Advanced Nonlinear Physics

international Master « Physics of Complex Systems »

Second Lecture

Traffic flow and the method of characteristics

tuesday, september 15$^{\rm th}$, 2020

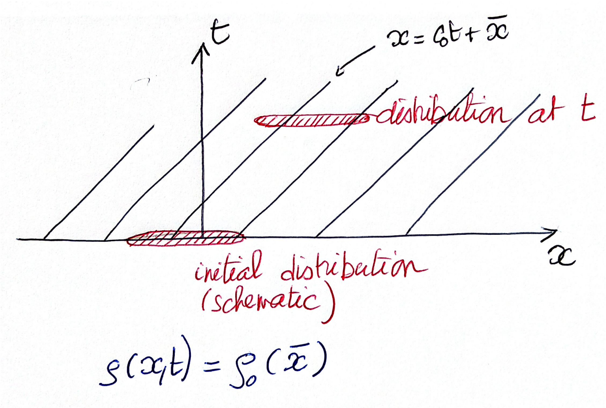
website of (the 1st part of) the course:
`http://lptms.u-psud.fr/nicolas_pavloff/enseignement/advanced-nonlinear-physics/`

One wishes to solve

$$\boxed{\rho_t + c(\rho)\rho_x = 0}\,, \tag{1}$$

i.e., to determine $\rho(x,t)$ for a given $\rho(x,0) \equiv \rho_0(x)$.

If $c(\rho) = c_0$ is a constant, the solution is clearly $\rho(x,t) = \rho_0(x - c_0 t)$. It means that in the (x,t) plane, the initial profile is parallel transported along straight lines of slope c_0^{-1}.

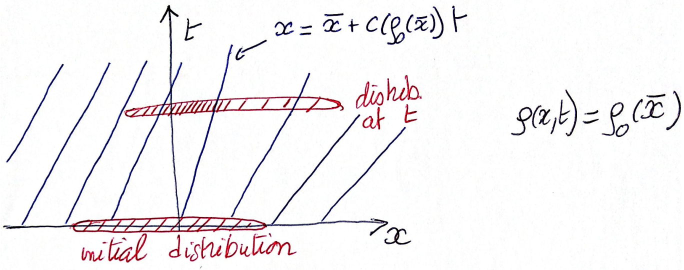


For non constant $c(\rho)$, trick: define $\psi(t) \equiv \rho(x(t), t)$, where $x(t)$ is a yet unknown function. One has

$$\frac{d\psi}{dt} = \rho_t + \frac{dx}{dt}\,\rho_x\ . \qquad (2)$$

If one chooses $x(t)$ such that $dx/dt = c(\psi)$, then one sees from (1) that $d\psi/dt = 0$: $\psi(t)$ is conserved along the curve $x(t)$, which is called a characteristic.

ψ being constant, $dx/dt = c(\psi)$ is constant along the characteristic which is thus a straight line[1]. At variance with the example of constant c_0, the characteristics' slopes are not all the same, because they depend on the initial distribution ρ_0.



[1] This is not a generic feature, cf. exercices in the first tutorial.

In practice, the solution is obtained in an implicit form:

$$\rho(x,t) = \rho_0(\bar{x}) \quad \text{where} \quad x = \bar{x} + c(\rho_0(\bar{x}))t$$

Example of Hopf equation:

$$\rho_t + \rho\,\rho_x = 0 \quad \text{with} \quad \rho_0(x) = \exp(-x^2)$$

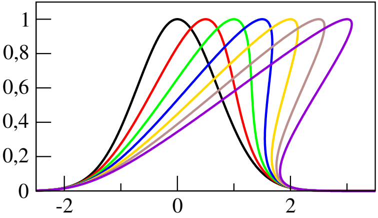
the numerical implementation of the method of characteristic is straightforward:

```
do i=1,imax
xt(i)=x(i)+rho(i)*t
enddo
```

then

```
plot  [xt(i),rho(i)]
```

One gets



simulation at times $t = 0, 0.5, 1, \cdots, 3$

One observes **wave-breaking** at time $t \gtrsim 1$

when 2 characteristics cross, the solution gets **multivalued** method of characteristics: $x = \bar{x} + f(\bar{x})t$ where $f(\bar{x}) = c(\rho_0(\bar{x}))$

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two neighboring characteristics (starting at $\bar{x}$ and $\bar{x} + d\bar{x}$) cross when:

$$x = \bar{x} + f(\bar{x})t = \bar{x} + d\bar{x} + f(\bar{x} + d\bar{x})t$$

in the limit $d\bar{x} \to 0$ this implies $0 = 1 + f'(\bar{x})t$.

Wave breaking occurs first at

$$\boxed{t_{WB} = \frac{1}{max[-f'(\bar{x})]}}$$

In the above example of Hopf equation, $-f'$ reaches ist maximum at $1/\sqrt{2}$ and its value is $\sqrt{2/e}$. Hence $t_{WB} = \sqrt{e/2} \simeq 1.1658$

In the following slides we consider "the Riemann problem":

$$\rho_0(x) = \begin{cases} \rho_2 & \text{if } x > 0 \\ \rho_1 & \text{if } x < 0 \end{cases} \quad \text{and thus} \quad f(x) = \begin{cases} c_2 = c(\rho_2) & \text{if } x > 0 \\ c_1 = c(\rho_1) & \text{if } x < 0 \end{cases} \tag{3}$$

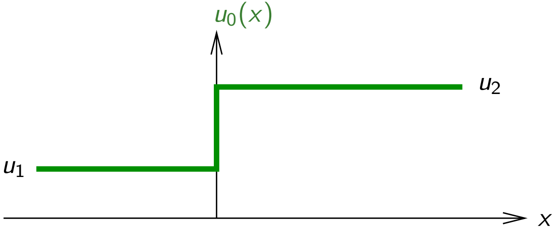
if $c_1 > c_2$ wave breaking occurs immediately.

Define $u(x,t) = c(\rho(x,t))$. From (1): u is solution of Hopf equation

$$\boxed{u_t + u\,u_x = 0}\,, \qquad (4)$$

Once (4) is solved, $\rho(x,t) = c^{-1}(u(x,t))$

- case $u_1 < u_2$: **rarefaction wave**



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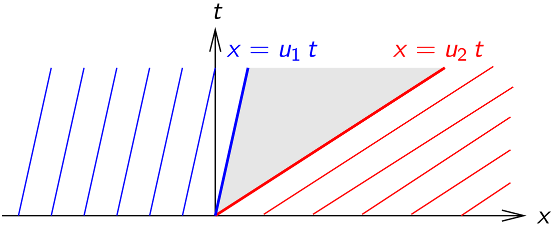

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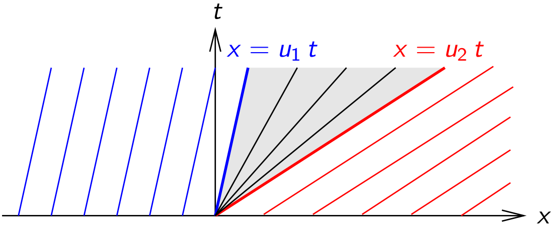
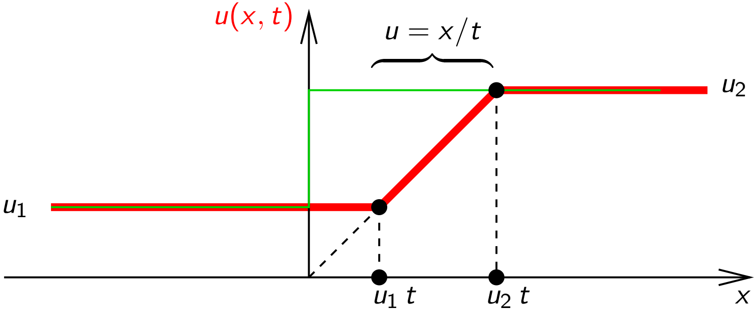


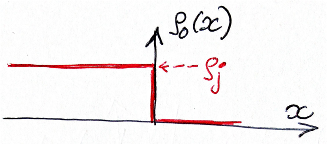
One fills the gray region by "smoothing" the distribution (3). Remember that $x = \bar{x} + u_0(\bar{x})\,t$ with, now, u_0 interpolating smoothly from u_1 to u_2 at $\bar{x} \simeq 0$: this leads to $x = u\,t$ with $u_1 < u < u_2$. Hence

$$u(x,t) = \begin{cases} u_2 & \text{if } u_2 < x/t \\ x/t & \text{if } u_1 < x/t < u_2 \\ u_1 & \text{if } x/t < u_1 \end{cases} \tag{5}$$



$u(x,t)$
$u = x/t$
u_2
u_1
$u_1\ t$
$u_2\ t$
x

Exercice: Describe what happens when a traffic light turns green



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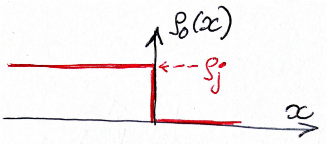

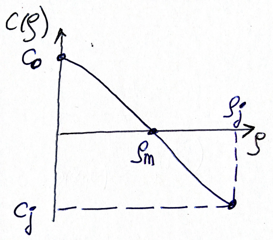

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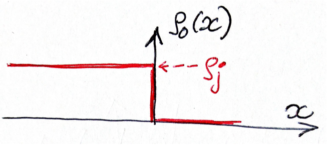

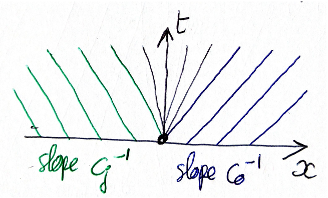

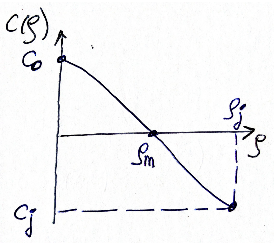

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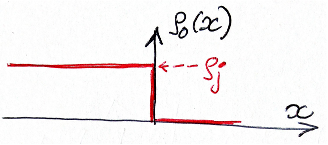
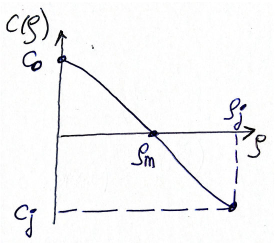
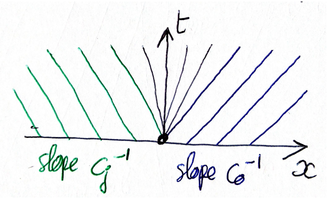
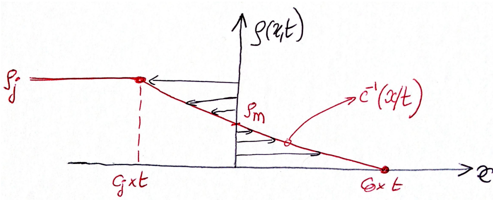
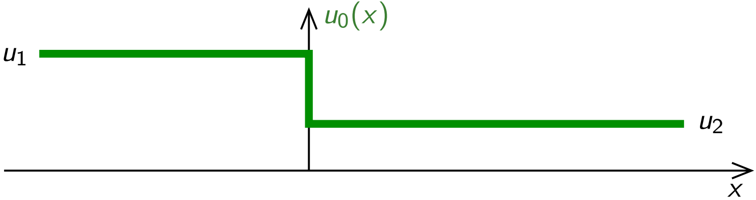
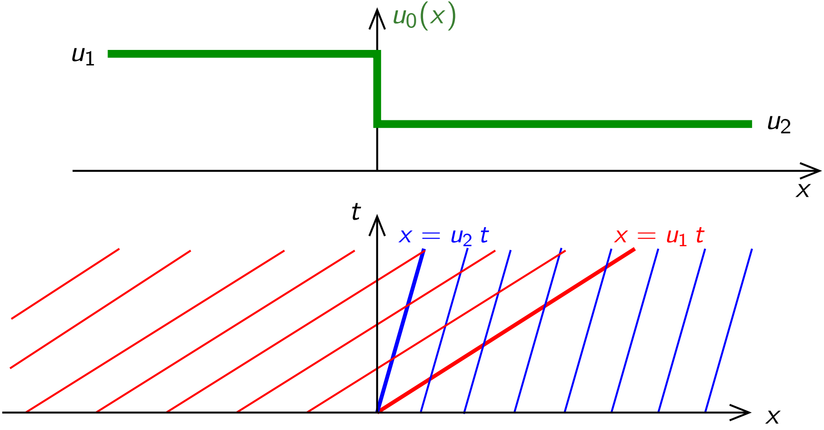




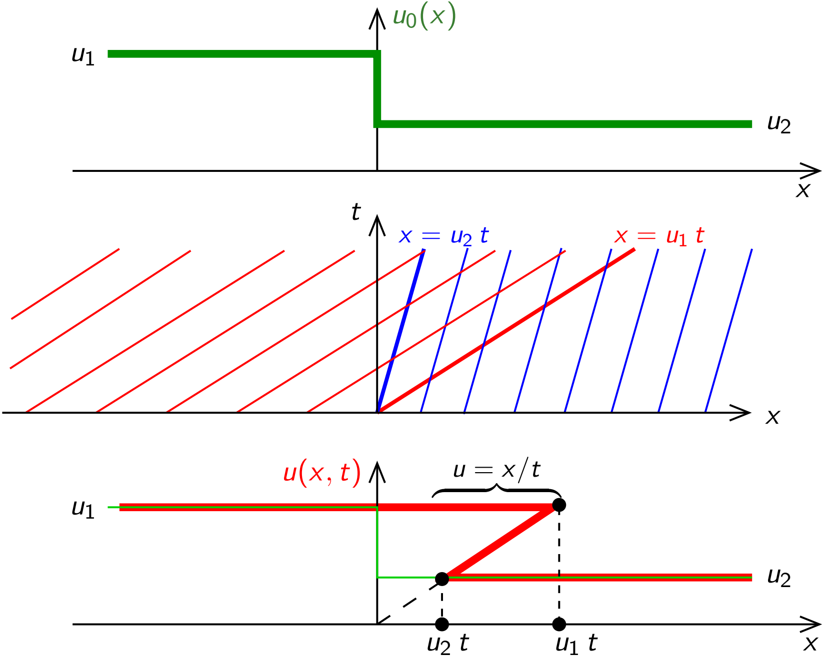





$u_0(x)$
u_1
u_2
x


$u_0(x)$
u_1
u_2
x
t
$x = u_2\, t$
$x = u_1\, t$
x

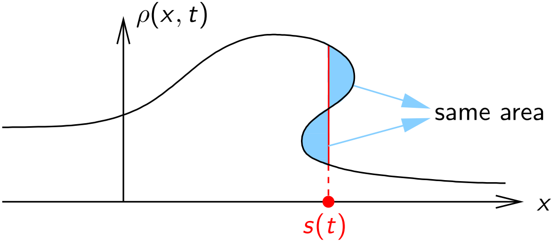
the Riemann problem: $u_1 > u_2$



multivalued solution and shock waves

Multivalued solution not acceptable: Replace  by 

Correct position of the discontinuity: current conservation



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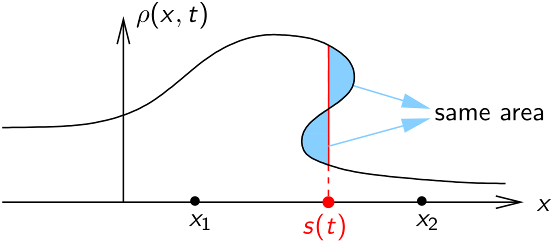

Velocity of the shock:

multivalued solution and shock waves

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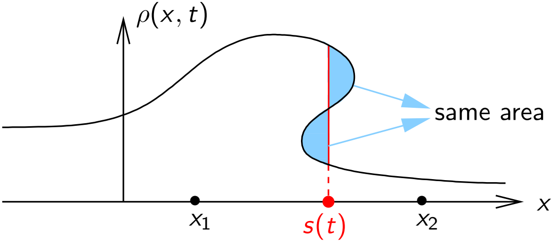

Velocity of the shock:

$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)\,dx = Q(x_1,t) - Q(x_2,t)$$

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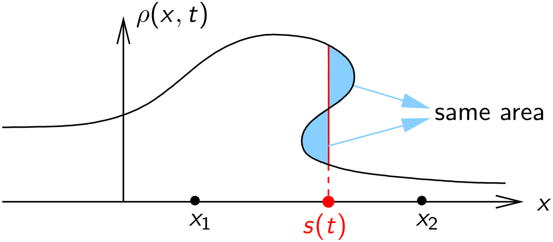
Correct position of the discontinuity: current conservation



Velocity of the shock:

$$\frac{d}{dt}\left(\int_{x_1}^{s(t)} \rho(x,t)\,dx + \int_{s(t)}^{x_2} \rho(x,t)\,dx\right) = Q(x_1,t) - Q(x_2,t)$$

$$\rho(s^{(-)}(t),t)\dot{s} - \rho(s^{(+)}(t),t)\dot{s} + \int_{x_1}^{x_2} \rho_t\,dx = Q(x_1,t) - Q(x_2,t)$$

Let $x_1 \to s \leftarrow x_2$:

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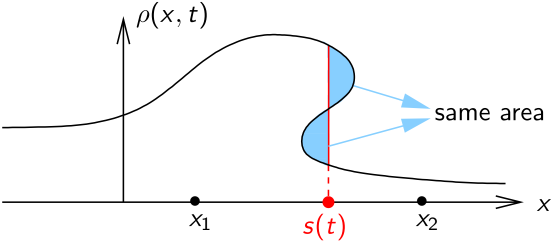
Correct position of the discontinuity: current conservation



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$$\rho(s^{(-)}(t),t)\dot{s} - \rho(s^{(+)}(t),t)\dot{s} + \int_{x_1}^{x_2} \rho_t\,dx = Q(x_1,t) - Q(x_2,t)$$

Let $x_1 \to s \leftarrow x_2$: $\boxed{\dot{s} = \frac{Q^{(+)} - Q^{(-)}}{\rho^{(+)} - \rho^{(-)}}}$ where $Q^{(\pm)} = Q(s^{(\pm)}(t), t)$ (idem ρ)