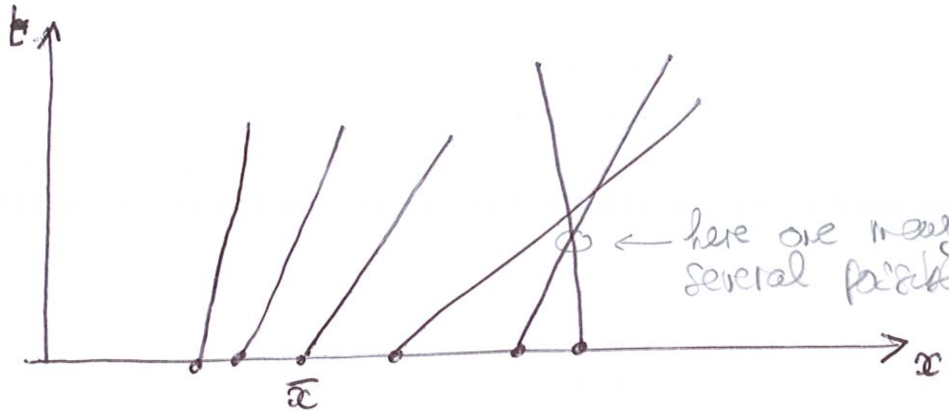
  the equation of the curve is then

  $$\boxed{x = \bar{x} + F(\bar{x})t}$$

in practice one has the solution in a ~~implicitte~~ form: $\rho(x,t) = \rho_0(\bar{x})$ where $x = \bar{x} + tF(\bar{x})$

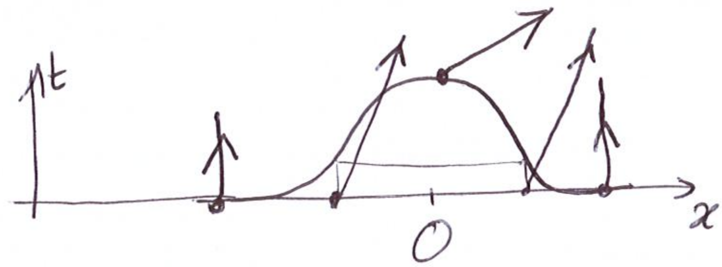
[cf Whitham page 61]

note: in the general case $\rho_t + c\rho_x = b$ where b and c are fcts of ρ, x and t the method of characteristics works: $\frac{d\rho}{dt} = b$ and $\frac{dx}{dt} = c$ but since $b \neq 0$ ρ is not cst along the characteristic and generally the characteristic are not straight lines one has a construction of the following type:



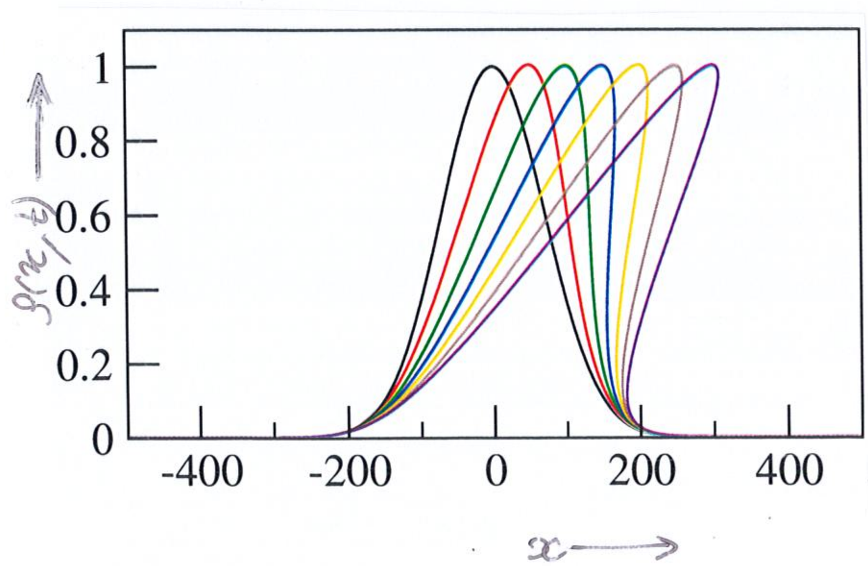
← here one may have a problem: several possible values of ρ

an example: Hopf's equation $\rho_t + \rho\rho_x = 0$ (ie $c(\rho) = \rho$).
I take the initial distribution: $\rho_0(x) = \exp(-x^2/\sigma^2)$
one has:



the numerical solution is ~~immedi~~ straighforward: one takes a swarm of initialy equi-distant $\bar{x}$ and they evolve according to $x = \bar{x} + \rho_0(\bar{x})\, t$. For each t one plots $x(t)$, $\rho_0(\bar{x})$:

plot for $\sigma = 100$



← numerical solution for $t = 0, 50, 100, \ldots, 300$

one observes WAVE BREAKING (at $t \gtrsim 100$)

(remark: notice that the evolution is trivial if the eq. is: $\rho_t + c_0 \rho_x = 0$)

when does breaking occur first? the condition that 2 neighbouring characteristics (starting at $\bar{x}$ and $\bar{x}+d\bar{x}$) intersect is:

$$x = \bar{x} + F(\bar{x})t \quad \text{and} \quad x = \bar{x}+d\bar{x} + F(\bar{x}+d\bar{x})t$$

in the limit $d\bar{x} \to 0$ this implies $0 = 1 + F'(\bar{x})t$ hence

$$t = -\frac{1}{F'(\bar{x})}$$ breaking occurs first at $$\boxed{t_B = \frac{1}{\max[-F'(\bar{x})]}}$$

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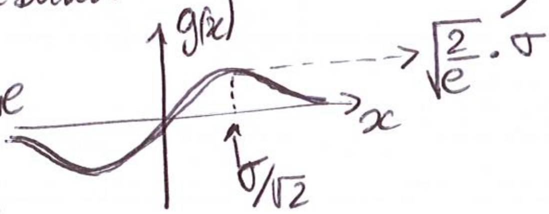
exercice: for $F(\bar{x}) = e^{-\bar{x}^2/\sigma^2}$ (Hopf equation with this initial distribution)

one gets

$$-F'(x) = \frac{2x}{\sigma^2} e^{-x^2/\sigma^2} \equiv g(x) \text{ where}$$



$$g'(x) = \frac{2}{\sigma^4} e^{-x^2/\sigma^2}(\sigma^2 - 2x^2) \text{ cancels}$$

for $x = \sigma/\sqrt{2}$ and the value of g is there $\sqrt{\frac{2}{e}}\,\frac{1}{\sigma}$

hence $t_B = \sqrt{\frac{e}{2}} \cdot \sigma \simeq 1.1658\ldots\,\sigma$ as observed numerically

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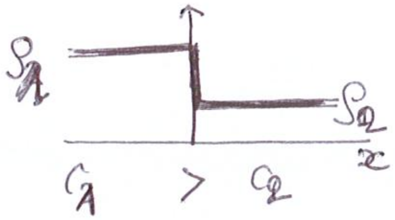
After wave breaking the multivalued solution is not acceptable. We will need a scheme for sorting this out. Before that we will study in more details an extreme case that mathematicians denote as the Riemann problem:

$$\rho_0(x) = f(x) = \begin{cases} \rho_2 & x>0, \\ \rho_1 & x<0, \end{cases} \quad \text{and thus } F(x) = \begin{cases} c_2 = c(\rho_2) & x>0, \\ c_1 = c(\rho_1) & x<0. \end{cases}$$

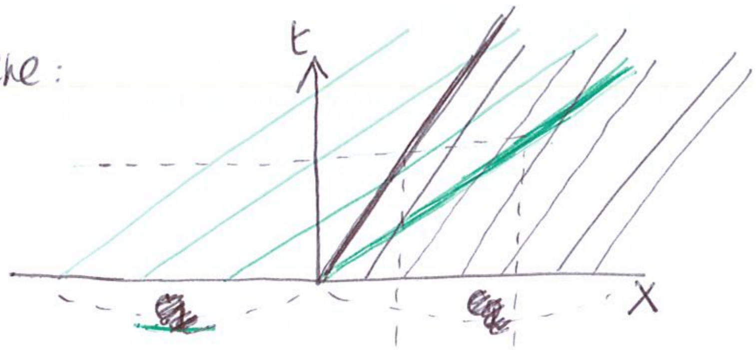
if $c_1 > c_2$ then breaking occurs immediatly. Let's consider for instance the case $\rho_1 > \rho_2$ and $c'(\rho) > 0$:

(in which case $c_1 > c_2$)

on has : $f(x) =$

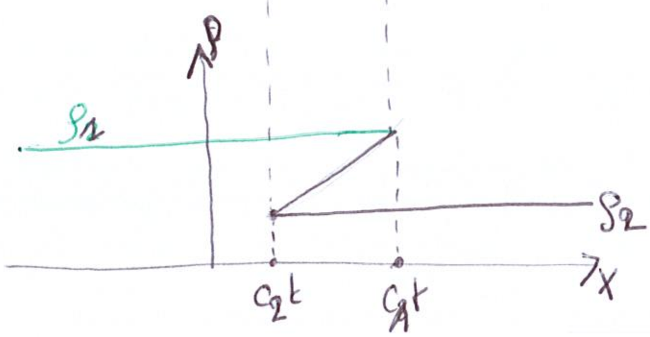


in the (x,t) plane:

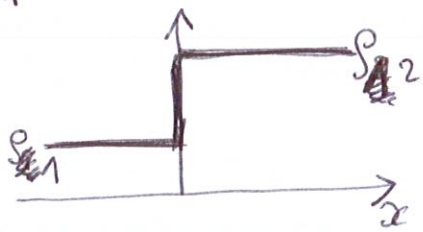


thicker lines boundary of the multi-valued region in the (x,t) plane

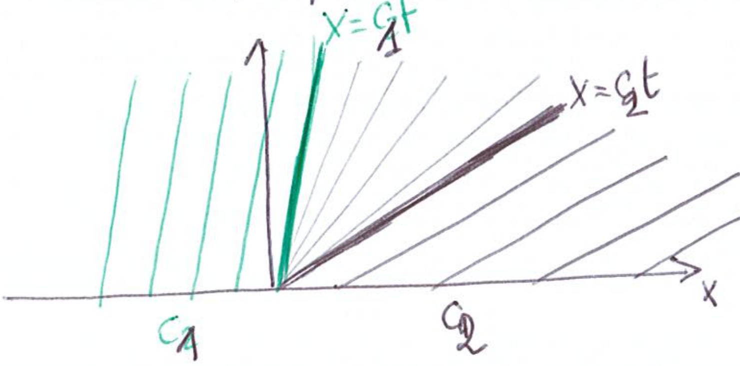
at fixet one has:



however if instead one has initially $f(x) =$

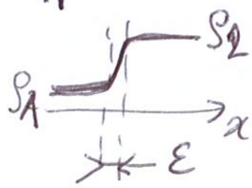


then, in the (x,t) plane =



($c_1 < c_2$ here)

the "empty" (gray) area is filled as follows: remember that $x = \bar{x} + F(\bar{x})t$ one takes $\bar{x} = 0$ and $F(\bar{x})$ going from c_2 to c_1: this leads to the gray lines of equation $x = Ft$ for $c_1 < F < c_2$ this procedure is exactly the sharp limit of



when $\varepsilon \to 0$.

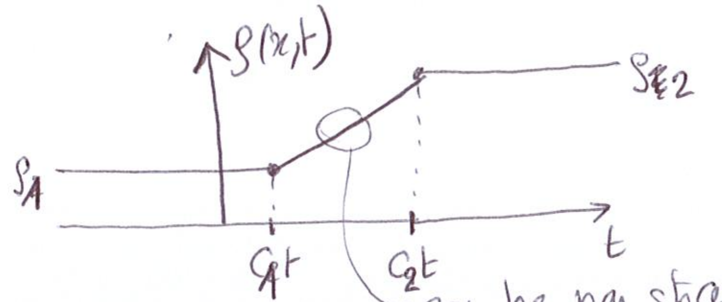
since here $c = F$ one has $\begin{cases} c = x/t \\ \text{for } c_1 < \frac{x}{t} < c_2 \end{cases}$

the complete solution for c is then:

$$c = \begin{cases} c_2 & \text{for } c_2 < x/t \\ x/t & \text{for } c_1 < x/t < c_2 \\ c_1 & \text{for } x/t < c_1 \end{cases}$$

then the value of ρ is obtained by inverting the relation $c = c(\rho)$.

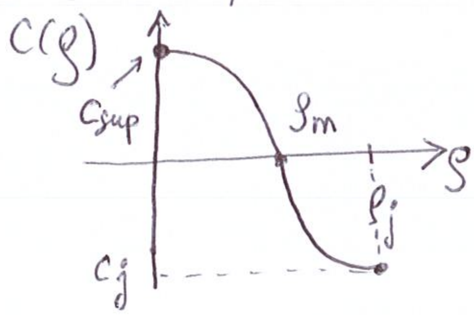
one gets smthg like: which is perfectly acceptable



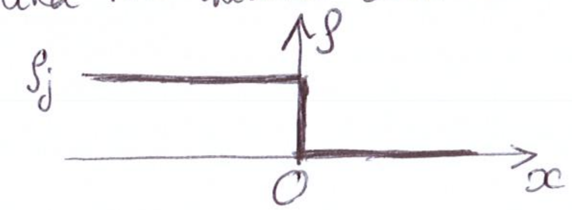
can be non straight line if the relation $c = c(\rho)$ is not linear

example: describe what happens after a traffic light turns green.

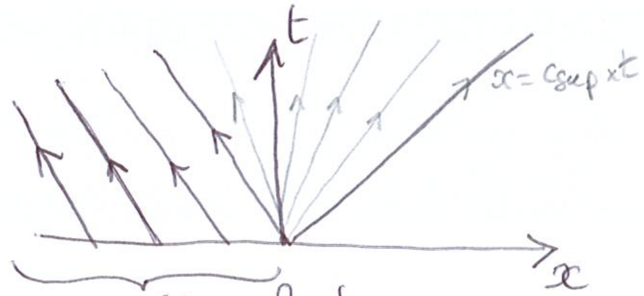
from the qualitative behavior of $Q(\rho)$ (cf page 5) one has



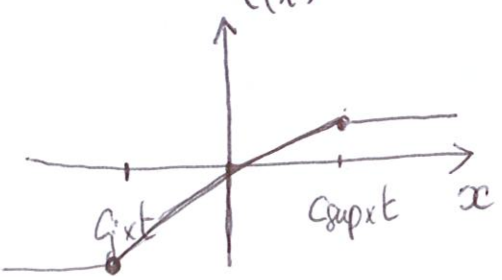
and the initial distribution is:



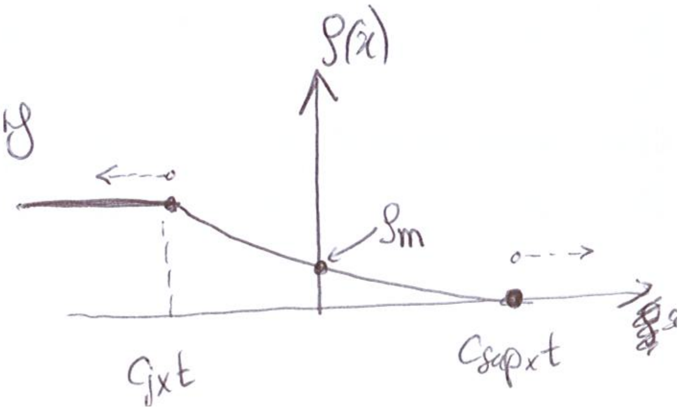
So one gets:



Thus one has at some finite t:



leading to:



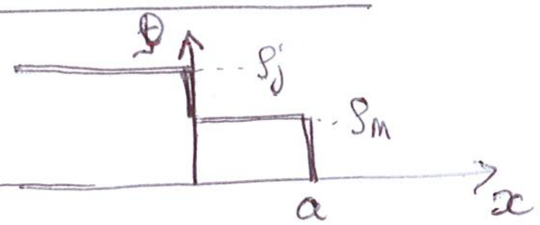
exercice:

assume that the initial distribution is

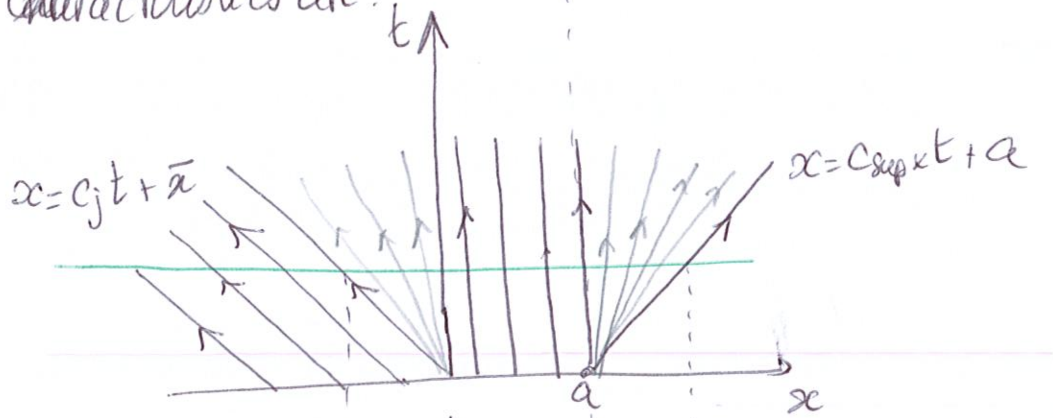
$$\rho = \begin{cases} \rho_j & \text{for } x<0 \\ \rho_m & \text{for } 0<x<a \\ 0 & \text{for } x>a \end{cases}$$

describe what happens at $t>0$

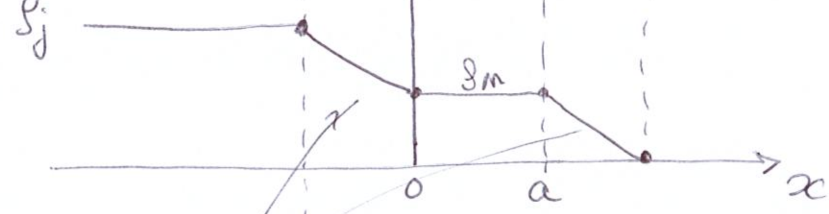
one has:



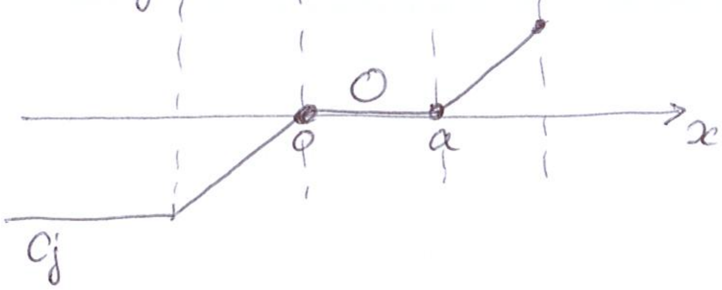
thus the characteristics are:



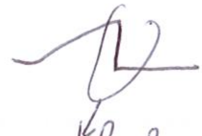
and the ~~flow~~ profile at time t



here the curve is bended because the relation $c = c(\rho)$ is not an exact straight line, but c plotted as a function of x is a series of straight lines:

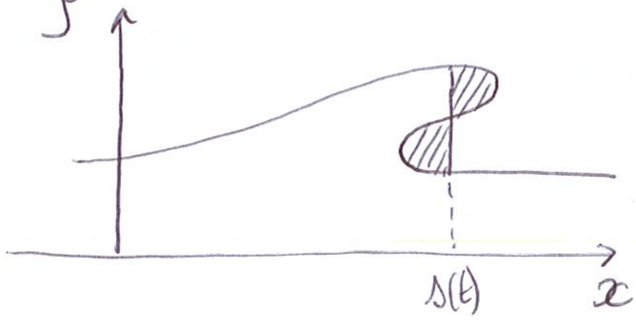


Shock waves. the multivalued solution (cf page 8) is (often) inacceptable  . ~~the~~ One has an

abrupt solution = replace ~~the~~  by 

shock

- the correct position for the discontinuity may be determined by the following argument: both the multivalued curve and the discontinuous curve & satisfy the conservation law. Therefore $\int \rho\, dx$ under each curve & must be the same: $\longrightarrow$ equal area law:



this determination is not very convenient for analytic work...

- <u>velocity of the shock</u>: one could argue that an eq. of the form $\rho_t + c(\rho)\rho_x = 0$ doesn't like discontinuity. But remember that it has an integrated form:

~~from~~ $\rho_t + Q_x = 0$ has been obtained from ~~one~~ ~~gets~~ $\displaystyle \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx + Q(x_2,t) - Q(x_1,t) = 0$

and this guy doesn't mind discontinuous ρ or Q.

(x_1 and x_2 fixed, ie independent of t)

let's denote by $s(t)$ the position of the shock. and ~~the~~ take in the above $x_1 < s(t) < x_2$.

One will have:

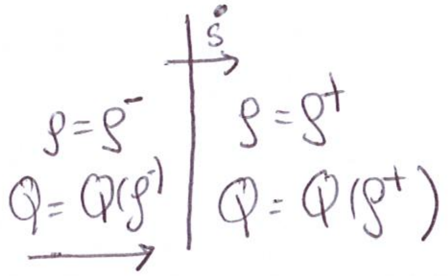
$$\frac{d}{dt}\left(\int_{x_1}^{s(t)} \rho(x,t)\,dx\right) + \frac{d}{dt}\left(\int_{s(t)}^{x_2} \rho(x,t)\,dx\right) = Q(x_1,t) - Q(x_2,t)$$

$$\Rightarrow \quad \rho(s^-,t)\,\dot{s} - \rho(s^+,t)\,\dot{s} + \int_{x_1}^{s(t)} \rho_t(x,t)\,dx + \int_{s(t)}^{x_2} \rho_t(x,t) = \text{''}$$

then take the limit $x_1 \to s^-$ and $x_2 \to s^+$, one gets

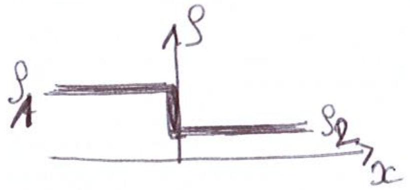
$$\boxed{\dot{s} = \frac{Q^+ - Q^-}{\rho^+ - \rho^-} = \frac{Q(\rho^+) - Q(\rho^-)}{\rho^+ - \rho^-}}$$

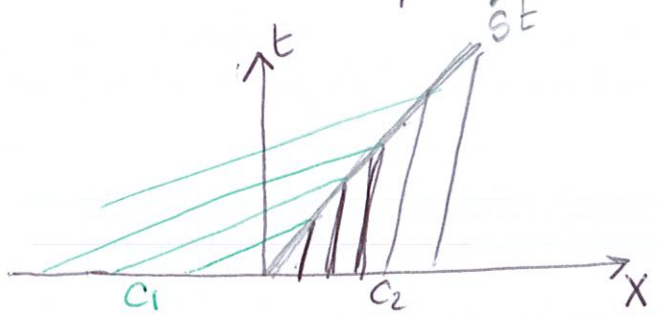
note that this reasoning is the same as the one which led to the equal area rule: one has:



and the shock moves at a velocity such that the conservation law is verified.

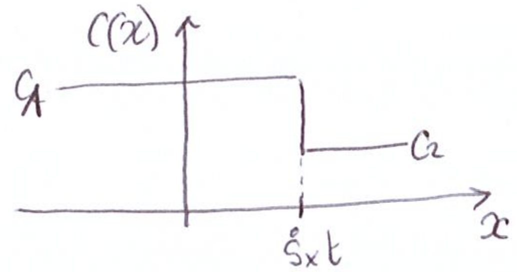
- one can then return to our Riemann problem (page 10)



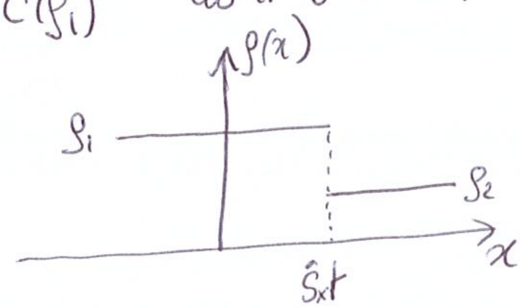


note: here $Q(\rho) = \int^\rho c$ and we have assume that $c'(\rho) > 0$ hence Q is convex. As a result $c(\rho_2) < \dot{s} < c(\rho_1)$ as it should.

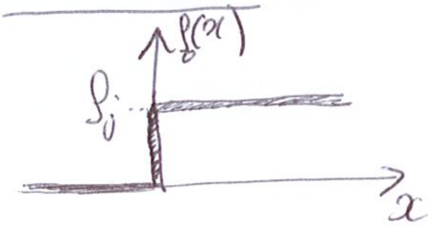
one has: at given t



and



question: how does evolve the end of a traffic jam?



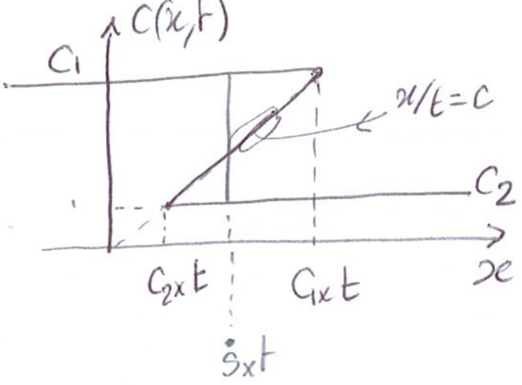
answer: remember that $Q(\rho)$

hence $\dot{s} = 0$

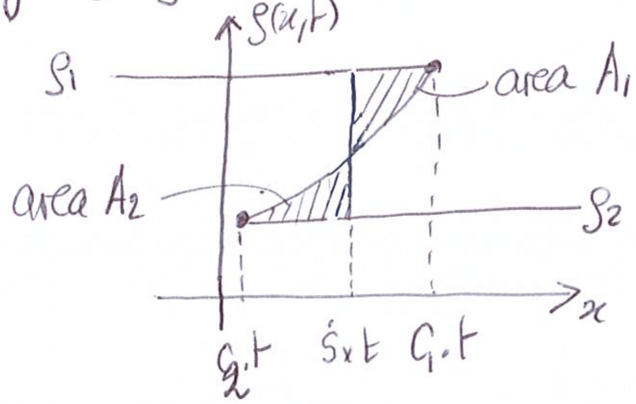
longue parenthèse, mais question légitime :

does our solution of the Riemann pb fulfill the equal area law?

– one has for $c(x)$ at time t :



and the shape of ρ is given by applying $\rho=\rho(c)$ to this curve :



$$A_1=\int_{\dot{s}\times t}^{c_1\times t}\left(\rho_1-\rho\left(c=\frac{x}{t}\right)\right)dx \qquad A_2=\int_{c_2 t}^{\dot{s}\times t}\left(\rho\left(c=\frac{x}{t}\right)-\rho_2\right)dx$$

change of variable : $c=\frac{x}{t}$ $\quad dx=t\,dc$:

$$A_1=t\int_{\dot{s}}^{c_1}(\rho_1-\rho(c))\,dc \qquad A_2=t\int_{c_2}^{\dot{s}}(\rho(c)-\rho_2)\,dc$$

notons $R(c)$ une primitive de $\rho(c)$ $\left[R(c)=\int^{c}\rho(c')dc' \Leftrightarrow \frac{dR}{dc}=\rho(c)\right]$

alors

$$\frac{A_1}{t}=\rho_1(c_1-\dot{s})-R(c_1)+R(\dot{s}) \text{ et } \frac{A_2}{t}=R(\dot{s})-R(c_2)-\rho_2(\dot{s}-c_2)$$

donc $A_1=A_2 \Leftrightarrow \dot{s}(\rho_2-\rho_1)=[\rho_2c_2-R(c_2)]-[\rho_1c_1-R(c_1)]$

et ça marche because the r.h.s is equal to $Q(\rho_2)-Q(\rho_1)$ (*) hence

$$A_1=A_2 \Leftrightarrow \dot{s}=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}$$ which is the definition of $\dot{s}$. so every thing's OK

(*) the rhs is $Q(\rho_2)-Q(\rho_1)$ because the function $G(\rho)=\rho\,c(\rho)-R(c(\rho))$ is equal to $Q(\rho)$ up to a constant : $\frac{d}{d\rho}G=c+\rho\frac{dc}{d\rho}-\underbrace{\frac{dR}{dc}}_{=\rho}\cdot\frac{dc}{d\rho}=c=\frac{dQ}{d\rho}$