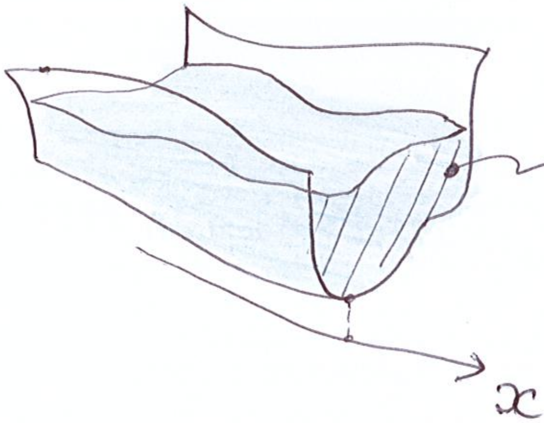
Flood waves



conservation equation:

$$\frac{d}{dt}\int_{x_1}^{x_2} A(x,t)\,dx + Q(x_1,t) - Q(x_2,t) = 0$$

hence $A_t + Q_x = 0$

where Q – flux in m^3/s.

Actually, on ipcal grounds one could add a source term:

$$\boxed{A_t + Q_x = M}$$

where M represents the supply to the river due to infiltration seepage and overland flow from the catchmen.

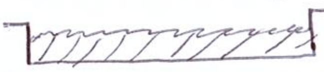
can one try to write a law $Q = Q(A)$? Let's define the average velocity $V = Q/A$. A slice of thickness dx is subjected to two forces: friction on the bottom and gravity.

$dF_f \propto V^2$, P and dx — dF_g

and not V as in high school or in Stokes flow: here V^2 is more realistic

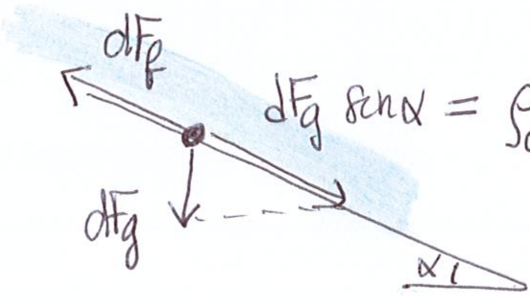
wetted perimeter $= \propto A^{1/2}$ for circular cross-section

and also independent of A for broad river:



$h \ll$ breadth (largeur)

slope α:



$dF_g \sin\alpha = \rho_0 g A\,dx \sin\alpha$ (ρ_0 = mass density of water) at quasi-equilibrium dF_f and $dF_g \sin\alpha$ should equilibrate (if this hypothesis is violated, no steady flow is possible!) FW2

writing $dF_f = C_f \rho_0 P V^2 dx$ yields then:

$$\begin{cases} V = \sqrt{\dfrac{gA\sin\alpha}{C_f\, P}} \\ Q = VA = \sqrt{\dfrac{A^3}{P}\,\dfrac{g\sin\alpha}{C_f}} \end{cases}$$

this yields $Q \propto A^{3/2}$ or $A^{5/4}$ for broad or circular river (assuming C_f and α constant)

other empirical laws give different power law dependences.

We will assume that $\boxed{Q = C\,\dfrac{A^{m+1}}{m+1}}$ where C is a constant and $m>0$ (also c^{st})

then the eq. becomes:

$$A_t + (dQ/dA)\,A_x = M \quad \text{with } dQ/dA = C.\,A^m$$

this can be nondimensionalized by defining: $\begin{cases} A = C^{-1/m}\mathcal{A} \\ M = C^{-1/m}\mathcal{M} \end{cases}$

then $C A^m = \mathcal{A}^m$

and the eq. becomes: $\boxed{\mathcal{A}_t + \mathcal{A}^m \mathcal{A}_x = \mathcal{M}}$

the simplest assumption is to take $\mathcal{M} \equiv 0$. Slightly better: $\mathcal{M} = c^{st}$.

In this case, define $a(t) = \mathcal{A}(x(t),t)$: along a characteristics

$$\frac{dx}{dt} = \mathcal{A}^m \quad \text{one has} \quad \frac{da}{dt} = \mathcal{M}$$

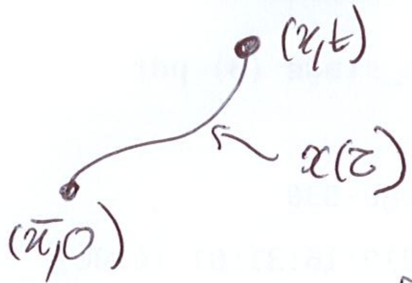
this means that $a(t) = \mathcal{M}t + a(0)$, ie

$\mathcal{A}(x,t) = \mathcal{M}t + \mathcal{A}(\bar{x},0)$ where $(\bar{x},0)$ is the starting point of the characteristics which reaches (x,t)

and whose eq. is

$$\frac{dx}{dt} = \mathcal{A}^m = \Big(\mathcal{M}t + \underbrace{\mathcal{A}(\bar{x},0)}_{\mathcal{A}_0(\bar{x})}\Big)^m$$

along the characteristics:



with $\dfrac{dx}{d\tau} = \left[\mathcal{R}\tau + \mathcal{A}_0(\bar{x})\right]^m \Rightarrow x(\tau) = \dfrac{\left[\mathcal{R}\tau + \mathcal{A}_0(\bar{x})\right]^{m+1}}{\mathcal{R}(m+1)} + C^{ste}$

(determined by imposing $x(0) = \bar{x}$)

and then, at time $\tau = t$:

$$x = \frac{\left[\mathcal{R}t + \mathcal{A}_0(\bar{x})\right]^{m+1}}{\mathcal{R}(m+1)} + \bar{x} - \frac{\left[\mathcal{A}_0(\bar{x})\right]^{m+1}}{\mathcal{R}(m+1)}$$

this eq. defines $\bar{x}$ implicitely as a function of x and t

then, one gets $\mathcal{A}(x,t)$ by writing:

$$\mathcal{A}(x,t) = \mathcal{R}t + \mathcal{A}_0(\bar{x}) = \mathcal{R}t + \mathcal{A}_0\left(x - \frac{\left[\mathcal{R}t + \mathcal{A}_0(\bar{x})\right]^{m+1}}{\mathcal{R}(m+1)} + \frac{\left[\mathcal{A}_0(\bar{x})\right]^{m+1}}{\mathcal{R}(m+1)}\right)$$

one can see this eq as an implicit equation for the quantity $\mathcal{A} = \mathcal{A}(x,t)$.

in this eq. one writes $\mathcal{A}_0(\bar{x}) = \mathcal{A}(x,t) - \mathcal{R}t$ and then:

$$\boxed{\mathcal{A} = \mathcal{R}t + \mathcal{A}_0\left(x + \frac{(\mathcal{A} - \mathcal{R}t)^{m+1} - \mathcal{A}^{m+1}}{\mathcal{R}(m+1)}\right)}$$

argument of $\mathcal{A}_0$

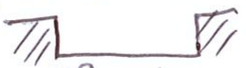
← implicitely defines $\mathcal{A} = \mathcal{A}(x,t)$

in the case $\mathcal{R} = 0$, on takes the limit $\mathcal{R} \to 0$ in the above eq. and this simply leads to: $\mathcal{A} = \mathcal{A}_0(x - \mathcal{A}^m t)$ as expected.

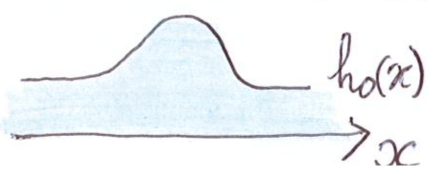
implicitely defines $\mathcal{A}$ as a fct of (x,t)

let's consider the case:

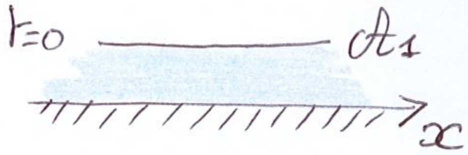
which is equivalent, in a river of cross-section



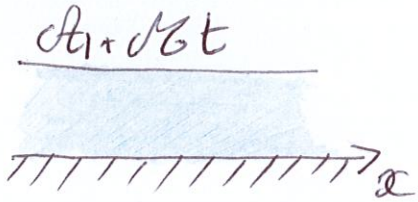
and constant breadth to:



before dealing with this complicated structure, let's first consider the constant initial flow:



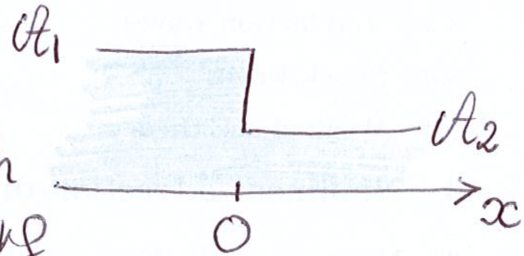
This clearly leads to:



and the eq. of a characteristics is:

$$\begin{cases} x(\tau) = \dfrac{[\mathcal{M}\tau+\mathcal{A}_1]^{m+1}-\mathcal{A}_1^{m+1}}{\mathcal{M}(m+1)} + \bar{x} \\ \simeq \mathcal{A}_1^m \tau + \bar{x} \quad \text{for small } \tau \end{cases}$$

the characteristics behave as (remember $m>0$)



* If one considers the initial condition one will certainly have a shock forming immediately.

A simple analysis of the conservation eq: $\dfrac{dN}{dt} = \displaystyle\int_{x_1}^{x_2} \mathcal{A}_t \, dx = Q(x_1,t) - Q(x_2,t) + \int_{x_1}^{x_2} \mathcal{M} \, dx$

shows (by taking $x_1 < s(t) < x_2$)

coordinate of the shock

that: $\dfrac{ds}{dt} = \dfrac{Q_R - Q_L}{\mathcal{A}_R - \mathcal{A}_L}$ where $\begin{cases} \mathcal{A}_R = \lim\limits_{x \to s(t)} \mathcal{A}(x,t) \\ \mathcal{A}_L = \lim\limits_{x \to s^+(t)} \mathcal{A}(x,t) \end{cases}$ (idem for Q)

$\mathcal{M}$ does not appear in this equation.

This gives here $\dfrac{ds}{dt} = \dfrac{(\mathcal{A}_1+\mathcal{M}t)^{m+1} - (\mathcal{A}_2+\mathcal{M}t)^{m+1}}{(\mathcal{A}_1+\mathcal{M}t) - (\mathcal{A}_2+\mathcal{M}t)} > 0$. Then the characteristics are:

at large t

$\dfrac{ds}{dt} \simeq (m+1)(\mathcal{M}t)^m$