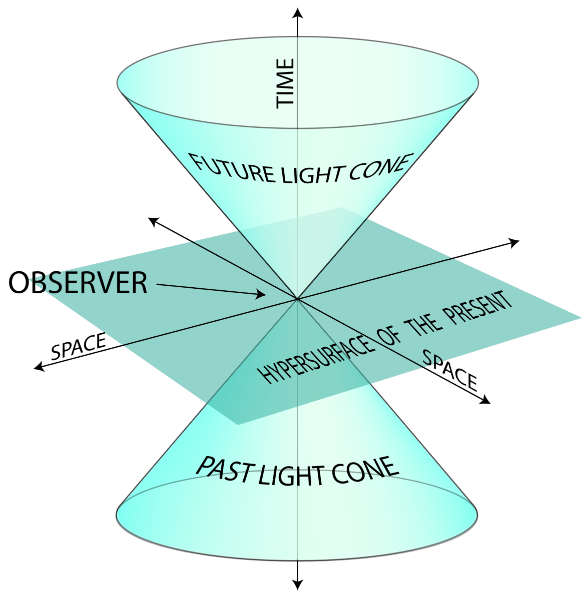
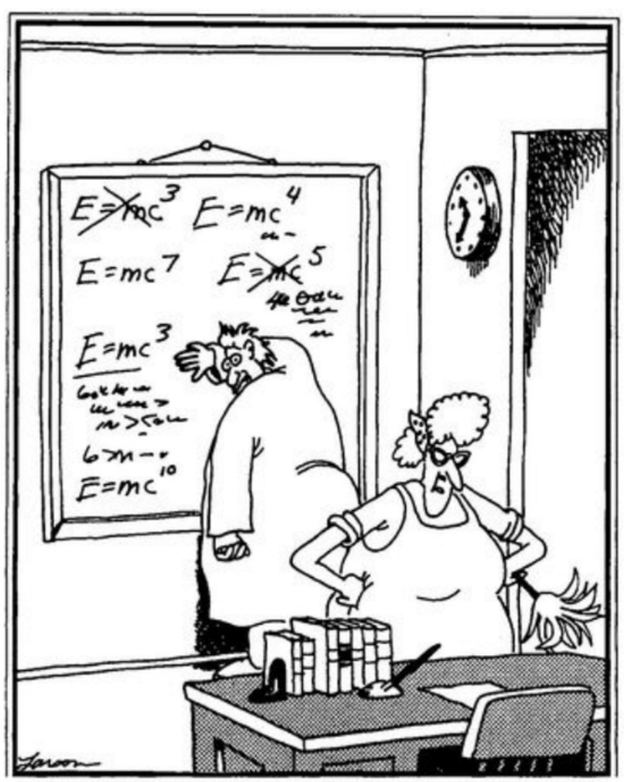
SPECIAL RELATIVITY – TUTORIALS





"*Now* that desk looks better. Everything's squared away, yessir, squaaaaaared away."

Tutorial # 1
Relativity principle and first consequences

1 Particle decay

The baryon Λ^0 decays in proton-pion pairs: $\Lambda^0 \to p\,\pi^-$. Given that its proper lifetime is 2.9×10^{-10} s, what is the mean distance covered in the lab frame by a Λ^0 of velocity $v = 0.994\,c$?

2 Length contraction

Let $\mathcal{R}$ and $\mathcal{R}'$ be 2 inertial frames in uniform motion with respect to each other. Let a ruler be at rest in $\mathcal{R}'$ and with its long axis aligned along the motion axis. Retrieve the length contraction law using two or even three different methods.

3 The ruler and the hole

In an inertial frame $\mathcal{R}$ of origin O, coordinates x, y and z and time t, a ruler R_1R_2 of length L_0 is moving at constant speed $\overrightarrow{V} = V\,\overrightarrow{e_x}$ where V is close to c (see Fig. 1). Similarly a plane with a circular hole of diameter L_0 is moving along the y-axis with a constant velocity $\overrightarrow{u} = u\,\overrightarrow{e_y}$ where $u \ll c$ so that the motion of the hole can be considered non-relativistic. The relative motion of the ruler and the hole is such that at $t = 0$ the midpoint of the ruler coincides with the origin O and with the centre of the hole.

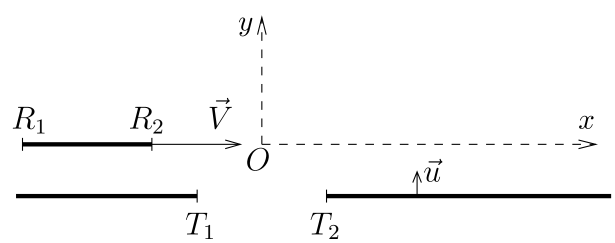


Figure 1: Set-up sketch. The ruler is limited by its endpoints R_1 and R_2 and T_1T_2 represents the diameter of the hole along the y-axis.

Because of its length contraction as seen in $\mathcal{R}$, the ruler is expected to easily pass through the hole. However as seen from the ruler's frame $\mathcal{R}'$, the hole size shrinks and possibly the ruler cannot fit in. Solve this paradox by considering the coordinates of the points $R_{1,2}$ and $T_{1,2}$ in the $\mathcal{R}$ and $\mathcal{R}'$ frames. *Tip: in $\mathcal{R}'$ study the trajectory of T_1 and T_2.*

4 Velocity composition

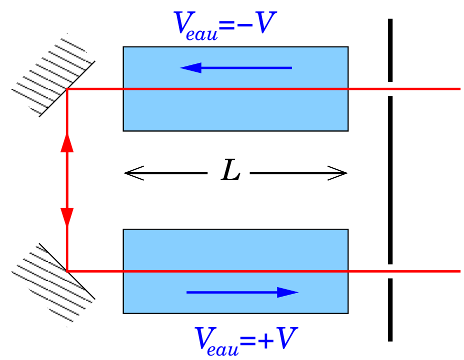
1/ Let $\mathcal{R}$ and $\mathcal{R}'$ be two inertial frames in relative uniform motion with $V\,\overrightarrow{e_x}$ the velocity of $\mathcal{R}'$ with respect to $\mathcal{R}$. Consider a material point M with velocity $\overrightarrow{v}$ in $\mathcal{R}$ and $\overrightarrow{v}'$ in $\mathcal{R}'$. Express then the coordinates of $\overrightarrow{v}$ as a function of those of $\overrightarrow{v}'$. This result will be derived in two different ways: first directly and then using the transformation of the 4-velocity.

2/ Let 1 and 2 be two particles with velocity $\vec{v}$ et $\vec{V}$ in the laboratory ($\mathcal{R}$). The relative velocity of 1 with respect to 2, $\overrightarrow{v_{rel}}$, is defined as the velocity of 1 in the frame where 2 is at rest[1]. Show that

$$v_{\rm rel}^2 = \frac{1}{\left(1 - \vec{v}\cdot\vec{V}/c^2\right)^2}\left\{\left(\vec{v}-\vec{V}\right)^2 - \frac{\left(\vec{v}\wedge\vec{V}\right)^2}{c^2}\right\}. \qquad (1)$$

5 Fizeau Experiment

We study an experiment performed by Hippolyte Fizeau in 1851 where the light from a source to the right is split in 2 rays (in red) which go through water tubes with opposite flows (see figure). Both rays then come out to the right and are superposed in order to interfere. The wavelength of the light is λ and the index of refraction of water is n.



We note v' the speed of light with respect to water and $v_\pm$ the speed of light measured in the laboratory frame $\mathcal{R}$. The index $\pm$ stands for the 2 possible flow velocities, $\pm V$.

(a) Show that $v_\pm = (c/n) \pm V(1 - 1/n^2)$ where the terms of order $(V/c)^2$ and higher have been neglected.
(b) Derive the phase difference $\Delta\phi$ of photons that have followed the 2 possible paths and compare to the results predicted by classical mechanics.
(c) In his experiment, Fizeau used a set-up where $L = 1.487$ m, $V = 7.059$ m/s, $\lambda = 0.526$ μm and $n = 1.333$. He then measured $(\Delta\phi/2\pi) = 0.23$: does this value allow to reject the classical approach ?

6 Hafele & Keating experiment

In 1971 Hafele & Keating [2] performed an experiment illustrating the "twins paradox". They synchronized several atomic clocks and flew them on airplanes. Then gathering the clocks, they compared the elapsed time measured by each of them in flight, i.e., the proper duration.
Numerical values: typical plane velocity with respect to ground $V = 900$ km/h, flight duration $T_f = 2\pi R/V = 45$ h, Earth radius $R = 6380$ km.

1/ Using the time dilation relationship, derive the time lag between a clock on the ground and a clock on a plane. How much lifetime does a pilot spare if he flies 1000 h per year during 30 years ? Same question for a student who performs 200 Paris-Orsay return trips per year.

2/ In this experiment, it is observed that clocks traveling to the east are lagged back and those traveling to the west are ahead of time with respect to a clock on the ground. Given that γ is independent of the velocity direction, how can you explain this result with special relativity?

[1] Beware that it is $\vec{v} - \vec{V}$ only in the non-relativistic case.
[2] J. C. Hafele et R. E. Keating, Science **177**, 166 (1972)

7 Happy Anniversary

The astronaut Alice (A) leaves her friend Bob (B) for a return trip to a stellar system located at a distance of 4 l.yrs from the Earth. For both trips we assume a constant velocity $v = 0.8\,c$ and neglect the time needed to U-turn[3].

1/ What is the duration of the first trip as measured by A ? And as measured by B?

2/ Each year, A celebrates her leave by sending a signal to B. For this latter what is the time between 2 subsequent signals sent during the first trip? Same question for the time between signals during the return trip. How many messages does B receive in total?

3/ If B also sends signals to A each year, how many messages will be received by A during the first trip? And during the return trip? How many messages will she receive in total?

8 Other exercices (in french)

Voici quelques exercices qui portent sur le même thème que cette feuille de travaux dirigés et que vous pouvez télécharger en vous reportant sur la page web de l'enseignement:

- ⋆ Premier partiel 2014/2015 : problème B (observation radio, corrigé).
- ⋆ Partiel 2015/2016 : problème A (contraction des longueurs, non corrigé).
- ⋆ Partiel 2018/2019 : problème B (Effet Sagnac, corrigé).
- ⋆ Partiel 2019/2020 : problème C (paradoxe du train et du tunnel, corrigé).
- ⋆ Examen 2019/2020 : problème A (encore un train et un tunnel, corrigé).
- ⋆ Partiel 2020/2021 : problème A (muons cosmiques, non corrigé) et problème C (Photographier n'est pas mesurer, corrigé).
- ⋆ Partiel 2021/2022 : problème A (le centre de gravité est-il un concept pertinent en relativité, corrigé) et problème B (train, tunnel et porte, corrigé).

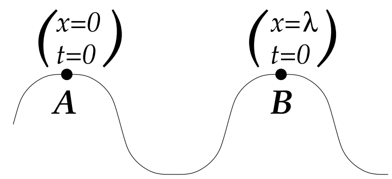
[3] C. G. Darwin [Nature **180**, 976 (1957)] used this example to answer to an opponent to the theory of special relativity.

Tutorial # 2
Optics and Particle Kinematics

1 Wave four-vector

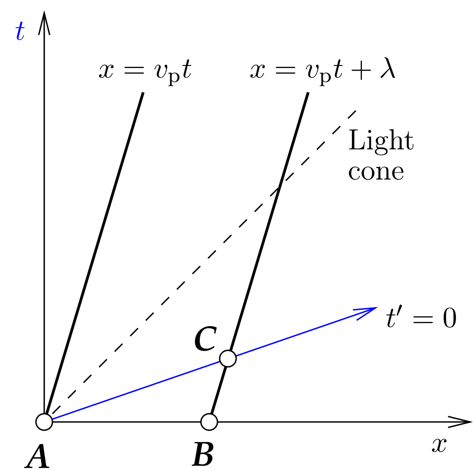
We aim at demonstrating that for a wave of angular frequency ω, wave vector $\vec{k}$ and phase velocity v_p, the quantity $(\omega/c, \vec{k}\,)$ is a "good" four-vector. For simplifying the demonstration we will work with a single space dimension.

In the lab frame $\mathcal{R}$, one defines the wavelength λ as the (usual) distance between two simultaneous events: two successive maxima of the wave (cf. figure). In $\mathcal{R}$, the wave maxima move at the phase velocity v_p, such that $v_\mathrm{p} = \omega/k$.



One considers a frame $\mathcal{R}'$ moving at velocity V with respect to $\mathcal{R}$ along axis Ox. One denotes as C the nearest maximum of A which is simultaneous to A in $\mathcal{R}'$. The corresponding space-time diagram is shown in the figure.

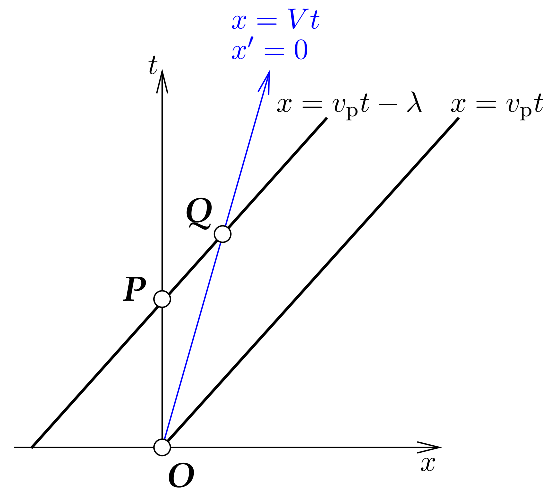
Determine the spatial coordinate of C in $\mathcal{R}'$, obtain the corresponding wave length λ'. Defining $k' = 2\pi/\lambda'$ and $k = 2\pi/\lambda$, show that k' can be expressed in terms of k and ω as expected from the spatial component of a four-vector $(\frac{\omega}{c}, k)$.



Check that the temporal component obeys also the correct transformation laws. For that matter, one will need to define the period as the time-interval between two events (which are they?) and to use a reasoning similar to the above.

indication : Here is a Minkowski diagram representing in $\mathcal{R}$ the world lines of two succesive maxima of the wave (the plot corresponds to the case $V < v_\mathrm{p}$).

It is clear that the temporal period in $\mathcal{R}$ is $T = t_\mathrm{P}$. What is T' in $\mathcal{R}'$?



2 Speed Measurement

A car, considered as a mirror, moves away from a policeman along a straight line, at constant velocity V. The policeman emits a light ray of angular frequency ω_{i} which, after reflexion on the car/mirror, return towards the policeman with an angular frequency ω_{r}. Express ω_{r} as a function of ω_{i}.

Indications : (1) Use the wave four-vector. (2) One will denote as ω'_{i} and ω'_{r} the angular frequencies in the frame $\mathcal{R}'$ attached to the mirror and one will justify by physical arguments that $\omega'_{\mathrm{i}} = \omega'_{\mathrm{r}}$.

U.S. Army soldier using a radar gun to catch speeding violators.

3 Angular distribution of photons emitted by a moving source

In its own frame $\mathcal{R}^*$, a light source emits photons isotropically. The number of photons dN emitted in the solid angle $d\Omega^*$ can be expressed as $dN/N_0 = d\Omega^*/(4\pi)$ where N_0 is the number of photons emitted in all directions. This light source moves at constant velocity $\vec{u}$ with respect to an inertial frame $\mathcal{R}$. In the frame $\mathcal{R}$, photons are seen to be emitted in a solid angle $d\Omega$ comprised between cones of half-angles θ and $\theta + d\theta$, where θ is the angle between the wave vector of the photon and $\vec{u}$.

Show that $dN/N_0 = f(\theta)\, d\Omega/(4\pi)$ where $f(\theta)$ represents the angular distribution of the emitted photons in $\mathcal{R}$ to be expressed as a function of β and θ. Represent $f(\theta)$ in a polar plot for $\beta = 1/2$. Show then that half of the photons are emitted in a cone with $\theta = 60°$.

4 Collision inélastique de deux protons

One studies a two protons collision ($m_p\, c^2 = 938.25$ MeV) resulting in a deuteron d^+ ($m_d\, c^2 = 1875.56$ MeV) and a π^+ meson ($m_\pi\, c^2 = 139.6$ MeV):

$$p^+ + p^+ \to d^+ + \pi^+$$

Determine the threshold energy of this reaction in the center of mass reference frame $\mathcal{R}^*$. Compute, at threshold, the energy of the incident proton in the lab frame $\mathcal{R}$ in which the traget proton is at rest. Give the analytical formula and the numerical value.

5 Head-on collision

One aims at studying the elastic head-on collision of an incident particle of mass M and velocity $\beta = v/c$ with a particle at rest of mass m.

1/ Write the initial energy-momentum four-vector of each particle in the lab frame.

2/ Write down the Lorentz transform between the lab frame and the frame $\mathcal{R}^*$ in which the particle of mass M is at rest. Derive then the initial four-momenta of each particle in $\mathcal{R}^*$.

3/ One considers the special case $M \gg m$.

(a) By studying the kinematics of a head-on collision in $\mathcal{R}^*$, determine without computation the *maximal* final momentum of the light particle.

(b) Going back to the lab frame, show that the final maximal kinetic energy transferred to the light particle in the collision is

$$K = 2\,m\,c^2\beta^2\gamma^2\ . \tag{1}$$

6 Elastic collisions

A particle of mass m and kinetic energy K moves along the x-direction (of unit vector $\overrightarrow{e_x}$) and collides with another particle of same mass at rest. After the collision the particles have different energies and velocity vectors, namely $\overrightarrow{v_1}' \cdot \overrightarrow{e_x} \neq \overrightarrow{v_2}' \cdot \overrightarrow{e_x}$. We note α the angle between $\overrightarrow{v_1}'$ and $\overrightarrow{v_2}'$.

1/ Show that in newtonian mechanics $\alpha = \pi/2$.

2/ Show that in special relativity α is a narrow angle (hint: express $\cos\alpha$ as a function of γ_1' and γ_2'). Discuss the newtonian and relativistic limits. In the case of a symmetrical collision ($\overrightarrow{v_1}' \cdot \overrightarrow{e_x} = \overrightarrow{v_2}' \cdot \overrightarrow{e_x}$), show that

$$\cos\alpha = \frac{K}{K + 4\,mc^2}\ .$$

7 Kinematics of a decay in 2 bodies

1/ A particle of mass M and momentum 4-vector $\underset{\sim}{P} = (\mathcal{E}/c, \vec{p}\,)$ disintegrates in 2 particles of mass m_1, m_2 and 4-momentum $\underset{\sim}{P}_1$, $\underset{\sim}{P}_2$ (we assume that $\vec{p}$ is along the z-axis).

(a) In the center of mass frame for the disintegration, express the energies of the particles $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ as well as their common momentum p^*.

(b) In the Lorentz transformation from the laboratory to the center of mass frame, show that β_{CM} is equal to $p_z c/\mathcal{E}$.

(c) Express $\mathcal{E}_1$ – the energy of particle 1 in the lab frame – as a function of β_{CM}, $\mathcal{E}_1^*$, p^* and of the angle θ^* between the z-axis and the momentum of particle 1 in the centre of mass frame.

(d) In the frame of the centre of mass, the decay is assumed to be isotropic so that: $\mathrm{d}N/\mathrm{d}(\cos\theta^*) = A$ (A a constant). From (c) deduce the distribution $\mathrm{d}N/\mathrm{d}\mathcal{E}_1$ in the lab frame.

2/ A particle π^+ decays in flight as : $\pi^+ \to \nu_\mu\,\mu^+$ ($m_\pi = 140$ MeV/c^2, $m_\mu = 106$ MeV/c^2 and $m_\nu = 0$ eV/c^2).

(a) Taking the kinetic energy K of the π^+ to be 140 MeV in the lab frame, what must be the value of the β_{CM} factor in the center of mass frame ?

(b) The z-axis is chosen to be along the flight line of the π in the lab frame. Let θ (resp. θ^*) be the angle between the μ line and the z-axis in the lab frame (resp. in the center of mass frame). Show that

$$\tan\theta = \frac{1}{\gamma_{\text{CM}}}\,\frac{\sin\theta^*}{\cos\theta^* + \beta_{\text{CM}}\,\dfrac{m_\pi^2 + m_\mu^2}{m_\pi^2 - m_\mu^2}}\ .$$

From this result show that the μ^+ is emitted in the lab frame in a cone of half-angle $\theta_{\max} \simeq 9°$.

8 Other exercices (in french)

Voici quelques exercices qui portent sur les même thématiques que celle traitées dans cette feuille de TD et que vous pouvez télécharger en vous reportant sur la page web de l'enseignement:

- ⋆ Partiel 2015/2016 : problème B (collisions, corrigé).
- ⋆ Partiel 2016/2017 : problème A (optique relativiste, corrigé).
- ⋆ Examen 2016/2017 : problème A (collisions, corrigé).
- ⋆ Partiel 2017/2018 : problème A (optique relativiste, corrigé) et problème B (collisions, corrigé).
- ⋆ Examen 2016/2017 : problème II (Compton inverse, corrigé).
- ⋆ Partiel 2018/2019 : Exercice A (collision de deux protons, non corrigé).
- ⋆ Examen 2018/2019 : Exercice A (collision photon – proton, corrigé).
- ⋆ Partiel 2019/2020 : problème A (optique: élargissement d'une raie, non corrigé) et exercice B (désintégration d'un méson, non corrigé).
- ⋆ Partiel 2020/2021 : problème B (désintégration d'un Kaon, corrigé).
- ⋆ Examen 2020/2021 : détail de la cinématique de l'effet Compton (corrigé).
- ⋆ Partiel 2021/2022 : problème C (perte de masse, corrigé).

Tutorial # 3
Relativistic dynamics

1 Mass = rest energy

1/ Two piece of clay, both of mass m, have a head-on collision at a speed $3c/5$. They form a unique clay ball after the collision (see picture below). What is the mass M of this particle?

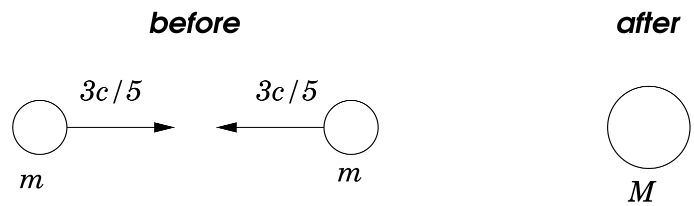


Figure 1: *Collision-fusion of two particles of same mass.*

2/ An example of bound state is the hydrogen atom. It consists of a positively charged proton and a negatively charged electron, the two being bound together by the electric force. The rest masses of the two particles are respectively: $m_p = 938.3$ MeV$/c^2$; $m_e = 0.5$ MeV$/c^2$. The corresponding binding energy is $\Delta\mathcal{E} = -13.5$ eV. Evaluate the rest energy Mc^2 of the hydrogen atom and the relative mass gain $(m_e + m_p - M)/(m_e + m_p)$.

3/ Mass defect: a particle α is the nucleus of ^{4}He atom, made of two neutrons and two protons. The different masses[1] are: Helium nucleus mass: 4.0026 u, proton mass: 1.0073 u, neutron mass: 1.0087 u. Calculate the binding energy of the α particle.

Same question for the $^{238}_{92}$U nucleus of mass 238.0022 u. What is the total electron binding energy of this element knowing that the $^{238}_{92}$U atomic mass is 238.0508 u (we will take $m_e = 5.49 \times 10^{-4}$ u)?

2 Motion with constant acceleration in its proper frame

0/ One considers a material point of velocity $v(t)$ in an inertial frame $\mathcal{R}$ with a single space coordinate. One defines the four-acceleration $\underset{\sim}{A} = \mathrm{d}\underset{\sim}{U}/\mathrm{d}t_0$, where t_0 and $\underset{\sim}{U}$ are the proper time and the four-velocity, respectively.

(a) Proove that it $\mathcal{R}$ one has:

$$\underset{\sim}{A} = \gamma\left(\frac{\mathrm{d}(\gamma c)}{\mathrm{d}t}, \frac{\mathrm{d}(\gamma v)}{\mathrm{d}t}\right) = \gamma^4\,\frac{\mathrm{d}v}{\mathrm{d}t}\,\left(\frac{v}{c}, 1\right) \quad \text{where} \quad \gamma(t) = (1 - v^2/c^2)^{-1/2}\,.$$

(b) Define the proper reference frame $\mathcal{R}_0$ of the material point. Show that in $\mathcal{R}_0$ the four-acceleration reads $\underset{\sim}{A}_0 = (0, a_0)$ where the proper acceleration a_0 relates to the coordinates in $\mathcal{R}$ through $a_0 = \gamma^3\,\mathrm{d}v/\mathrm{d}t$.

[1] The atomic mass unit "u" is 1/12 of the ^{12}C atomic mass. 1 u $= (1/N_A)$ g $= 1.6605 \times 10^{-27}$ kg $= 931.49$ MeV$/c^2$.

1/ A rocket leaves the Earth with a constant proper acceleration a_0. We will suppose that at $t = 0$ in the Earth frame $\mathcal{R}$, assumed inertial, the rocket is at the origin of the coordinates with a speed equal to zero.

(a) Give the expression in $\mathcal{R}$ of the velocity $v(t)$ and of the position $x(t)$ of the space ship.
(b) Express the time t in the Earth frame $\mathcal{R}$ as a function of the proper time t_0 of the space ship (the spacetime origins are the same in $\mathcal{R}$ and $\mathcal{R}_0$)[2].
(c) Give the proper time t_0 as a function of the distance x covered by the rocket and measured in the Earth frame $\mathcal{R}$.
(d) Taking as time unit the year (yr) and the light-year as unit of distance (lyr), calculate the proper time t_0 and the corresponding time t for an observer on Earth, for the cosmonaut to reach by keeping the proper acceleration $a_0 = 9.52$ m.s^{-2} constant (close to the standard acceleration of free fall), the nearest star from the Sun, Proxima Centauri, located at 4.25 $lyrs$. Same question to reach the centre of our galaxy located at 30000 $lyrs$, the Andromeda galaxy located at $2.2.10^6$ $lyrs$ and at last to reach the limits of the observable universe estimated at 15.10^9 $lyrs$

3 Minkowski force

A relativistic generalization of the fundamental principle of the dynamics is given by the Minkowski equation:

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = \mathcal{F}^\mu \ , \tag{1}$$

where $\mathcal{F}^\mu$ is called the four-force. In the following, we are going to study what are the properties of this four-force.

1/ By studying the spatial component of (1) justify that we can write $\vec{\mathcal{F}} = \gamma \vec{F}$ where $\vec{F}$ is the analogous of the Newtonian force and $\gamma = (1 - v^2/c^2)^{-1/2}$.

2/ Show that the dot product of $\underset{\sim}{P}$ and $\underset{\sim}{\mathcal{F}}$ is zero. Deduce from that the relation (How to call it?)

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\gamma c^2) = \vec{F} \cdot \vec{v} \ . \tag{2}$$

3/ We suppose that the force is conservative, associated to a potential $V(\vec{r})$. Write the conservation of the total energy. Then show that the dynamics of this system corresponds to the Lagrangian

$$L(\vec{r}, \vec{v}) = -mc^2\sqrt{1 - \vec{v}^{\,2}/c^2} - V(\vec{r}) \ . \tag{3}$$

Explain why the concept of force and potential seems to be difficult to define correctly in relativistic mechanics.

4/ From the orthogonality relation $\mathcal{F}^\mu U_\mu = 0$ demonstrated in question 2/, the four-force can't be independent from the four-velocity. The simplest dependence corresponds to a linear relation between the two quantities, of the type $\mathcal{F}^\mu = F^{\mu\nu} U_\nu$ where $F^{\mu\nu}(\vec{r}, t)$ is a "tensor field".

(a) Show that the orthogonality relation is verified if $F^{\mu\nu}$ is antisymmetric under the exchange of its indexes. This will be assumed in the following.

[2] One has

$$\int_0^X \frac{\mathrm{d}x}{\sqrt{1 + (\alpha x)^2}} = \frac{1}{\alpha} \operatorname{argsh}(\alpha X) \ ,$$

where argsh is the reciprocal function of $\sinh(x) = \frac{1}{2}[\exp(x) - \exp(-x)]$.

(b) We decide to parametrize $F^{\mu\nu}$ as [3]:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} . \tag{4}$$

Write the corresponding Minkowski equations. Don't you feel the desire to add a multiplicative parameter to $F^{\mu\nu}$?

(c) Justify without calculation that for $\mathcal{F}^\mu$ to be a regular four-vector, $F^{\mu\nu}$ must have specific properties under Lorentz transformations. Verify that under a change of frame $F^{\mu\nu}$ is transformed like:

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha \, \Lambda^\nu{}_\beta \, F^{\alpha\beta} . \tag{5}$$

4 Charged particle motion in a uniform electric field

Let $\mathcal{R}$ a Galilean frame where a uniform electric field $\vec{E_S}$, independent of the time, is directed along the Ox axis. At $t = 0$ a charged particle is sent from the origin O of the frame with an initial velocity $\vec{v_0}$ parallel to the Oy axis.

Study the motion of this charged particle in the frame $\mathcal{R}$ and give the equation of its trajectory. Study the the non-relativistic limit.

5 Proton synchrotron

A synchrotron is a particle accelerator in which particles follow a fixed circular trajectory. This is achived by using a set of electromagnets which create a uniform magnetic field $\vec{B}$, perpendicular to the plane of the trajectory and which vary slowly with time so that the radius R stays constant while the particle's energy increases. These protons are accelerated in an accelerating cavity where the applied electric field varies with the frequency f, multiple of the rotation frequency ν of the particles. We will write $k = K/mc^2$ the reduced kinetics energy of the particles (protons), with $mc^2 = 0.938$ GeV for a proton.

(a) For a given value of k, detremine the values of the magnetic field B and of the frequency ν that hold the protons on a circular trajectory of given radius R.

(b) We suppose that at each turn, the proton's energy increases by a constant value ΔE_0. By assimilating the variation of the energy to a continuous variation, establish the law of variation of the magnetic field with the time. Deduce from this the laws for k and ν in function of the time.

(c) N.A.: At the Super Proton Synchrotron (SPS) at CERN in Geneva, the protons are injected with an initial momentum $p_\mathrm{i} = 26$ GeV/c, the radius of the synchrotron is $R = 1100$ m, $\Delta E_0 = 50$ keV and the final magnetic field $B_\mathrm{f} = 1.21$ T. Calculate the final proton energy, their final momentum p_f, the duration of the increase of the energy, the number of turns done, the distance travelled by the protons during this acceleration stage and the extreme frequencies.

[3] We take the convention: the first index is the row index and the second is the column index. Take care that the indexes vary from 0 to 3: so the column of index 1 is the second column.

6 Other exercices (in french)

Voici quelques exercices qui portent sur les même thématiques que celle traitées dans cette feuille de TD et que vous pouvez télécharger en vous reportant sur la page web de l'enseignement:

- ⋆ Deuxième examen 2014/2015: problème B (particule dans champ électromag, corrigé).
- ⋆ Examen 2015/2016: problème A (fusée relativiste, corrigé).
- ⋆ Examen 2016/2017: problème B (paquets d'électrons, non corrigé).
- ⋆ Examen 2019/2020: problème C (faisceau laser, corrigé).
- ⋆ Examen 2021/2022: problème B (voilier cosmique, corrigé).

Tutorial # 4
Relativistic electrodynamics

1 The Coulomb gauge

We consider an electromagnetic field corresponding to the potentials $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$. Determine the scalar field $G(\vec{r},t)$ such that the gauge transformated potentials $\phi^* = \phi - \partial_t G$, and $\vec{A}^* = \vec{A} + \vec{\nabla} G$ satisfie the Coulomb gauge condition: $\vec{\nabla}\cdot\vec{A}^* = 0$.

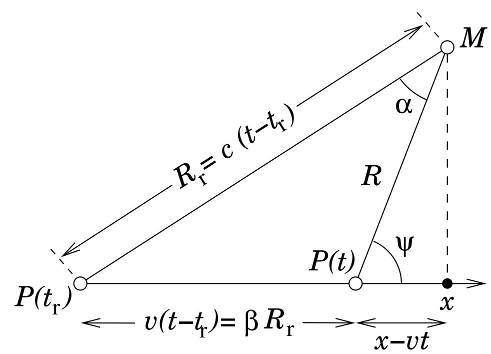
2 Point charged particle in uniform rectilinear motion

Let a particle P of charge q move along Ox at uniform velocity $\vec{v} = v\,\vec{e}_x$ in an inertial frame $\mathcal{R}$. P is at the origin at $t = 0$. We denote as $\mathcal{R}_0$ the proper frame associated to this particle. Let M a point of coordinates (x, y, z) in $\mathcal{R}$.

1/ Calculate in $\mathcal{R}_0$ the four-potential at a point M. Deduce from this the expressions of $\vec{E}_0$ and $\vec{B}_0$. Express in $\mathcal{R}$ the four-potential as a function of (x, y, z, t).

2/ We henmceforth work in the frame $\mathcal{R}$. Show that the scalar potential can be written as follows (the notations are defined on the figure):

$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{R\,(1-\beta^2\sin^2\psi)^{1/2}}\,. \qquad (1)$$



3/ The "retarded time" t_r is defined as follows: a photon emitted by P at time t_r arrives in M at time t. We note $\vec{R}_r = \overrightarrow{P(t_r)M}$. Show that $\sin\alpha = \beta\sin\psi$. Deduct from this that:

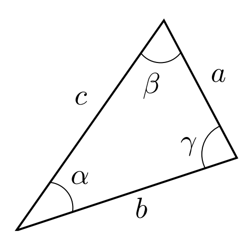
$$\phi(\vec{r},t) = \frac{q/4\pi\epsilon_0}{R_r\left(1-\hat{R}_r\cdot\vec{\beta}\right)}\,, \quad \text{where} \quad \begin{cases} \hat{R}_r = \vec{R}_r/R_r\,, \\ \vec{\beta} = \frac{v}{c}\,\vec{e}_x\,. \end{cases}$$

4/ Express $\vec{E}$ as a function of x, y, z, t. Study the variations of E_x and of $E_{y(z)}$ as a function of $\xi = x - vt$, in particular for $\beta \simeq 0$ and for $\beta \to 1$. Comment on. Draw the field lines. Express $\vec{B}$. Show that $\vec{B} = \vec{\beta}\wedge\vec{E}/c = \hat{R}_r\wedge\vec{E}/c$.

Note: To answer to question 3/ you may use the geometric relation

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}\,,$$

which is valid for the triangle in the figure to the right.



3 An electromagnetic paradox

One considers a charged particle at rest in a frame $\mathcal{R}$ in the presence of a uniform and constant magnetic field $\vec{B} = B_x\,\vec{e}_x + B_z\,\vec{e}_z$ (this problem is general enough and there is no need to consider a field with also a component along $\vec{e}_y$).

1/ What is the force experienced by the particle ?

2/ If one now works in a frame $\mathcal{R}'$ moving at the velocity $\vec{V} = V\,\vec{e}_x$ with respect to $\mathcal{R}$, the particle is no more at rest. It should then undergo a force... Solve this paradox.

4 Force between two electrons moving side by side.

In an inertial frame $\mathcal{R}$, two electrons M_1 and M_2 are moving side by side at a constant speed v on two parallel straight lines distant by d. Let $\mathcal{R}'$ be the frame in which they are both at rest.

1/ Calculate in $\mathcal{R}'$ the fields $\vec{E}\,'$ and $\vec{B}\,'$ created by M_1 at the point M_2. Deduce the force exerted on M_2.

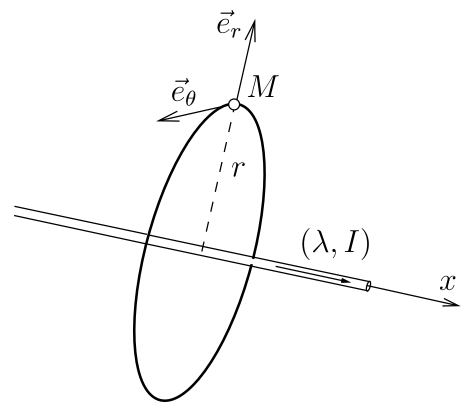
2/ Answer to the same questions in $\mathcal{R}$. Calculate the fields $\vec{E}$ and $\vec{B}$ created by M_1 at point M_2 by two different methods. Verify the covariant nature of the Lorentz force.

5 Current in a wire

• Show that a infinitely long and thin conducting wire –assimilated to the axis Ox– travelled by a current I and carrying a linear charge λ, creates in a point M located at a distance r of the wire, electric and magnetic fields which have the following expressions:

$$\vec{E}(M) = \frac{1}{2\pi\varepsilon_0}\,\frac{\lambda}{r}\,\vec{e}_r\ , \quad \text{et} \quad \vec{B}(M) = \frac{\mu_0}{2\pi}\,\frac{I}{r}\,\vec{e}_\theta\ . \qquad (2)$$

In these expressions λ and I are algebraic quantities, $\vec{e}_r$ and $\vec{e}_\theta$ are defined on the nearby figure and are independent of the sign of λ or I.



Let's consider a rectilinear wire of axis Ox. The wire contains positive stationary charges (ions) corresponding to an electric charge density $\rho_{\rm i}$ and a current density $\vec{J}_{\rm i} = \vec{0}$. There are also electrons (charge $\rho_{\rm e}$) moving at a velocity $\vec{u} = u\,\vec{e}_x$ and creating a current density $\vec{J}_{\rm e} = \rho_{\rm e}\,\vec{u}$. The wire being neutral on average, the whole charge density is $\rho = \rho_{\rm i} + \rho_{\rm e} = 0$.

1/ The **total** four-current will be noted $\underset{\sim}{J}$. Write its components in the frame $\mathcal{R}$ in wich the wire is at rest.

2/ We consider a frame $\mathcal{R}'$ moving at a constant velocity $V\,\vec{e}_x$ with respect to $\mathcal{R}$. Determine the total charge and current densities in $\mathcal{R}'$.

3/ A charged test q is stationary in $\mathcal{R}$ at a distance r from the wire. What is the force felt by this particle ?

4/ We will try to derive this result working in the frame $\mathcal{R}'$.

(a) By using the results to question **1**/, determine in $\mathcal{R}'$ the fields $\vec{E}\,'$ and $\vec{B}\,'$ created by the wire. *Indication: Let S be the cross section of the wire. It is appropriate to introduce the linear charge of the wire in $\mathcal{R}'$ ($\lambda' = \rho'\, S$), the current $I' = J'_x\, S$ and to use* (2).

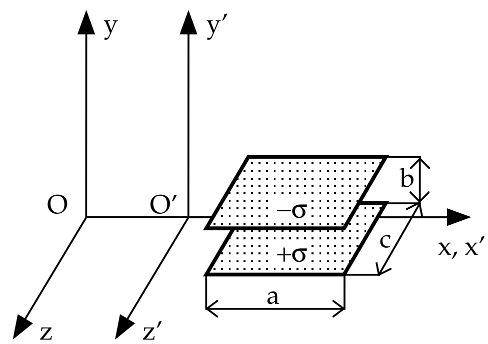
(b) Determine, always in $\mathcal{R}'$, the force induced on the particle (which is moving at a constant velocity in $\mathcal{R}'$).

(c) Find the same result by using the concept of four-force (see exercise 3 of tutorial # 3). We recall that the four-force is given by: $\mathcal{F}^\mu = \gamma\,(\vec{F}_{\text{Lorentz}} \cdot \vec{v}/c\,, \vec{F}_{\text{Lorentz}})$ where $\vec{F}_{\text{Lorentz}} = q\,(\vec{E} + \vec{v} \wedge \vec{B}\,)$.

6 Moving capacitor

Let $\mathcal{R}$ and $\mathcal{R}'$ be two inertial frames . An event has coordinates $X^\mu = (ct, x, y, z)$ in $\mathcal{R}$ and $X'^\mu = (ct', x', y', z')$ in $\mathcal{R}'$. The origin $(0,0,0,0)$ corresponds to the same event in the two frames. $\mathcal{R}'$ is moving with respect to $\mathcal{R}$ with a constant velocity $\vec{v}$ parallel to Ox axis (Lorentz boost).

A parallel plate capacitor ($b \ll a, c$) is stationary in $\mathcal{R}$. Determine the field $\vec{E}\,'$, $\vec{B}\,'$ in the frame $\mathcal{R}'$ by using two different methods:



1. by transformation of the field after having determined the field in $\mathcal{R}$.
2. by a direct calculation using Gauss and Ampère theorems in the frame $\mathcal{R}'$. You will need to determine with care the charge and current densities in this frame.

7 Other exercices (in french)

Voici quelques exercices qui portent sur les même thématiques que celle traitées dans cette feuille de TD et que vous pouvez télécharger en vous reportant sur la page web de l'enseignement:

- ⋆ Examen 2015/2016 : exercice B (Force entre deux faisceaux, non corrigé).
- ⋆ Examen 2017/2018 : problème III (courant dans un fil, corrigé).
- ⋆ Examen 2018/2019 : exercice C (transformation d'une onde plane, corrigé).
- ⋆ Examen 2019/2020 : exercice D (sélecteur de vitesse, corrigé).
- ⋆ Examen 2020/2021 : Plan chargé en mouvement (corrigé).
- ⋆ Examen 2021/2022: problème A (Invariants du champ électromagnétique, corrigé).