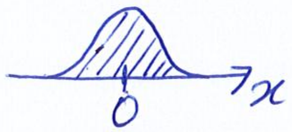
$$\boxed{\text{solve} \quad \phi_t + e^{-t}\phi_x = 0}$$

(1/1)

for $x \in \mathbb{R}$, $t \geq 0$ and $\phi(x,0) = \phi_0(x) \simeq$ 

• define $\varphi(t) = \phi(x(t),t)$ where $x(t)$ is yet unspecified.

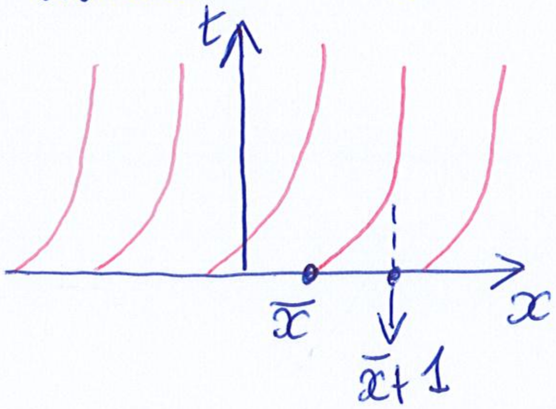
$$\frac{d\varphi}{dt} = \phi_t + \frac{dx}{dt}\phi_x$$

if one defines the characteristics by $\frac{dx}{dt} = e^{-t}$, then φ will be constant along a characteristic.

$$\frac{dx}{dt} = e^{-t} \rightsquigarrow x(t) - x_0 = \left[-e^{-t}\right]_0^t : x(t) = \bar{x} + 1 - e^{-t}$$

notation for $x(0)$ [as in main course]

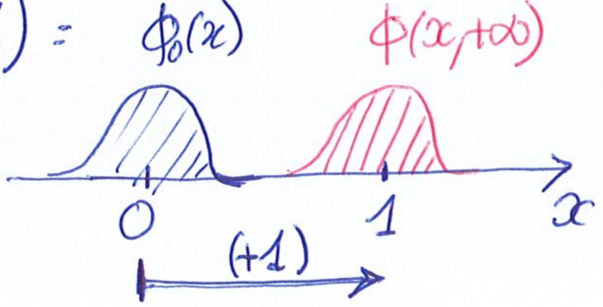
the characteristics are:



and $\phi(x,t) = \phi_0(\bar{x}) = \phi_0(x - 1 + e^{-t})$

it is easy to check that this expression
(i) fulfills the eq. $\phi_t + e^{-t}\phi_x = 0$
(ii) verifies $\phi(x,0) = \phi_0(x)$

× remark = $\phi(x,+\infty) = \phi_0(x-1)$:



✓

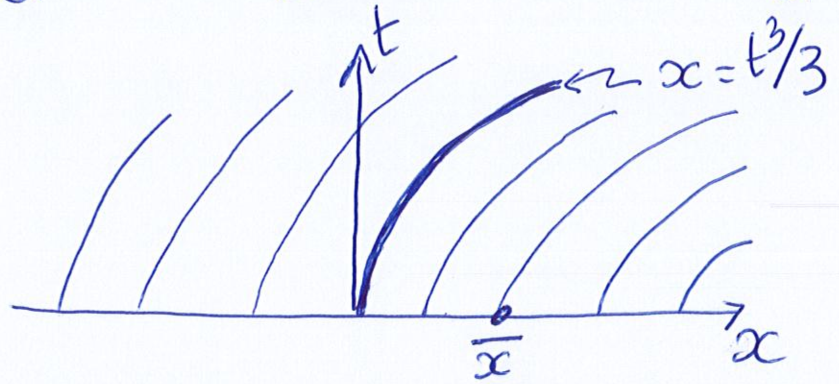
$$\boxed{\text{solve } \phi_t + t^2 \phi_x + x\phi = 0}$$

(I) for $x \in \mathbb{R}$, $t \geqslant 0$ and $\phi(x,0) = f(x)$

(II) for $x \geqslant 0$, $t \geqslant 0$ and $\begin{cases} \phi(x \geqslant 0, 0) = f(x) \\ \phi(0, t \geqslant 0) = g(t) \end{cases}$

determination of the characteristics: $\psi(t) \stackrel{def}{=} \phi(x(t), t)$

$$\frac{d\psi}{dt} = \phi_t + \frac{dx}{dt}\phi_x \quad \text{impose } \frac{dx}{dt} = t^2 \text{, that is: } x(t) = \frac{1}{3}t^3 + C^{ste}$$



in case (I): one writes $x(t) = \bar{x} + t^3/3$ and, from the above framed equation, one obtains:

$$\frac{d\psi}{dt} = -x\psi = -(\bar{x} + t^3/3)\psi$$

that is: $\frac{d\psi}{\psi} = -(\bar{x} + t^3/3)\,dt \rightsquigarrow \ln\psi - \ln\psi(0) = -\left[\bar{x}t + \frac{t^4}{12}\right]_0^t$

thus $\phi(x,t) = f(\bar{x})\exp(-\bar{x}t - t^4/12)$ where $\bar{x} = x - t^3/3$

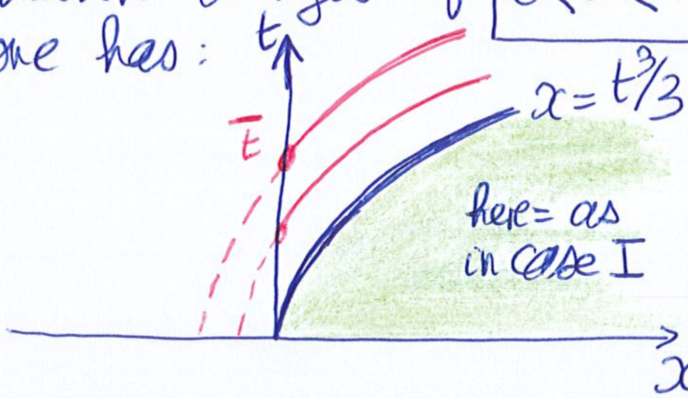
replacing in the above expression yields:

$$\boxed{\phi(x,t) = f(x - t^3/3)\exp(-xt + t^4/4)}$$

simple check: $\phi_t + t^2\phi_x = \left\{-t^2 f' + (-x + t^3)f + t^2[f' - tf]\right\} e^{-xt + t^4/4}$

$= -x\phi$ as it should.

in case II The situation changes if $\boxed{0 < x < t^3/3}$ - In this case one has:



for the red characteristics one writes $x = 1/3(t^3 - \bar{t}^3)$
this yields $$\frac{d\Psi}{dt} = -x\Psi = \frac{1}{3}(\bar{t}^3 - t^3)\Psi$$

and, upon integration $$\ln\Psi - \ln\Psi(\bar{t}) = \frac{1}{3}\left[\bar{t}^3 t - \frac{t^4}{4}\right]_{\bar{t}}^{t}$$
$$= -\frac{1}{4}\bar{t}^4 + \frac{1}{3}\bar{t}^3 t - t^4/12$$

thus $\phi(x,t) = g(\bar{t})\exp\left(-\frac{\bar{t}^4}{4} + \frac{1}{3}\bar{t}^3 t - \frac{t^4}{12}\right)$
where $\bar{t} = (t^3 - 3x)^{1/3}$ - replacing in the above yields =

$$\boxed{\phi(x,t) = g((t^3 - 3x)^{1/3})\exp\left[-\frac{(t^3-3x)^{4/3}}{4} + \left(-x + \frac{t^3}{3}\right)t - \frac{t^4}{12}\right]}$$

for $(0 < x < t^3/3)$ - in the case $x > t^3/3$ the solution is the same as in case I.

<u>simple check</u> using $$\begin{cases} \frac{d}{dt}(t^3 - 3x)^{1/3} = \frac{t^2}{(t^3-3x)^{2/3}} \\ \frac{d}{dx}(t^3 - 3x)^{1/3} = \frac{-1}{(t^3-3x)^{2/3}} \end{cases}$$

one gets $$\phi_t + t^2\phi_x + x\phi = \left\{\frac{t^2}{(t^3-3x)^{2/3}}g' + g\times\left[-(t^3-3x)\frac{t^2}{(t^3-3x)^{2/3}} - x + t^3\right]\right.$$
$$\left. + t^2\left[-\frac{g'}{(t^3-3x)^{2/3}} + g\times\left(\frac{t^3-3x}{(t^3-3x)^{2/3}} - t\right)\right] + xg\right\}e^{blabla} = 0$$

✓

Exercice 1 (e) $\quad \phi_0(x) =$

$\phi_t + \phi\,\phi_x + \alpha\phi = 0$

define $\psi(t) = \phi(x(t),t)$ where $x(t)$ is still unknown.

$$\frac{d\psi}{dt} = \frac{dx}{dt}\phi_x + \phi_t \quad \text{if one chooses} \quad \left.\begin{aligned} \frac{dx}{dt} &= \psi \\ \text{one has:} \quad \frac{d\psi}{dt} &= -\alpha\psi \end{aligned}\right\}$$

the second eq. is easily solved: $\psi(t) = \psi(0)\,e^{-\alpha t}$. Inserting this expression in the 1st eq, on gets $\frac{dx}{dt} = \psi(0)e^{-\alpha t}$, hence:

$$x(t) = x_0 + \frac{\psi(0)}{\alpha}\left(1 - e^{-\alpha t}\right) \quad \text{where } \psi(0) = \phi(x(0),0) = \phi_0(x_0)$$

x_0: value of x at $t=0$

Solving the PB, ie obtaining $\phi(x,t) =$

→ first, for given (x,t) compute x_0 solution of

$$\boxed{x = x_0 + \frac{\phi_0(x_0)}{\alpha}\left(1 - e^{-\alpha t}\right)}$$

→ then one has $\phi(x,t) = \phi_0(x_0)\,e^{-\alpha t}$

(one can also write, from the 1st eq. $\phi_0(x_0) = \alpha\frac{x - x_0}{1 - e^{-\alpha t}}$
this yields
$\phi(x,t) = \frac{\alpha(x - x_0)}{e^{\alpha t} - 1}$)

the wave breaking occurs when $\partial_x\phi \to \pm\infty$

one has $\partial_x\phi = e^{-\alpha t}\phi_0'(x_0)\frac{\partial x_0}{\partial x}$

can be computed from the framed eq. above:

$\frac{\partial x_0}{\partial x} = \frac{dx_0}{dx}$ evaluated at fixed t

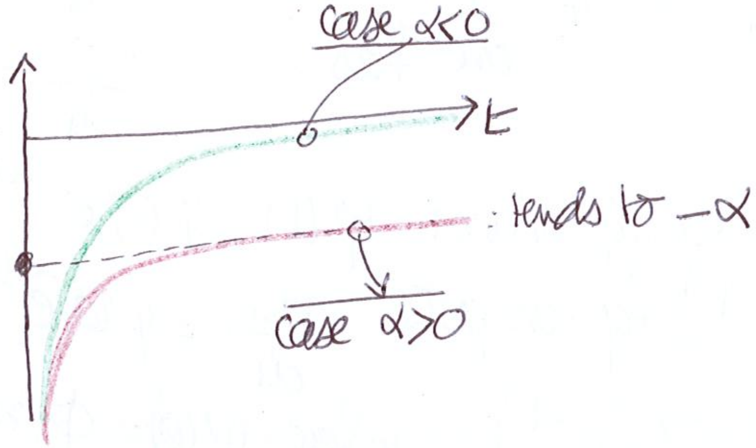
hence $dx = dx_0 + \frac{1}{\alpha}\phi'(x_0)\,dx_0\left(1 - e^{-\alpha t}\right)$

and $\dfrac{\partial x_0}{\partial x} = \dfrac{1}{1 + \frac{1}{\alpha}\phi'(x_0)(1-e^{-\alpha t})}$

↓

this diverges if there exist t and x_0 such that $\phi'(x_0) = \dfrac{-\alpha}{1-e^{-\alpha t}}$

<u>plot</u> = $\dfrac{-\alpha}{1-e^{-\alpha t}}$



so, if $\alpha<0$ one will always find a solution

but, if $\alpha>0$ and if $(\forall x_0 \in \mathbb{R})\ \phi'(x_0) > -\alpha$ wave breaking never occurs.

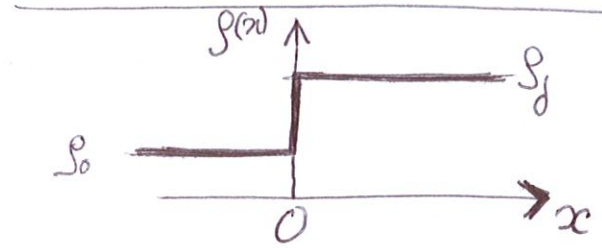
PS = one can (as was done during the course) write the condition of wave-breaking as the condition that 2 neighboring characteristics (issued from x_0 and x_0+dx_0) intersect at point (x,t) =

$$x = x_0 + \phi_0(x_0)\frac{1-e^{-\alpha t}}{\alpha} = x_0 + dx_0 + \phi_0(x_0+dx_0)\frac{1-e^{-\alpha t}}{\alpha}$$

this yields $1 + \phi'(x_0)\dfrac{1-e^{-\alpha t}}{\alpha} = 0$ = same condition as above

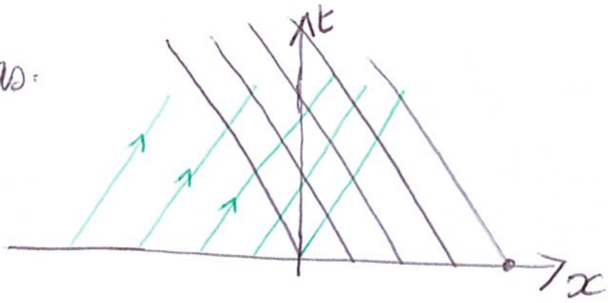
Exercice : comment, et à quelle vitesse progresse l'arrière d'un bouchon (voitures à l'arrêt $\rho = \rho_j$ pour $x \geq 0$) lorsqu'un flux continu (densité ρ_0) l'alimente par l'arrière?

initially :

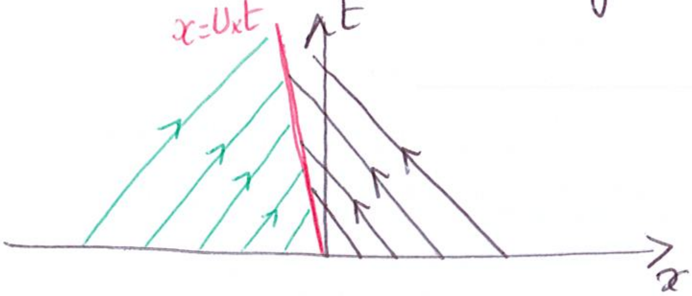


let's assume that $c(\rho_0) > 0$ (ie $0 < \rho_0 < \rho_m$ cf page 11).
(the situation is not ≠ if $c(\rho_0) < 0$)

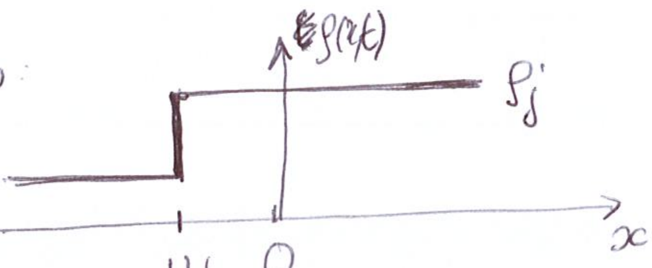
then one has:



one thus has a shock with velocity $\dot{s} = U = \dfrac{Q(\rho_j) - Q(\rho_0)}{\rho_j - \rho_0} = \dfrac{-Q(\rho_0)}{\rho_j - \rho_0} < 0$

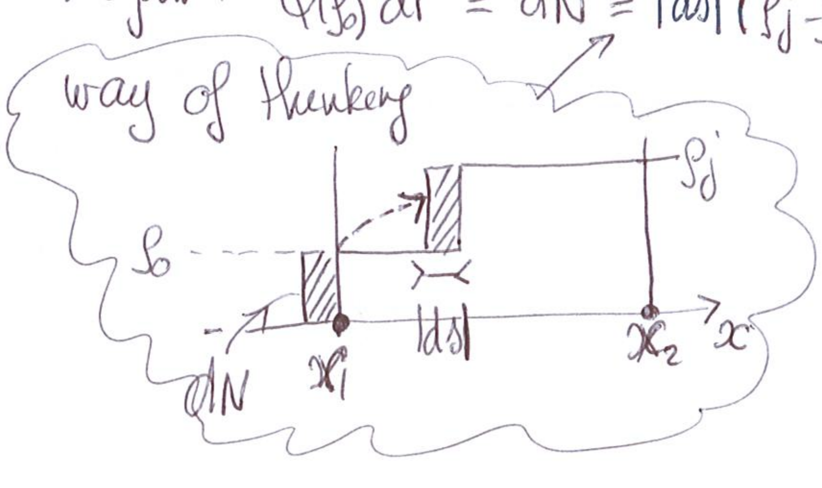


and thus:

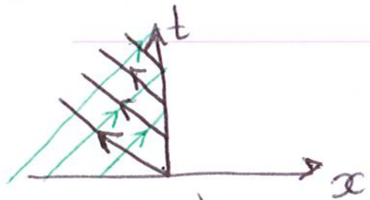


and here one understands easily the formula for U on the basis of the conservation of current = during dt one has dN cars arriving at the rear of the jam: $Q(\rho_0)\, dt = dN = |ds| (\rho_j - \rho_0)$.

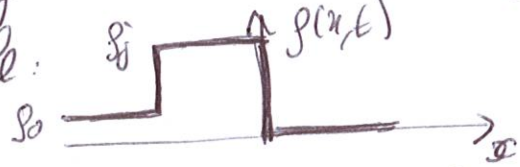
attention, ~~ne pas~~ do not believe that all shocks move along a straight line in (x,t) they bend if $\rho(s^-)$ or $\rho(s^+)$ depend on t.
cf example next page



note: a red traffic light makes the same effect. Except that $\rho \equiv 0$ for $x>0$, and we can draw the characteristics starting from the axis or ordinates:

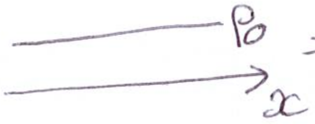


and one will have the same value of U for the shock, with a profile:



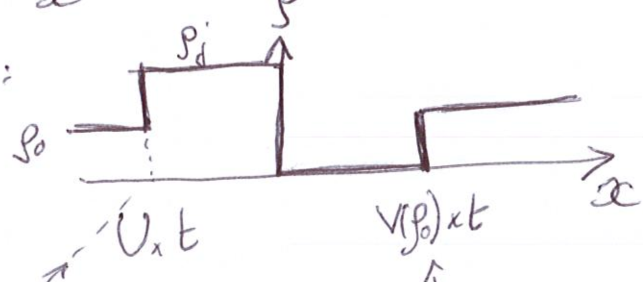
Exercise: what is the effect of a ~~red light~~ traffic light passing red on a uniform traffic flow (density ρ_0)?

one has initially



= everybody has the same velocity $V(\rho_0) = \frac{Q(\rho_0)}{\rho_0}$

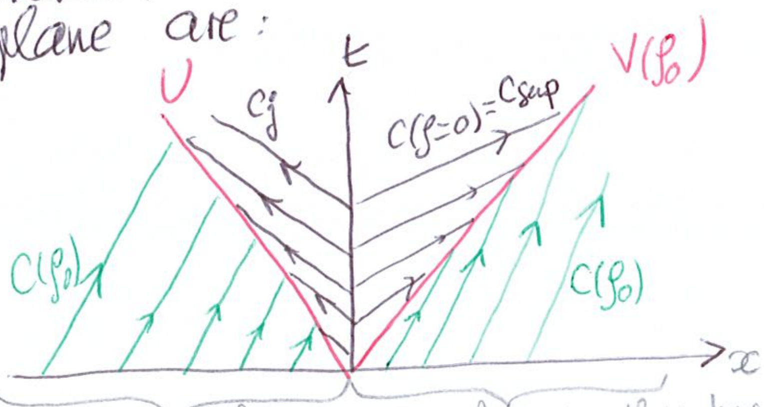
then (the traffic light is at $x=0$):



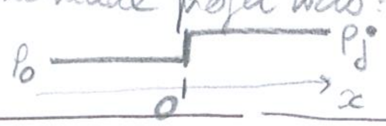
here we already performed the computation:
$$U = \frac{Q(\rho_j)-Q(\rho_0)}{\rho_j-\rho_0} = -\frac{Q(\rho_0)}{\rho_j-\rho_0}$$

here there is also a shock, ~~but~~ with
$$\dot{s} = \frac{Q(\rho_0)-Q(0)}{\rho_0-\rho} = V(\rho_0)$$
as expected.

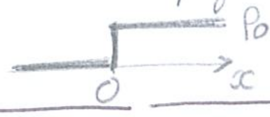
the characteristics in the (x,t) plane are:



here everything happens as if the initial profile was:



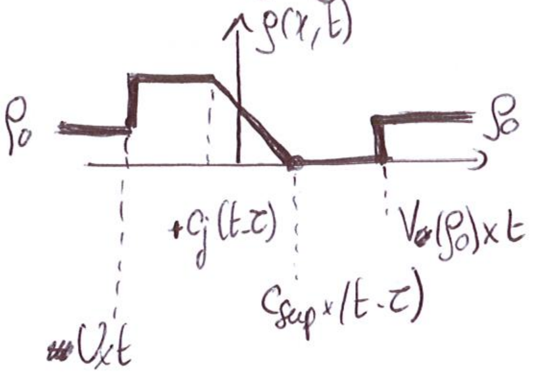
here everything happens as if the initial profile was



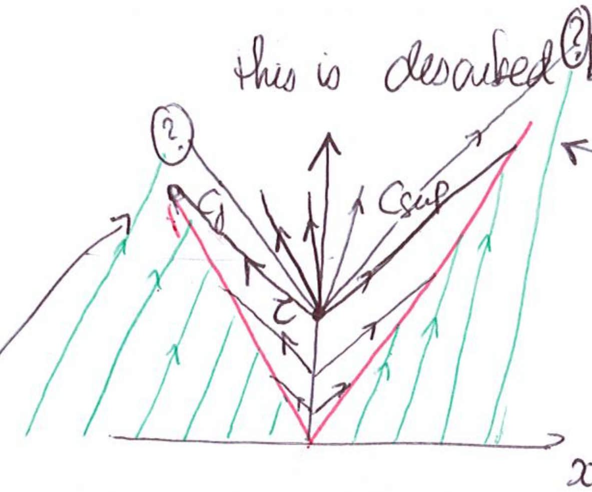
plot for low traffic, ie $\rho_0 < \rho_m$ and $C(\rho_0) > 0$

exercice: describe what happens in the previous situation when the light turns green after a time τ

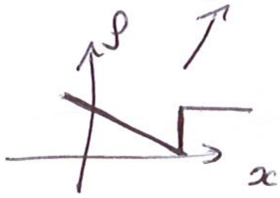
one has roughly (at time t close to τ, but $>\tau$):



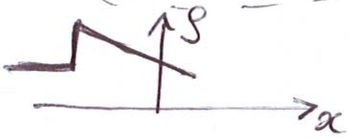
this is described (?) by:



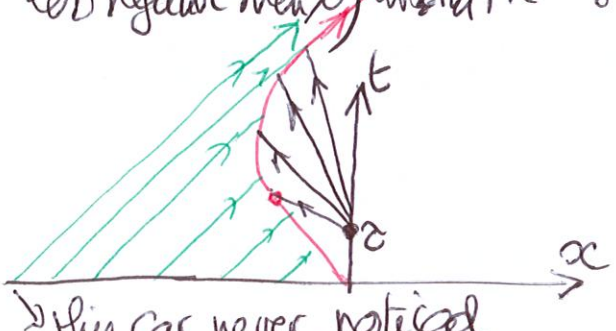
here at time $t_r = \dfrac{c_{sup}\,\tau}{c_{sup} - V(\rho_0)}$ the cars issued from the green light catch up the main flow ahead



at time $t_\ell = \dfrac{c_j\,\tau}{c_j - U}$ the traffic jam start disappearing.



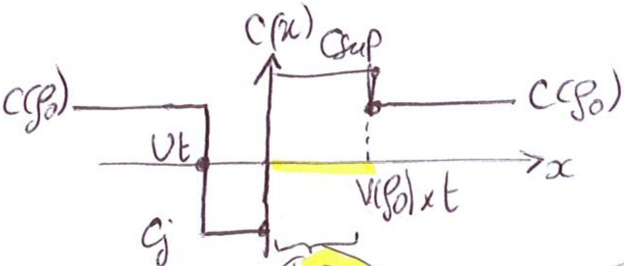
hence the discontinuity of ρ decreases, so the shock velocity decreases (i.e becomes less negative then >0) until the shock disappears:



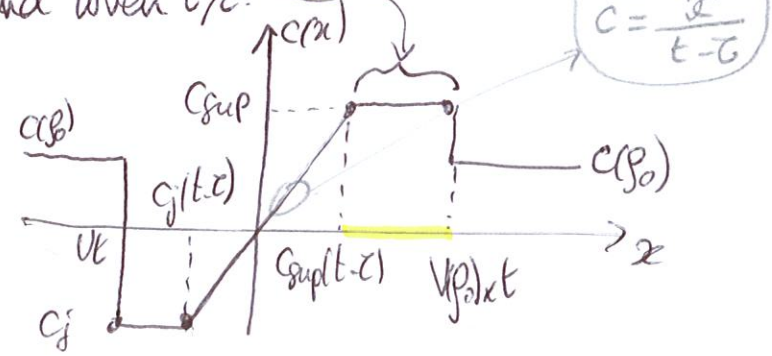
This car never noticed the red light ahead...

(no, this is not what happens --- see below)

one has for $t<\tau$:



and when $t>\tau$:

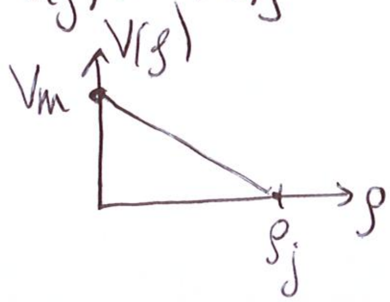


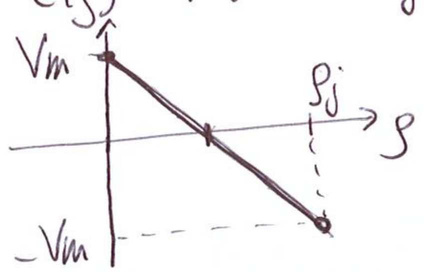
in order to get explicit ~~reg~~ relations I work with a model: quadratic $Q(\rho)$:

$$Q(\rho) = 4\frac{Q_m}{\rho_j}\left(\rho - \rho^2/\rho_j\right) \quad \text{in this case } \rho_m = \rho_j/2$$

and $V(\rho) = Q(\rho)/\rho = V_m(1 - \rho/\rho_j)$ with $V_m = 4Q_m/\rho_j$

$c(\rho) = V_m(1 - 2\rho/\rho_j)$





and I take $\rho_0 < \rho_m = \rho_j/2$, hence $c(\rho_0) > 0$

then one gets "light traffic"

$$\begin{cases} c(\rho_0) = V_m(1 - 2\rho_0/\rho_j) > 0 \\ V(\rho_0) = V_m(1 - \rho_0/\rho_j) \\ U = -\dfrac{Q(\rho_0)}{\rho_j - \rho_0} = -\dfrac{V_m\rho_0(1 - \rho_0/\rho_j)}{\rho_j - \rho_0} = -V_m\dfrac{\rho_0}{\rho_j} \\ c_{sup} = V_m \qquad c_j = -V_m \end{cases}$$

velocity of the left shock at time $0 < t < t_\ell$

the time at which the cars issued from the green light catch up with the warm flow is $= t_r = \dfrac{c_{sup}\tau}{c_{sup} - V(\rho_0)} = \dfrac{V_m\tau}{V_m - V_m(1 - \rho_0/\rho_j)} = \dfrac{\rho_j\tau}{\rho_0}$

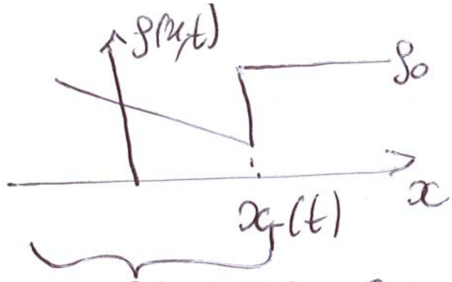
and the time at which the traffic jam starts to disappear at the left

is $= t_\ell = \dfrac{c_j\tau}{c_j - U} = \dfrac{-V_m\tau}{-V_m + V_m\rho_0/\rho_j} = \dfrac{\rho_j\tau}{\rho_j - \rho_0}$

since I chose the light traffic condition
$\rho_j - \rho_0 > \rho_0$
$t_r < t_\ell$

so let's discuss the right case first, but this does not really matter = the 2 shocks will not interfere.

at the right shock one will have:



in this region $c = \dfrac{x}{t-\tau}$

and since $c = V_m(1 - 2\rho/\rho_j)$ this gives

$$\rho = \frac{1}{2}\rho_j\left(1 - \frac{x}{V_m(t-\tau)}\right)$$

the velocity of the shock is

$$\frac{dx_r}{dt} = \frac{\rho V(\rho) - \rho_0 V(\rho_0)}{\rho - \rho_0} \quad \text{where } \rho = \rho(x_r(t))$$

within our model this reads

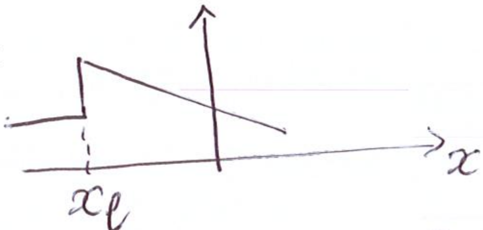
$$\frac{dx_r}{dt} = \frac{V_m(\rho - \rho^2/\rho_j) - V_m(\rho_0 - \rho_0^2/\rho_j)}{\rho - \rho_0}$$

$$= V_m - \frac{V_m}{\rho_j}\,\frac{\rho^2 - \rho_0^2}{\rho - \rho_0} = V_m\left(1 - \frac{\rho + \rho_0}{\rho_j}\right)$$

using the expression for ρ gives:

$$\boxed{\frac{dx_r}{dt} = V_m\underbrace{\left(\frac{1}{2} - \frac{\rho_0}{\rho_j}\right)}_{>0 \text{ (for light traffic flow)}} + \frac{x_r}{2(t-\tau)}}$$

actually one has the same equation at the left shock:



but with $\neq$ initial values ρ_0 and only for $t > t_\ell$, whereas at the right it is only for $t > t_r$

the solution is

$$\boxed{x_{(r/\ell)} = B_{(r/\ell)}(t-\tau)^{1/2} + V_m\underbrace{\left(1 - \frac{2\rho_0}{\rho_j}\right)}_{>0 \text{ for light traffic}}(t-\tau)}$$

indeed, one has to solve sthg of the form: $\dfrac{dx}{d\theta} = C + \dfrac{x}{2\theta}$ (where $\theta = t - \tau$)

this is written as: $\theta\dfrac{dx}{d\theta} - \dfrac{x}{2} = \theta \times C$

the general solution of the homogeneous eq. is $x = B\theta^{1/2}$

looking for a particular solution of the form $x = A \times \theta$

one finds $A = 2C \longrightarrow x = B\sqrt{\theta} + 2C\theta$

* the discontinuity at the shock is =

$$\rho(x) - \rho_0 = \frac{\rho_j}{2}\left(1 - \frac{x}{V_m(t-\tau)}\right) - \rho_0 = -\frac{\rho_j B_{(r/\ell)}}{2V_m\sqrt{t-\tau}} \xrightarrow[t\to\infty]{} 0$$

(where $x = x_r$ or x_ℓ)

* also

$$\frac{dx}{dt} = \underbrace{V_m\left(1-\frac{2\rho_0}{\rho_j}\right)}_{c(\rho_0)} + \frac{B_{(r/\ell)}}{2\sqrt{t-\tau}}$$

so the asymptotic shock velocities are the same (equal to $c(\rho_0)$) but the distance between the shock tends to ∞ since

$$x_r - x_\ell = (B_r - B_\ell)\sqrt{t-\tau}$$

* let's determine the actual value of B_ℓ. One should have at time $t = t_\ell = \rho_j\tau/\rho_j-\rho_0$

$$x_\ell = c_j(t_\ell-\tau) = -V_m(t_\ell-\tau) = w\,t_\ell = -V_m\frac{\rho_0\tau}{\rho_j-\rho_0}$$

this gives

$$-\frac{V_m\rho_0\tau}{\rho_j-\rho_0} = B_\ell\left(\frac{\rho_j\tau}{\rho_j-\rho_0}-\tau\right)^{1/2} + V_m\left(1-\frac{2\rho_0}{\rho_j}\right)\left(\frac{\rho_j\tau}{\rho_j-\rho_0}-\tau\right)$$

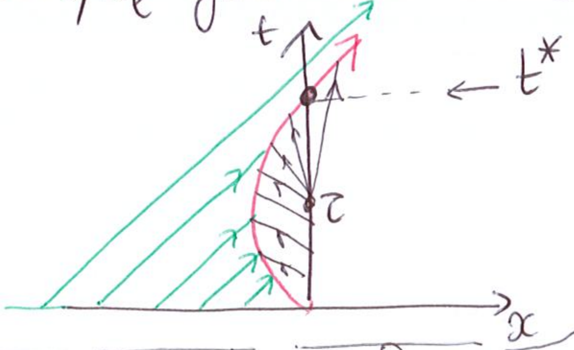
$$= B_\ell\sqrt{\tau}\sqrt{\frac{\rho_0}{\rho_j-\rho_0}} + V_m\tau\left(1-\frac{2\rho_0}{\rho_j}\right)\frac{\rho_0}{\rho_j-\rho_0}$$

$\downarrow \times\left(\frac{\rho_j-\rho_0}{\rho_0}\times\frac{1}{\sqrt{\tau}}\right)$

$$-V_m\sqrt{\tau} = B_\ell\sqrt{\frac{\rho_j-\rho_0}{\rho_0}} + V_m\sqrt{\tau}\left(1-\frac{2\rho_0}{\rho_j}\right) \xrightarrow[\text{computation}]{\text{easy}} \boxed{B_\ell = -2V_m\sqrt{\tau}\times\sqrt{\frac{\rho_0}{\rho_j}\left(1-\frac{\rho_0}{\rho_j}\right)}}$$

$B_\ell<0$, but for light traffic the other contribution to $x(t)$ is >0 and at some time t^*, x_ℓ goes back to zero:

(note: $B_\ell<0$ so, from the formula at the top of this page, $\rho(x_\ell)>\rho_0$ = normal)



an observer at the traffic light would mark t^* as the time where the jam has finally cleared away

note: if one wants to determine B_r one imposes that at time $t_r = \rho_j\tau/\rho_0$ one has $x_r = V(\rho_0)\,t_r = V_m(\rho_j/\rho_0 - 1)\tau$ this gives $B_r = -B_\ell$

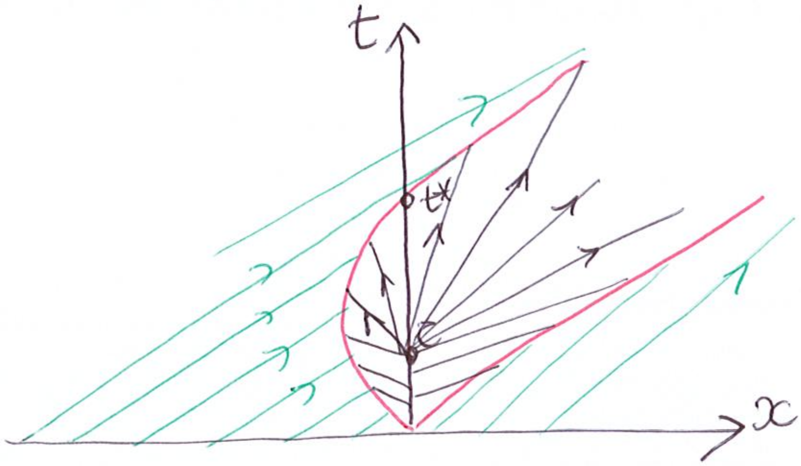
determination of t^*: $x_\ell(t^*)=0$ reads

$$t^* = \tau + \frac{B_\ell^2}{V_m^2\left(1-\frac{2\rho_0}{\rho_j}\right)^2} = \tau\left\{1+\frac{4\,\rho_0/\rho_j\,(1-\rho_0/\rho_j)}{(1-2\rho_0/\rho_j)^2}\right\}$$

hence $$\boxed{t^* = \frac{\tau}{(1-2\rho_0/\rho_j)^2}}$$

for instance, if $\rho_0/\rho_j = 1/4$, then $t^* = 4\tau$: a 15 mns accident produces a perturbation that lasts 45 mn after the cars have been removed.

one has a plot in the (x,t) plane:

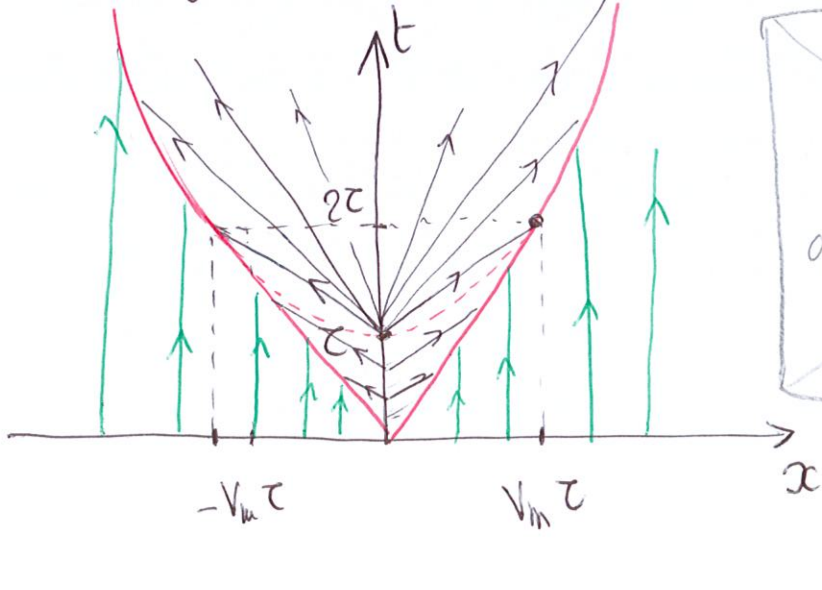


particular case: $\rho_0 = \rho_m = \rho_j/2$ — in this case

$$\begin{cases} V(\rho_0) = \frac{V_m}{2} = -U \\ t_r = 2\tau = t_\ell \\ x_r = -x_\ell = V_m\tau \\ B_r = V_m\sqrt{\tau} = -B_\ell \end{cases}$$

and one has the plot:

$$\begin{cases} x_{r/\ell}(t>2\tau) = \pm V_m\sqrt{\tau(t-\tau)} \\ x_{r/\ell}(t<2\tau) = \pm V_m\, t/2 \end{cases}$$



~~an other particular case: $\rho_0 = \rho_j \Rightarrow t^* = \tau$~~
~~of course: there nobody who notices that the light has turned red!~~