ADVANCED NONLINEAR PHYSICS

Duration: 3 hours. Problems A and B correspond to the first part of the course, problem C to the second part. They are all independent of each other. Please use different sheets for writing up the solution of each problem.

Dictionaries, handwritten notes on the courses and tutorials are allowed. Books as well as computers, telephones and other electronic devices are forbidden.

A Flood rarefaction wave

A flood wave is described by the area $A(x,t)$ of a cut of a vertical slice of the flow (x being the longitudinal coordinate along the stream). In the presence of a constant and uniform water supply $M(x,t) = M$ ($M > 0$) to the river, a simplified dynamical model of flood is described –after re-scaling to dimensionaless units– by the equation

$$A_t + A\,A_x = M \ . \tag{A1}$$

1/ Describe the time evolution of an initially uniform flow: $A(x,0) = A_0$. Give the expression of the characteristics, and draw some of them in the (x,t) plane.

2/ One considers the Riemann initial condition

$$A(x,0) = \begin{cases} A_1 & \text{if } x < 0 \ , \\ A_2 & \text{if } x > 0 \ , \end{cases} \qquad \text{with} \quad 0 < A_1 < A_2 \ . \tag{A2}$$

Draw the corresponding characteristics in the (x,t) plane and give the expression of $A(x,t)$ for $t > 0$.

B Spread of a disease

We consider the spread (in one dimension) of a rabies epidemic in a population of foxes[1]. The population is divided into two classes, the healthy ones, called the "susceptibles" [with density $S(x,t)$] and the infectives [density $I(x,t)$]. The fraction of susceptibles getting infected increases at a rate γI (where $\gamma > 0$). The infectives never recover and die at a constant rate μ. Finally, we postulate that infectives diffuse, possibly because of disorientation, but that susceptibles do not. The model equations are therefore

$$S_t = -\gamma\, I\, S \ , \quad \text{and} \quad I_t = D\, I_{xx} + \gamma\, I\, S - \mu\, I \ , \tag{B1}$$

where D is the diffusion coefficient. Initially the population is healthy with uniform density: $I(x,0) = 0$ and $S(x,0) = S_0$.

1/ Show that by an appropriate re-scaling of I, S, x and t, one can re-write Eq. (B1) replacing μ by $b = \mu/(\gamma S_0)$ and taking otherwise $D = \gamma = 1$. In all the following we consider only the dimensionless version of (B1). We do not change notation for the dimensionless quantities (except for $\mu \to b$).

[1] rabies=rage=rabbia ; fox=renard=volpe.

2/ One looks for traveling wave solutions of the dimensionless version of Eq. (B1). In this case $I = I(z)$, $S = S(z)$, with $z = x - ct$, where $c > 0$ is the velocity of propagation of the disease. The boundary conditions for $S(z)$ and $I(z)$ are

$$S(+\infty) = 1\ ,\quad I(+\infty) = 0 \quad \text{and} \quad S'(-\infty) = I'(-\infty) = 0\ , \tag{B2}$$

where differentiation with respect to z is noted with a prime ($I' = \mathrm{d}I/\mathrm{d}z$ for instance). Eq. (B2) means that the population is healthy ahead of the front ($z \to +\infty$) and in a stationary uniform state behind the front ($z \to -\infty$).

(a) Show that the corresponding flow equations in the (S, I) phase space are

$$S' = S\,I/c \qquad I' = c\,(-I - S + b\,\ln S + 1)\ . \tag{B3}$$

(b) Identify two fixed points. One will denote by $S = \sigma$ the solution of $S - 1 = b\ln(S)$ different from unity. One will not try to express σ as a function of b.

(c) Show that both fixed points are physically admissible only if $0 < b < 1$, which will be assumed in all the following. Show that in this case one has[2] $0 < \sigma < b$.

(d) Characterize the two fixed points (nodes, saddles, spirals...).

(e) Show that one can find a solution corresponding to a front with boundary conditions (B2) provided c is larger than a threshold, the value of which you will give as a function of b. Sketch the corresponding trajectory in the (S, I) plane and also the profiles $S(z)$ and $I(z)$.

[2] Hint: a possible proof uses the remark that: $(\forall b \in \mathbb{R})\ b\ln b \geq b - 1$.