Notes on Fisher's equation[1]

A Physical discussion

Let $u(\vec{r},t)$ be the density of population of a given species. A simple model: u does not depend on $\vec{r}$ and obeys a continuous version of the logistic map $u_t = r\,u(1-u/K)$, where r is the growth rate and K the "carrying capacity", i.e., the maximum density of population that the environment can sustain (if $K=\infty$ the population grows exponentially).

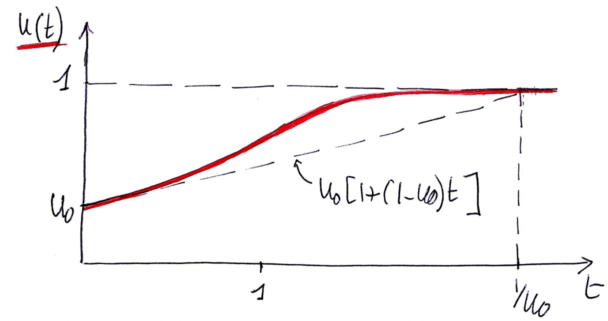
A simple way to introduce a position dependence consists in adding a diffusive term (the members of the population wander randomly): $u_t = D\,\vec{\nabla}^2 u + r\,u(1-u/K)$. Define $\tilde{t} = t\cdot r$, $\tilde{x} = x\sqrt{r/D}$ (idem for y and z), $\tilde{u} = u/K$, then rename $\tilde{t}=t$, $\tilde{x}=x$ and $\tilde{u}=u$. In 1D this leads to Fisher's equation

$$u_t - u_{xx} = u(1-u)\ . \tag{1}$$

For a uniform initial condition $u(x,0) = u_0$, where $0 < u_0 < 1$, the solution is simple: u remains x-independent and the time-integration of (1) yields

$$u(t) = \left[1 + \frac{1-u_0}{u_0}\exp(-t)\right]^{-1}\ .$$

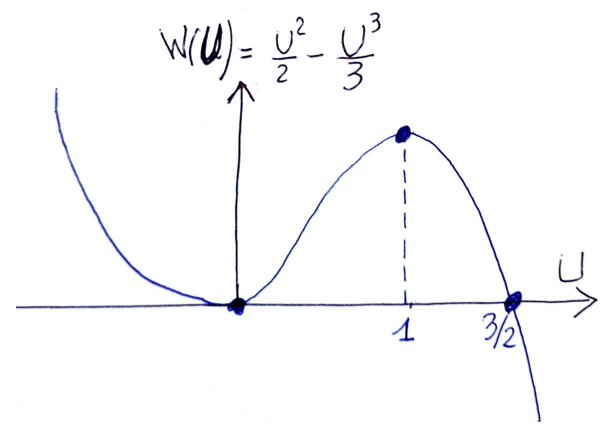
If $u_0 \ll 1$, this expression indeed corresponds to an initially exponential growth rate: $u(t) \simeq u_0\exp(t)$, can you see that ?



One then looks for more interesting solutions where $u(x,t) = U(z)$ with $z = x - c\,t$ with $c>0$ (these are called "traveling wave" solutions) and $U(z\to\pm\infty) = K^{\pm}$ (yet unknown constants). One gets ($U' = \mathrm{d}U/\mathrm{d}z$)

$$U'' = -c\,U' - U(1-U)\ . \tag{2}$$

This equation is not so easily integrated as the one encountered for instance when studying traveling wave solutions of KdV (lecture 4). Things are simpler if $c\equiv 0$. In this case, multiplying (2) by U', one can find a first integral: $\frac{1}{2}(U')^2 + W(U) = C^{\rm st}$, where $W(U) = U^2/2 - U^3/3$. This is analogous to the dynamics of a particle evolving in a 1D potential. One can find solutions to this equation, but for all of them, at some point, U becomes negative, this is unphysical for a population density.



When $c \neq 0$, Eq. (2) can be interpreted as the equation of motion of a classical particle in a potential well $W(U)$ in the presence of a dissipative force proportional to the velocity ($-c\,U'$). If we are lucky enough, we will be able to find a solution that starts at $U=1$ with "energy" $W(1)$, falls down the maximum, and, thanks to dissipation, ends up exactly at $U=0$ with zero velocity.

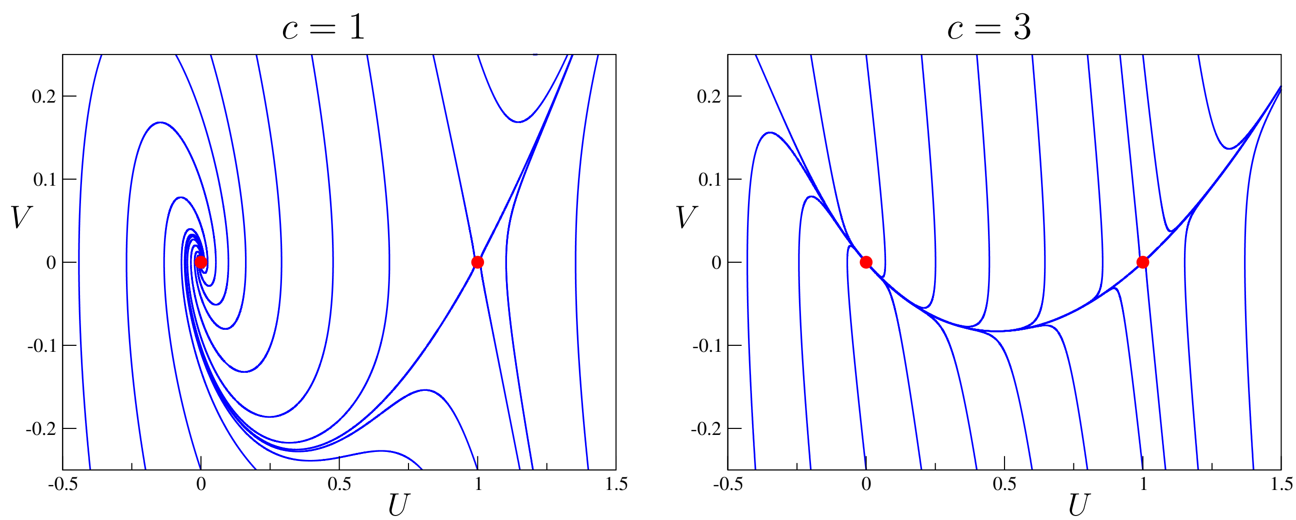
[1]Notes largely inspired by the book by J. D. Logan, "an introduction to nonlinear PDEs", chapter 5.

B Phase portrait of traveling wave solutions

In order to check if the above discussed lucky scenario is possible, let's denote $V = U'$ and study the dynamics in the (U, V) phase plane. From (2) we see that it is governed by the following equations:

$$U' = V \ , \quad V' = -cV - U(1-U) \ . \tag{3}$$

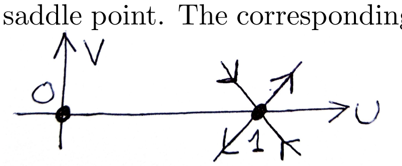
There are 2 fixed points: $(1, 0)$ and $(0, 0)$. The corresponding phase portrait is represented below in the cases $c = 1$ and $c = 3$.



The Jacobian matrix[2] around a fixed point (U, V) is

$$J(U, V) = \begin{pmatrix} 0 & 1 \\ 2U - 1 & -c \end{pmatrix} \ . \tag{4}$$

$J(1, 0)$ has eigenvalues $\lambda_\pm = \frac{1}{2}(-c \pm \sqrt{c^2 + 4})$, they are real and of opposite sign, therefore $(1, 0)$ is a saddle point. The corresponding stable and unstable directions are $\begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$. This looks like:



$J(0, 0)$ has eigenvalues $\frac{1}{2}(-c \pm \sqrt{c^2 - 4})$. If $c^2 < 4$ the eigenvalues are complex with negative real parts, and $(0, 0)$ is thus a stable spiral. In this case U would tend toward the asymptotic value 0 in an oscillatory way, which is impossible for a population density (always positive). If $c^2 > 4$ both eigenvalues are negative: $(0, 0)$ is a stable node; this is the only acceptable situation. In this case there exists a heteroclinic orbit leaving the saddle along the unstable manifold and reaching the node, such that $U \geq 0$ all along the way. The corresponding $U(z)$ goes from 1 (for $z \to -\infty$) to 0 (for $z \to +\infty$)[3]; it is called a "domain wall". It describes a population which has reached its carrying capacity in some region of space ($U = 1$ when $x \to -\infty$), and which spreads at constant velocity towards a region initially not occupied ($U = 0$ when $x \to +\infty$).

[2] If you need to refresh your memory on dynamical systems, I advise you read the 9 pages appendix of section 5 of Logan's book.

[3] This is the above mentioned "lucky scenario".

C Approximate domain wall solution

Let's now try to see if we can get an approximate expression $U(z)$ for the profile of the domain wall solution. One identifies a small parameter $\varepsilon = c^{-2} < 0.25$. Define $s = \sqrt{\varepsilon}\, z$ and $g(s) = U(z)$. One will denote $g' = \mathrm{d}g/\mathrm{d}s$. From Eq. (2) one sees that g is solution of

$$\varepsilon g'' + g' + g(1-g) = 0\ , \quad \text{with the boundary conditions} \lim_{s\to-\infty} g(s) = 1 \text{ and } \lim_{s\to+\infty} g(s) = 0\ . \tag{5}$$

One looks for a solution of the form $g(s) = g_0(s) + \varepsilon\, g_1(s) + \varepsilon^2\, g_2(s) + \cdots$ This gives at leading order $g_0' + g_0(1-g_0) = 0$, which is a separable first order differential equation. Its solution reads

$$-\int \mathrm{d}s = \int \frac{\mathrm{d}g_0}{g_0(1-g_0)} = \int \mathrm{d}g_0 \left(\frac{1}{g_0} + \frac{1}{1-g_0}\right) \rightsquigarrow g_0(s) = \frac{1}{1+e^s}\ . \tag{6}$$

In the above expression the integration constant has been fixed so that $g_0(0) = 1/2$ (arbitrary choice). Note that g_0 verifies the expected boundary conditions (5). The next terms in the expansion of g will thus have to fulfill $\lim_{(s\to\pm\infty)} g_n(s) = 0$.
At next order in ε one gets

$$g_1' + g_1(1-2g_0) = -g_0''\ . \tag{7}$$

This is an inhomogeneous linear ODE. The corresponding homogeneous equation is obtained by replacing the right-hand side of (7) by 0. It reads

$$\frac{\mathrm{d}g_1}{g_1} = (2g_0 - 1)\mathrm{d}s = \frac{1-e^s}{1+e^s}\,\mathrm{d}s = \left(1 - \frac{2\,e^s}{1+e^s}\right)\mathrm{d}s. \tag{8}$$

Its solution is $g_1(s) = Ke^s(1+e^s)^{-2}$, where K is an integration constant. One then looks for a solution of the full inhomogeneous equation (7) under the form[4] $g_1(s) = K(s)e^s(1+e^s)^{-2}$. Inserting this form into (7) yields, after a little algebra, $K'(s) = (1-e^s)(1+e^s)^{-1}$. The relevant integral has already been computed in (8). One gets

$$K(s) - K(0) = s - 2\ln(1+e^s) \quad \rightsquigarrow \quad K(s) = \ln\left[\frac{C^{\mathrm{st}}\cdot e^s}{(1+e^s)^2}\right].$$

One fixes the integration constant in such a way that $g_1(0) = 0$ (so that one still has $g(0) = 1/2$). This imposes $C^{\mathrm{st}} = 4$ and the final result reads

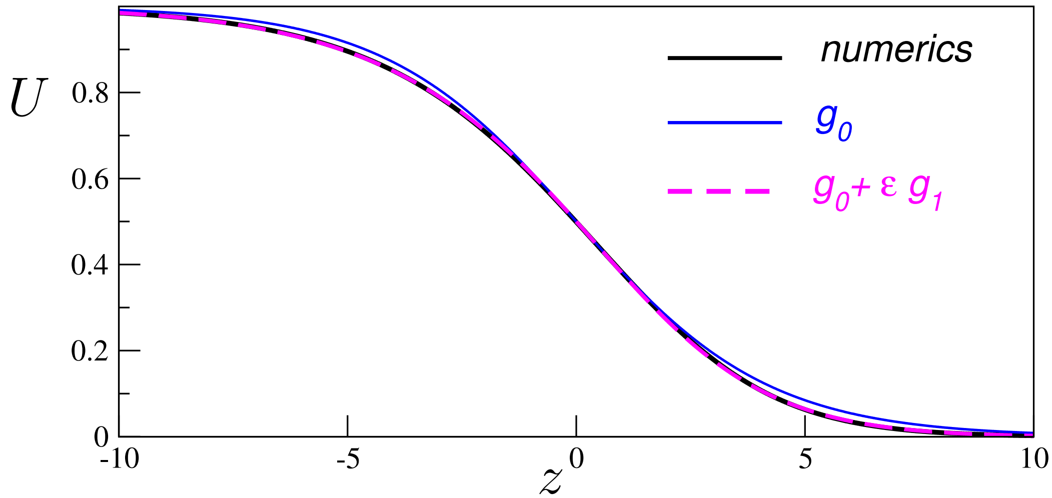
$$g_1(s) = \frac{e^s}{(1+e^s)^2}\ln\left(\frac{4\,e^s}{(1+e^s)^2}\right)\ . \tag{9}$$

$g_1(s\to\pm\infty) = 0$, so that $g_0 + \varepsilon g_1$ verifies the expected boundary conditions. Going back to the original function:

$$U(z) = \frac{1}{1+e^{z/c}} + \frac{1}{c^2}\frac{e^{z/c}}{(1+e^{z/c})^2}\ln\left(\frac{4\,e^{z/c}}{(1+e^{z/c})^2}\right) + \mathcal{O}\left(\frac{1}{c^4}\right)\ . \tag{10}$$

The accuracy of expansion (10) is illustrated by the figure below which is drawn for the case $c = 2.1$. For larger values of c the agreement is even better.

[4] This is called the method of "variation of the constant".



D Stability of the domain wall and of traveling waves

In this section we study the stability of the domain wall solution. To this end, let's write Fisher's equation (1) in a moving coordinate frame, by changing variables to $t = t$ and $z = x - c\,t$ (where $c > 2$). One gets

$$u_t - u_{zz} - c\,u_z = u(1-u)\ . \tag{11}$$

Let's seek for a solution of the form $u(z,t) = U(z) + w(z,t)$, where $|w| \ll U$. We further assume that

$$w(z,t) = 0 \quad \text{for} \;\; |z| \geq L\ , \quad \text{for some} \;\; L > 0\ . \tag{12}$$

This implies that the perturbation vanishes outside a bounded domain in the moving frame[5]. We now want to study the dynamics of the small perturbation w: will it eventually decay or blow up? This will decide if the solution $U(z)$ is stable or not. Using (2) one sees that the linearized version of (11) reads

$$w_t - w_{zz} - c\,w_z = (1 - 2\,U)w\ . \tag{13}$$

We look for a solution under the form $w(z,t) = \Phi(z)\exp(-\lambda\,t)$, with $\Phi(\pm L) = 0$. If all possible values of λ are positive, the traveling wave solution will be considered as stable. It is convenient to write $\Phi(z) = \varphi(z)\exp(-cz/2)$ in order to eliminate the first derivative term in (13). One obtains

$$-\varphi'' + \left[2\,U(z) + \frac{c^2}{4} - 1\right]\varphi = \lambda\,\varphi\ . \tag{14}$$

This has the form of a Schrödinger equation, since $U(z) > 0$ and $c > 2$ the effective potential is positive. Consequently, the eigenvalue problem with boundary conditions $\varphi(\pm L) = 0$ has only positive eigenvalues λ. This means that any perturbation of finite size support eventually decays: the domain wall solution is stable. The argument only relies on the fact that $U(z) > 0$ and $c > 2$: it applies for any traveling wave solution with large enough velocity.

[5] This is not a natural assumption, but it makes the discussion simpler.