Advanced Nonlinear Physics

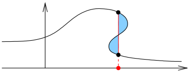
international Master « Physics of Complex Systems »

Fourth Lecture

**Beyond advection equations:
advection-diffusion and KdV**

tuesday, september 29$^{\mathrm{th}}$, 2020

website of (the 1st part of) the course:
`http://lptms.u-psud.fr/nicolas_pavloff/enseignement/advanced-nonlinear-physics/`

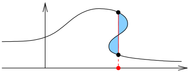
does wave breaking always leads to  ?

Answer is no in general. In the presence of viscosity this is indeed $\simeq$ the case.

Modify traffic flow model: $Q(\rho) \to \tilde{Q}[\rho] = Q(\rho) - \nu \rho_x$ (with $\nu > 0$)

The conservation equation $\rho_t + \tilde{Q}_x = 0$ now reads

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx} \qquad \text{advection-diffusion equation} \qquad (1)$$

does wave breaking always leads to  ?

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The diffusive term induces damping:

$\rho(x,t) = \rho_0 + \rho_1(x,t) \quad (|\rho_1| \ll \rho_0)$

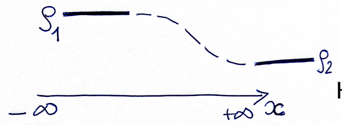
linearization of (1) $\quad \rho_{1t} + c_0 \rho_{1x} = \nu \rho_{1xx} \quad$ where $c_0 = c(\rho_0)$

Look for $\rho_1(x,t) = A \exp[i(kx - \omega t)]$

this leads to $-i\omega + c_0 ik = -\nu k^2 \quad (\omega = c_0 k - i\nu k^2)$

The imaginary part of the frequency leads to a damping $\exp(-\nu k^2 t)$.

Within model (1), for a Riemann initial condition, one looks for a shock-like

solution 

moving at constant velocity U.
Here: case c increasing function of ρ

Denote as $\xi = x - Ut$ and look for $\rho(x,t) = \rho(\xi)$.

$\rho_x = \rho'$ and $\rho_t = -U\,\rho'$ where $\rho' = \frac{d\rho}{d\xi}$

(1) reads $\quad [-U + c(\rho)]\,\rho' = \nu\rho'' \qquad 1^{st}$ integral: $\boxed{-U\rho + Q(\rho) + A = \nu\rho'}$

at $\xi \to \pm\infty$: $\quad -U\rho_1 + Q(\rho_1) + A = 0 = -U\rho_2 + Q(\rho_2) + A$

Hence $\boxed{U = \dfrac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}}$

as for the schematic solution of the previous lecture
(indeed: independent of ν)

Profile of the shock

Usual technique:

$$d\xi = \frac{d\rho}{\rho'} \quad \rightsquigarrow \quad \frac{\xi}{\nu} = \int^{\rho} \frac{d\varrho}{Q(\varrho) - U\varrho + A} \equiv F(\rho) \quad \rightsquigarrow \quad \rho = F^{-1}(\xi/\nu)$$

ρ_1 and ρ_2 : zeros of the denominator, hence when $\rho \to \rho_{1/2}$, ξ diverges and tends to $\pm\infty$ as it should.

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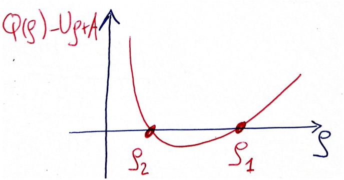
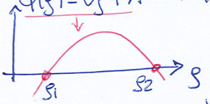
Here $d^2Q/d\rho^2 = dc/d\rho > 0$ and one has:

$Q(\rho) - U\rho + A < 0$ between roots:
$\rho' < 0$ as it should



assume that $\Phi''(\rho) < 0$ (as in traffic flow) and that Φ is quadratic.
solve exactly $\xi/v = \int \frac{d\rho}{\Phi(\rho) - v\rho + A}$

solution: one has here $\Phi(\rho) - v\rho + A = \alpha(\rho - \rho_1)(\rho_2 - \rho)$ with α = positive c^{st}



writing $\frac{1}{(\rho-\rho_1)(\rho_2-\rho)} = \frac{1}{\rho_2-\rho_1}\left(\frac{1}{\rho-\rho_1} + \frac{1}{\rho_2-\rho}\right) \rightsquigarrow \int \frac{d\rho}{(\rho-\rho_1)(\rho_2-\rho_1)} = \frac{1}{\rho_2-\rho_1}\left(\ln(\rho-\rho_1) - \ln(\rho_2-\rho_1)\right)$

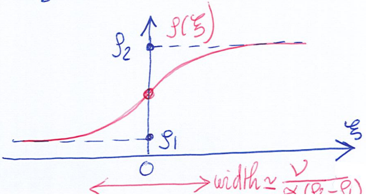
$$= \frac{1}{\rho_2-\rho_1} \ln \frac{\rho-\rho_1}{\rho_2-\rho}$$

I choose to fix $\xi = 0$ at $\rho = \frac{\rho_1+\rho_2}{2}$

then $\frac{\alpha}{v}\xi = \int_{\frac{\rho_1+\rho_2}{2}}^{\rho} \frac{d\rho}{(\rho-\rho_1)(\rho_2-\rho)} = \frac{1}{\rho_2-\rho_1}\left[\ln \frac{\rho-\rho_1}{\rho_2-\rho}\right]_{\frac{\rho_1+\rho_2}{2}}^{\rho} = \frac{1}{\rho_2-\rho_1} \ln \frac{\rho-\rho_1}{\rho_2-\rho}$

invert the above expression:

$$\rho(\xi) = \frac{\rho_1 + \rho_2 \exp[(\rho_2-\rho_1)\alpha\,\xi/v]}{1 + \exp[(\rho_2-\rho_1)\alpha\,\xi/v]}$$



dispersion vs viscosity I

In some situations viscosity is negligeable, and realistic shocks involve dispersive effets. This is the case if (1) is replaced by

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xxx} \tag{2}$$

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$$\rho_t + c(\rho)\rho_x = \nu \rho_{xxx} \qquad (2)$$

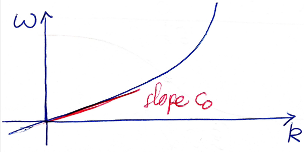
The new term induces dispersion:

$\rho(x,t) = \rho_0 + \rho_1(x,t)$ linearize: $\rho_{1t} + c_0\,\rho_{1x} = \nu\rho_{1xxx}$

look for plane wave solution: $\rho_1(x,t) = A\exp[i(kx - \omega t)]$

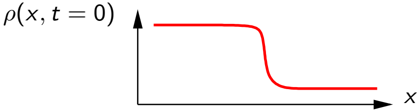
this leads to $-i\omega + c_0(ik) = \nu(ik)^3$

$$\omega = c_0 k + \nu k^3 \qquad (3)$$



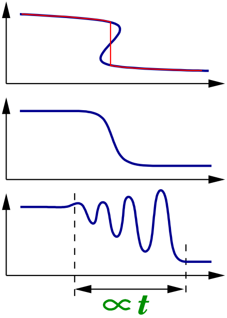
plot for positive ν

dispersion vs viscosity II: different scenarii



model equations, $c(\rho) = \rho$

$$\rho_t + \rho\rho_x = \begin{cases} 0 & \text{Hopf} \\ \epsilon\, \rho_{xx} & \text{Burgers} \\ \epsilon\, \rho_{xxx} & \text{KdV} \end{cases}$$



steepening then breaking

Taylor profile

dispersive shock wave

Eq. (2): $\rho_t + c(\rho)\rho_x = \nu \rho_{xxx}$ is the simplest way to include dispersion in the advection equation.

For weak nonlinearity $c(\rho) \simeq c_0 + c_1 \rho$ by the way $c_1 \lessgtr 0$ and $\nu \lessgtr 0$

$\xi = x - c_0 t$, $\tau = t$ $(\partial_x = \partial_\xi,\ \partial_t = -c_0 \partial_\xi + \partial_\tau)$: $\rho_\tau + c_1 \rho \rho_\xi - \nu \rho_{\xi\xi\xi} = 0$

Define $\begin{cases} t = \alpha \tau \\ x = \beta \xi \\ \rho = \gamma u \end{cases} \quad \rightsquigarrow \quad \alpha\gamma u_t + c_1 \gamma^2 \beta u u_x - \nu \gamma \beta^3 u_{xxx} = 0$

Choose $\begin{cases} \alpha = 1 \\ c_1 \beta \gamma = 1 \\ -\nu \beta^3 = 1 \end{cases} \quad \rightsquigarrow \quad \boxed{u_t + u u_x + u_{xxx} = 0}$

Korteweg-de Vries equation (shallow water waves, 1895)

Solution propagating without deformation $u(x,t) = u(\xi)$ with $\xi = x - Vt$ $(V > 0)$.

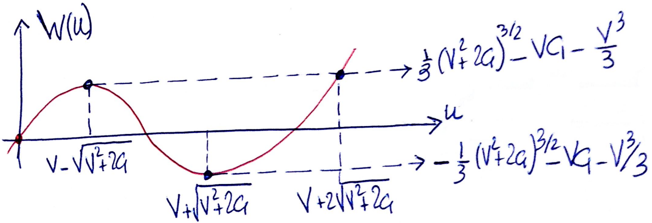
$-Vu' + uu' + u''' = 0$ first integral $-Vu + \frac{1}{2}u^2 + u'' = c_1$.

multiply by u' and integrate again $-\frac{1}{2}Vu^2 + \frac{1}{6}u^3 + \frac{1}{2}(u')^2 = c_1 u + c_2$.

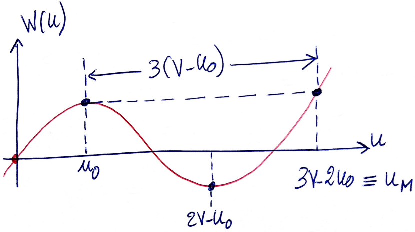
This is of the form

$$\frac{1}{2}(u')^2 + W(u) = c_2 \quad \text{with} \quad W(u) = \frac{1}{6}u^3 - \frac{1}{2}Vu^2 - c_1 u \qquad (4)$$

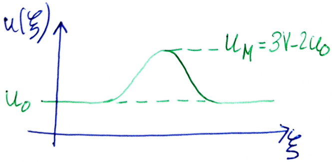
formally equivalent to $\frac{1}{2}m\dot{x}^2 + V(x) = E_{tot}$. The effective potential $W(u)$ is a 3^{rd} order polynomial. If one wants a bounded solution, $W(u)$ should not be monotonous. $W'(u)$ has real zeros if $V^2 + 2c_1 > 0$, and then cancels for $u = V \pm \sqrt{V^2 + 2c_1}$ (sketch below for $c_1 < 0$)



Define $u_0 = V - \sqrt{V^2 + 2c_1}$.



Situation where $c_2 = W(u_0)$:



Eq. (5) reads $\quad \frac{1}{2}(u')^2 = W(u_0) - W(u) = \frac{1}{6}(u-u_0)^2(u_M - u)$.

$u(\xi) = u_0 + v(\xi) \rightsquigarrow (v')^2 = \frac{1}{3}v^2(v_M - v)$ where $v_M = u_M - u_0 = 3(V - u_0)$

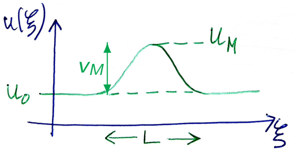
$\frac{d\xi}{\sqrt{3}} = \pm\frac{dv}{v\sqrt{v_M - v}}$ change of variable: $v = \frac{v_M}{\cosh^2\theta} \rightsquigarrow \theta = \frac{1}{2}\xi\sqrt{v_M/3}$

$$u(\xi) = u_0 + \frac{3(V-u_0)}{\cosh^2\left(\frac{1}{2}\sqrt{V-u_0}\,\xi\right)} \qquad \text{where} \quad \xi = x - Vt \tag{5}$$

works also when $u_0 < 0$ (which is equiv. to $c_1 > 0$)

Unique condition $V > u_0$ (which is satisfied when $V^2 + 2c_1 > 0$)

linear dispersion relation (3): $\omega = u_0 k - k^3 \leadsto V_\varphi = \frac{\omega}{k} = u_0 - k^2$
V_φ is k-dependent: spreading. This spreading is compensated by non-linearity.
Had hoc description: $V_\varphi \sim u_0 + v_M - k^2|_{typ}$



$k|_{typ} \sim 1/L$

nonlinearity and dispersion compensate:

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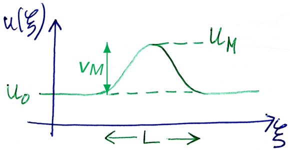

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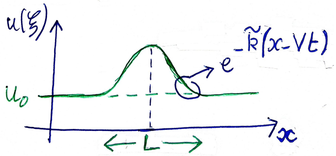
The outskirt of the solitary wave: small perturbation. linear expansion similar to what is done for $\exp[i(kx - \omega t)]$. dispersion relation: $\omega = fct(k)$



clearly $\tilde{k} \sim 1/L$

$$\begin{cases} \tilde{k} \leftrightarrow -ik \\ \tilde{k}V \leftrightarrow -i\omega \end{cases}$$

$i\tilde{k}V = fct(i\tilde{k}) \rightsquigarrow \tilde{k}V = u_0\tilde{k} + \tilde{k}^3$
one gets

$V \sim u_0 + 1/L^2$

Both estimates are in agreement with the exact result (5)