Advanced Nonlinear Physics

international Master « Physics of Complex Systems »

First Lecture

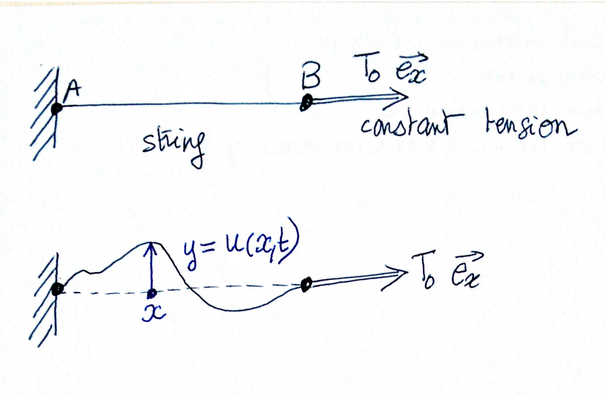
Linear and nonlinear wave equations

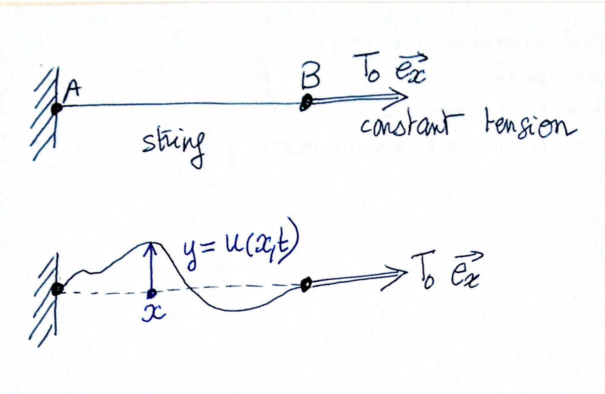
tuesday, september $8^{\rm th}$, 2020

Bibliography:

- "Linear and nonlinear waves" by G. B. Whitham (chaps. 2 and 3)
- "an introduction to nonlinear partial differential equations" by J. D. Logan (chaps. 2, 3 and 5).
- "Mathematical models" by R. Haberman (part 3).

website of (the 1st part of) the course:
`http://lptms.u-psud.fr/nicolas_pavloff/enseignement/advanced-nonlinear-physics/`


A
B
$T_0 \vec{e_x}$
string
constant tension
$y = u(x,t)$
$T_0 \vec{e_x}$
x



Assuming that $|u_x| \ll 1$ (notation: $u_x = \partial u/\partial x = \partial_x u$), and denoting by μ_0 the linear mass density of the string, the dynamics is governed by

$$\boxed{u_{tt} - c^2 u_{xx} = 0}\,, \qquad (1)$$

where $c = (T_0/\mu_0)^{1/2}$ is the velocity of propagation of a deformation of the string (see below).

$0 = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\,\partial_x)(\partial_t - c\,\partial_x)u$ Let's define $\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$

$$\begin{cases} \dfrac{\partial \bullet}{\partial t} = \dfrac{\partial \bullet}{\partial \eta}\dfrac{\partial \eta}{\partial t} + \dfrac{\partial \bullet}{\partial \xi}\dfrac{\partial \xi}{\partial t} = c\,\partial_\eta - c\,\partial_\xi \\ \dfrac{\partial \bullet}{\partial x} = \dfrac{\partial \bullet}{\partial \eta}\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \bullet}{\partial \xi}\dfrac{\partial \xi}{\partial x} = \partial_\eta + \partial_\xi \end{cases} \rightsquigarrow \begin{cases} \partial_t + c\,\partial_x = 2\,c\,\partial_\eta \\ \partial_t - c\,\partial_x = -2\,c\,\partial_\xi \end{cases} \tag{2}$$

and d'Alembert equation reads $\partial_{\xi\eta} u = 0$.

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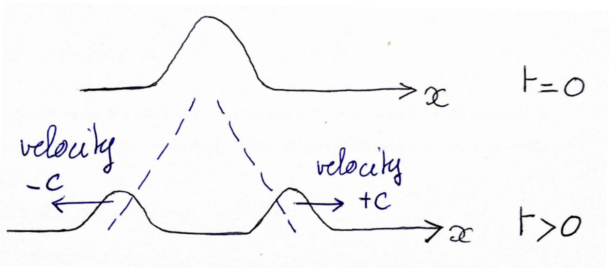
and d'Alembert equation reads $\partial_{\xi\eta} u = 0$. General solution:
$u = f(\eta) + g(\xi) = f(x+ct) + g(x-ct)$. The first component propagates to the left, the second to the right, both with velocity c, without deformation.

f and g can be expressed in terms of $u(x,0)$ and $u_t(x,0)$. In the simple case $u_t(x,0) = 0$ and $u(x,0) \equiv u_0(x)$ one gets $f = g = \frac{1}{2}u_0$, cf. sketch below:



How to include nonlinear effects in a simplified version of d'Alembert equation?

$$\boxed{u_t + c(u)u_x = 0}\,, \tag{3}$$

where $c(u)$: nonlinear velocity specified by the problem under consideration.
One speaks of advection equation.

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Example: simplified model of traffic flow. $\rho(x,t)$: linear car density. $Q(x,t)$: flow rate (number of car per unit time). Car number conservation:

$$\partial_t \rho + \partial_x Q = 0\,. \qquad (4)$$

the rate of change of the number of cars in a section $[x, x+\Delta x]$ is balanced by the net inflow accross x and $x+\Delta x$:

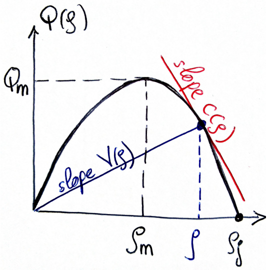
$$\frac{d}{dt}\int_x^{x+\Delta x} \rho(x', t)\, dx' = Q(x,t) - Q(x+\Delta x, t)\,.$$

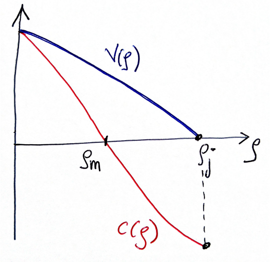
Taking the limit $\Delta x \to 0$ yields Eq. (4).

Assume that the Q depends on ρ only: $Q(x,t) = Q(\rho(x,t))$, then (4) reads $\rho_t + c(\rho)\rho_x = 0$ where $c(\rho) = dQ/d\rho$ can be > 0 or < 0.

$Q(\rho)$ for traffic flow

According to our assumption, the velocity of the flow $V = Q/\rho$ depends only on ρ. Empirical facts: it is a decreasing function, which starts at a maximum value at $\rho = 0$ and reaches 0 at some ρ_j, with $\rho_j \simeq (\text{length of a car})^{-1}$ (j stands for "jam"). Typical behavior of Q, V and c :





$\rho_j \simeq 140$ car/km

$\rho_m \simeq 50$ car/km

$Q_m/\rho_m \simeq 30$ km/h

$c = dQ/d\rho$ is the velocity of propagation of a deformation of ρ. On physical grounds, it is clear that $c = V$ at $\rho = 0$. Mathematically:
$c = dQ/d\rho = d(\rho V)/d\rho = V + \rho\,(dV/d\rho)$: since V is a decreasing function, one has $c \leq V$ and, indeed, $c = V$ at $\rho = 0$.