Flood rarefaction wave

$\mathcal{A}_t + \mathcal{A}\mathcal{A}_x = \mathcal{M}$ method of characteristic: $\varphi(t) \equiv \mathcal{A}(x(t),t)$

$$\frac{d\varphi}{dt} = \mathcal{A}_t + \frac{dx}{dt}\mathcal{A}_x$$

if one choose $\frac{dx}{dt} = \mathcal{A}(x(t),t) = \varphi(t)$ one sees that $\frac{d\varphi}{dt} = \mathcal{M}$

hence $\varphi(t) = \varphi(0) + \mathcal{M}t$ or $\mathcal{A}(x(t),t) = \mathcal{A}(x(0),0) + \mathcal{M}t$
denoting as $\mathcal{A}_0(x) = \mathcal{A}(x,t=0)$ this reads:

$$\mathcal{A}(x(t),t) = \mathcal{A}_0(x(0)) + \mathcal{M}t$$

For a given (x,t) one looks for a characteristic starting at $t=0$ from some $\bar{x}$ such that $x(t) = x$. The eq. of the characteristic is:

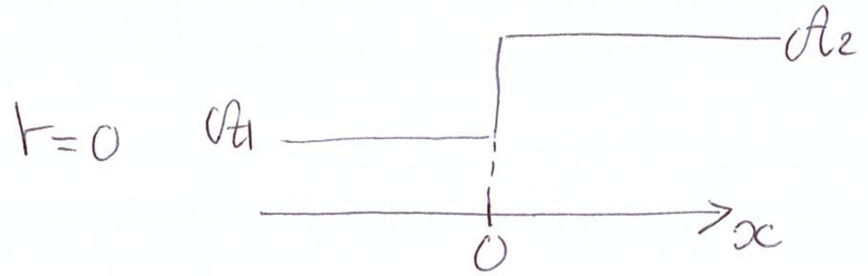
$$\frac{dx}{dt} = \mathcal{A} = \mathcal{M}t + \mathcal{A}_0(\bar{x})$$

thus $\boxed{x(t) = \mathcal{M}\frac{t^2}{2} + \mathcal{A}_0(\bar{x})t + \bar{x}}$

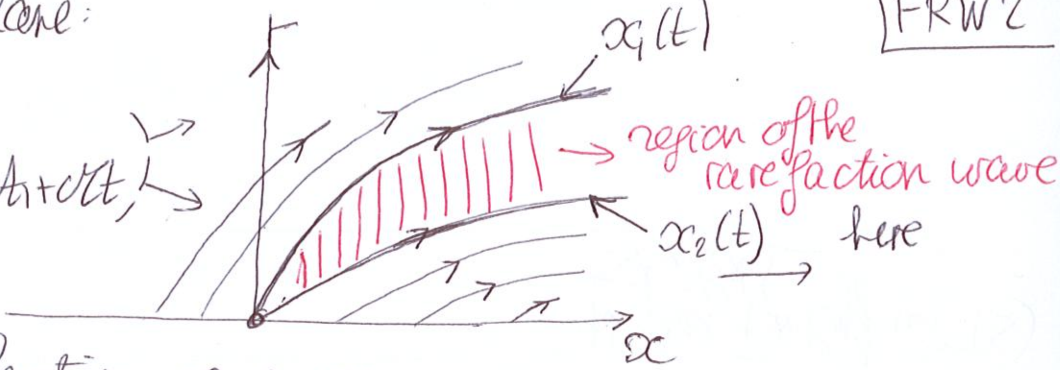
for a given (x,t) this eq. implicitly defines $\bar{x}$. Once $\bar{x}$ is known one just has: $\boxed{\mathcal{A}(x,t) = \mathcal{A}_0(\bar{x}) + \mathcal{M}t}$

▨ simple case of a uniform initial condition: $\mathcal{A}_0(x) = \mathcal{A}_0$ (c^{st})
then $\mathcal{A}(x,t) = \mathcal{A}_0 + \mathcal{M}t$ and the characteristics are parabola in the (x,t)-plane.

▨ case of a Riemann initial condition:



on has in the (x,t) plane:



at fixed t, the rarefaction wave occurs between $x_1(t)$ and $x_2(t)$ with

$$\begin{cases} x_1(t) = \mathcal{R}\frac{t^2}{2} + \mathcal{A}_1 t \\ x_2(t) = \mathcal{R}\frac{t^2}{2} + \mathcal{A}_2 t = x_1(t) + (\mathcal{A}_2-\mathcal{A}_1)t \end{cases}$$

the region of the rarefaction wave is filled by characteristics starting from $x=0$ with initial $\bar{\mathcal{A}}$ interpolating between $\mathcal{A}_1$ and $\mathcal{A}_2$.

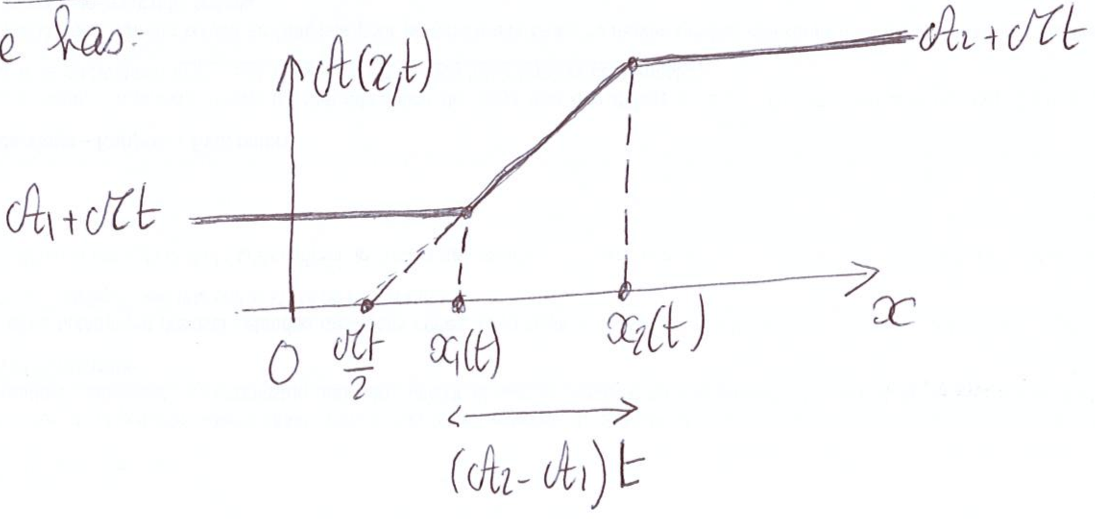
At given t an x in the region of the RW will be reached by a characteristic: $x = \mathcal{R}\frac{t^2}{2} + \bar{\mathcal{A}}t$ this determines the initial $\bar{\mathcal{A}}$ and, then for this x: $\mathcal{A}(x,t) = \bar{\mathcal{A}} + \mathcal{R}t$

this gives, for $x \in [x_1(t), x_2(t)]$:

$$\mathcal{A}(x,t) = \underbrace{\frac{x}{t} - \frac{\mathcal{R}t}{2}}_{\text{this is } \bar{\mathcal{A}}} + \mathcal{R}t = \frac{x}{t} + \mathcal{R}\frac{t}{2}$$

check: if $x = x_1(t)$ or $x_2(t)$ one can easily verify that this formula gives $\mathcal{A}_1+\mathcal{R}t$ or $\mathcal{A}_2+\mathcal{R}t$

so, one has:



Spread of a disease

① $S_t = -\gamma I S \qquad I_t = D I_{xx} + \gamma I S - \mu I$

scaling: $\tilde{S} = S/S_0 \quad \tilde{I} = I/S_0 \quad \tilde{t} = t/t_0 \quad \tilde{x} = x/x_0$

(t_0 and x_0 not yet determined)

$$\begin{cases} \dfrac{S_0}{t_0}\dfrac{\partial \tilde{S}}{\partial \tilde{t}} = -\gamma S_0^2 \tilde{I}\tilde{S} \\[2ex] \dfrac{S_0}{t_0}\dfrac{\partial \tilde{I}}{\partial \tilde{t}} = \dfrac{D S_0}{x_0^2}\dfrac{\partial^2 \tilde{I}}{\partial \tilde{x}^2} + \gamma S_0^2 \tilde{I}\tilde{S} - \mu S_0 \tilde{I} \end{cases}$$

if one chooses $\gamma t_0 S_0 = 1$ and $\dfrac{D t_0}{x_0^2} = 1$ this writes:

$$\frac{\partial \tilde{S}}{\partial \tilde{t}} = -\tilde{I}\tilde{S} \qquad \frac{\partial \tilde{I}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{I}}{\partial \tilde{x}^2} + \tilde{I}\tilde{S} - \mu t_0 \tilde{I}$$

one has $\mu t_0 = \dfrac{\mu}{\gamma S_0} \equiv b$ and removing henceforth all the "~"

one gets:

$$\boxed{S_t = -IS \qquad I_t = I_{xx} + IS - bI} \quad (1)$$

② $S = S(z)$ and $I = I(z)$ with $z = x - ct$ ($c>0$: the velocity spreads in the positive x-direction)

one gets:

$$\begin{cases} cS' = SI \\ -cI' = I'' + SI - bI \end{cases}$$

from the first equation one can rewrite $I = cS'/S$ in the r.h.s of the second equation.

this yields: $-cI' = I'' + cS' - bc\, S'/S$

this integrates to: $-cI = I' + cS - bc \ln S + C^{st}$

the value of this integration constant is fixed by the boundary condition at $+\infty$: $C^{st} = -c$

hence

$$\boxed{\begin{aligned} S' &= SI/c \\ I' &= c(-I - S + b\ln S + 1) \end{aligned}} \quad (2)$$

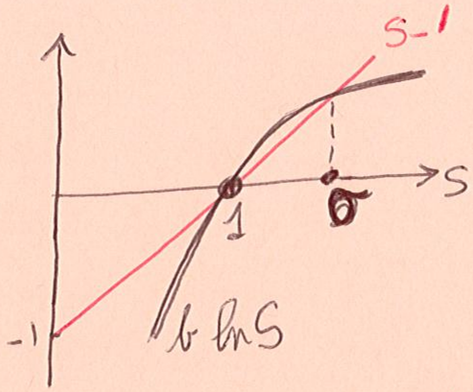
look for fixed points. In (2a) one could take $S=0$, but then in (2b) because $\ln S$ would $\to -\infty$, one would need to take $I \to \infty$ = impossible ($0 \leq I \leq 1$).

other possibility = $I = 0$, then from (2b) one should take =

$$\boxed{S - 1 = b \ln S}$$

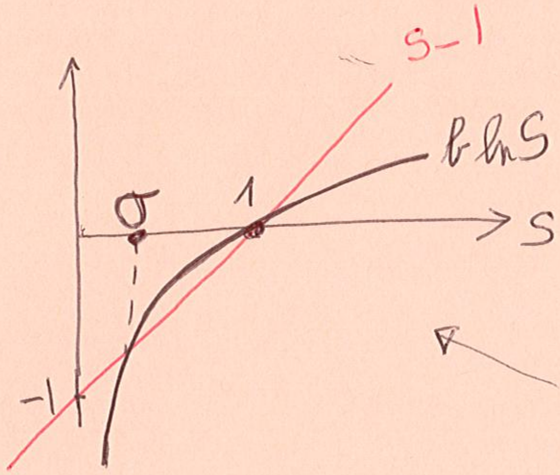
there are 2 cases:

$b > 1$



in this case $\sigma > 1$ = impossible $S \in [0, 1]$

$b < 1$



here $\sigma \in]0,1[$ = OK

in the case $b<1$ one can check that $b \in [\sigma, 1[$:

proof = $(\forall b \in \mathbb{R})$ $b \ln b \geq b - 1$ (easy to check)

and, when $b<1$ $b \ln S \geq S - 1$ only for $S \in [\sigma, 1]$ cf

thus, one must have $b \in [\sigma, 1]$ and $\sigma \in [0, b]$

study of the critical points = $(S,I) = (\sigma, 0)$ and $(1,0)$

the jacobian matrix is :

$$J(S,I) = \begin{pmatrix} \partial S'/\partial S & \partial S'/\partial I \\ \partial I'/\partial S & \partial I'/\partial I \end{pmatrix} = \begin{pmatrix} I/c & S/c \\ -c + \dfrac{bc}{S} & -c \end{pmatrix}$$

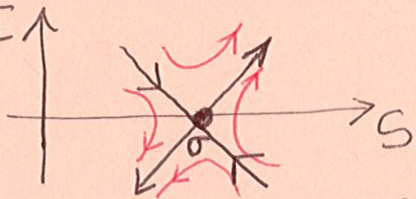
$S=\sigma, I=0$ eigenvalue of J solution of

$$\lambda^2 + \lambda c + \underbrace{\sigma - b}_{<0} = 0$$

(as seen in the previous page).

the discriminant = $c^2 + 4(b-\sigma)$ is thus >0. the product of the two roots $= (\sigma - b)$ is <0 = one has two real eigenvalues of opposite sign = this is a saddle.

It is easy to see that the contracting (expanding) manifold has a negative (positive) slope, and that locally =

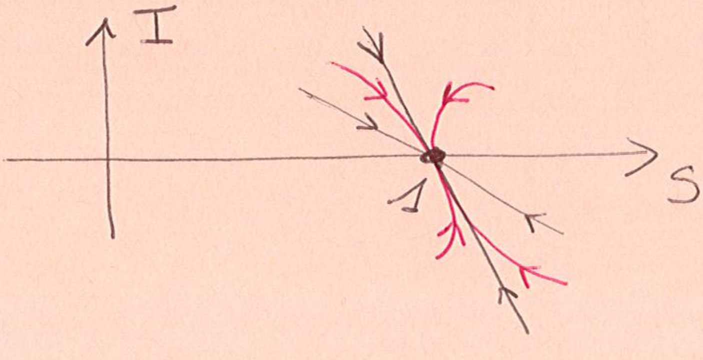


$S=1, I=0$ eigenvalues are solutions of:

$$\lambda^2 + c\lambda + \underbrace{1 - b}_{>0} = 0 \rightarrow \text{discriminant} = c^2 - 4(1-b)$$

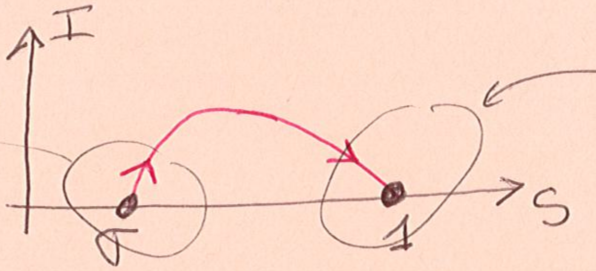
if $c \leq 2\sqrt{1-b}$ = complex eigenvalues with negative real part = stable spiral = should be rejected on physical grounds (I must remain >0)

if $c > 2\sqrt{1-b}$ both eigenvalues are real <0 = stable node



so, when $0<b<1$ and $c>2\sqrt{1-b}$ one can find an heteroclynic orbit which makes:

leaves along the unstable manifold



arrives along the manifold associated to the less negative eigenvalue

$$\lambda_+ = \frac{1}{2}\left(-c+\sqrt{c^2-4(1-b)}\right)$$

since here one has

$$\begin{pmatrix} 0 & \sigma/c \\ c\left(\frac{b}{\sigma}-1\right) & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix}$$

the associated slope is $c\lambda_+/\sigma$

where

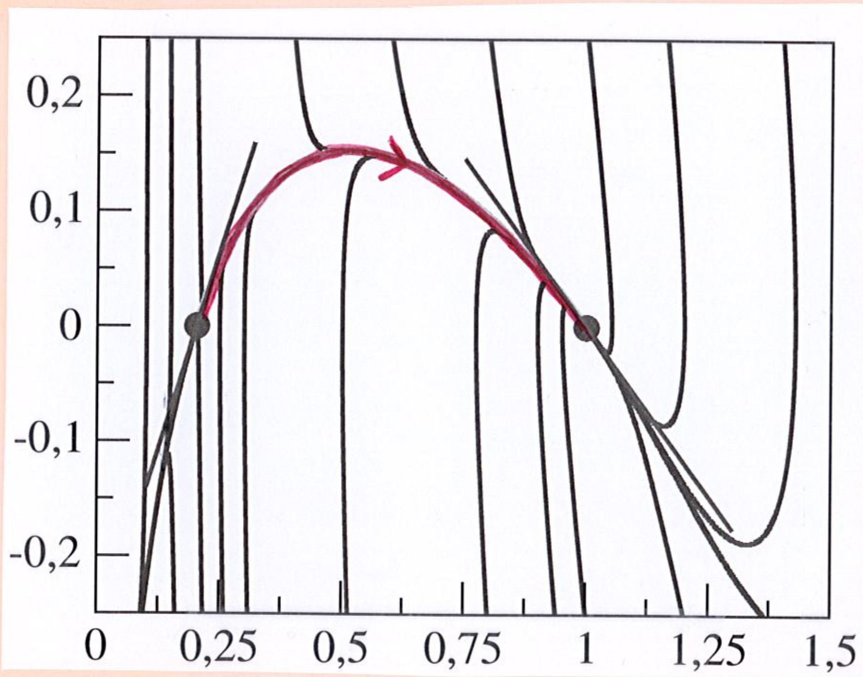
$$\lambda_+ = \frac{1}{2}\left(-c+\sqrt{c^2+4(b-c)}\right)$$

since here one has

$$\begin{pmatrix} 0 & 1/c \\ (b-1)c & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix}$$

the associated slope is $c\lambda_+$

plot for $b=0.5$
$c=2$
(leading to $\sigma \simeq 0.203$)



the heteroclynic orbit is in red.

and one has schematically: