Topological constraints on vortex formation in a bi-dimensional quantum fluid

Nicolas Pavloff, LPTMS, Université Paris-Saclay

Newcastle, Sept. 2023

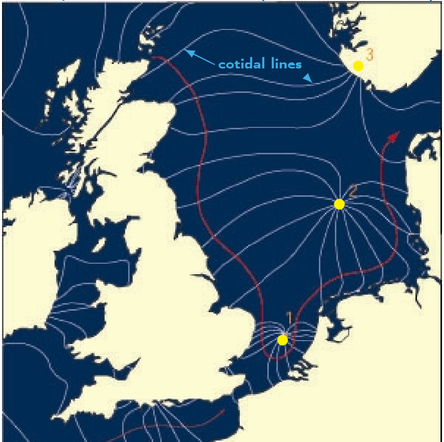
T. Congy
Northumbria Univ.

P. Azam
Nice Phys. Inst.

R. Kaiser
Nice Phys. Inst.

W. Whewell ⟶
1794–1866

Principal tidal constituent (M2: semi-diurnal)



Topological constraints on vortex formation in a bi-dimensional quantum fluid

Nicolas Pavloff, LPTMS, Université Paris-Saclay

Newcastle, Sept. 2023

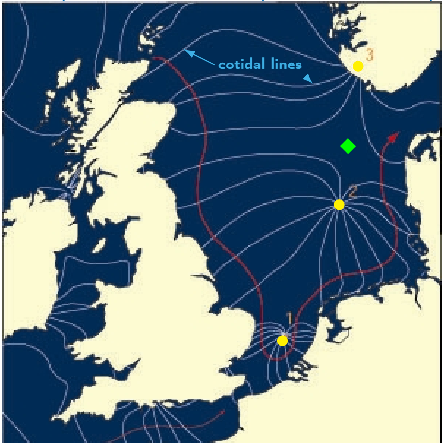
T. Congy
Northumbria Univ.

P. Azam
Nice Phys. Inst.

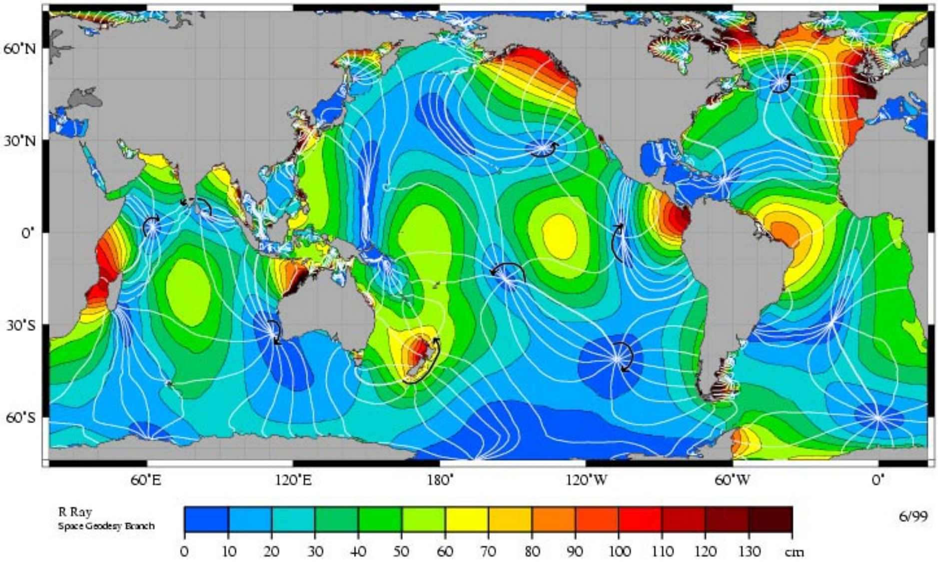
R. Kaiser
Nice Phys. Inst.

W. Whewell ⟶
1794–1866

Principal tidal constituent (M2: semi-diurnal)



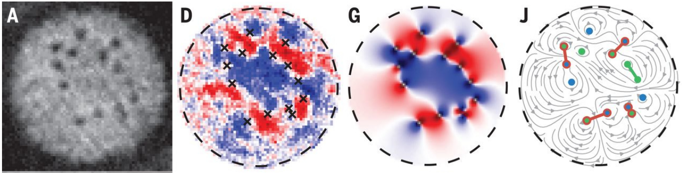
Amphidromic M2 points



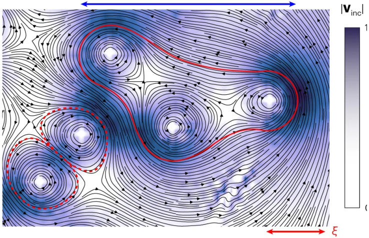
$T_{M2} = 12$ h 25 min, $\Delta t_{cotidal} = 1$ h 2 min, $\omega_{M2} \times \Delta t_{cotidal} = \pi/6$

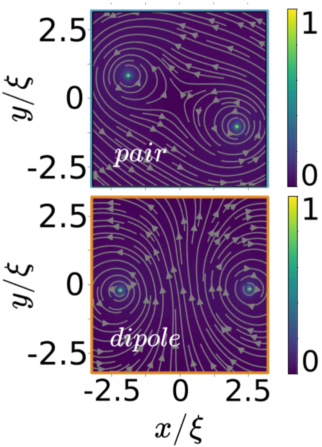
$\Longrightarrow$ quantized circulation of tidal current of the M2 component

Melbourne group 2D ^{87}Rb (Science 2019)

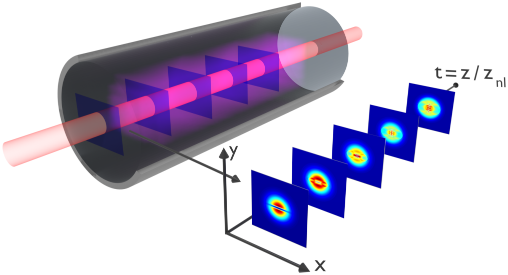


Lecce group 2D polaritons (Nat. Phot. 2023)

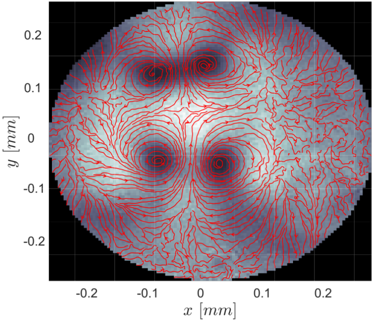


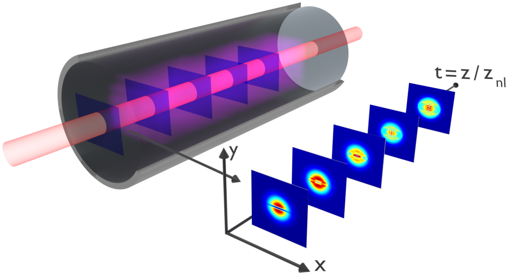


LKB group nonlinear light (arXiv 2023) ⟶

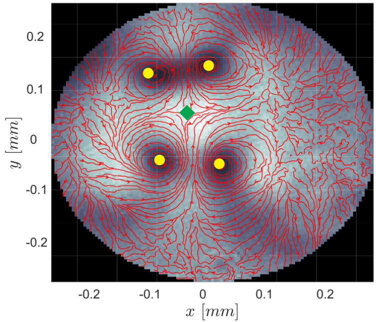


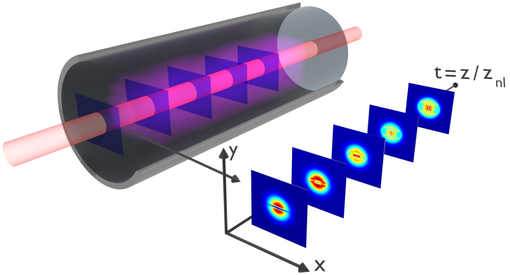
output intensity and streamlines



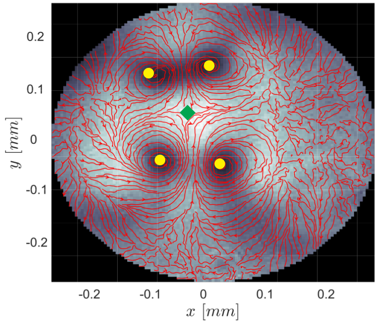


output intensity and streamlines





output intensity and streamlines

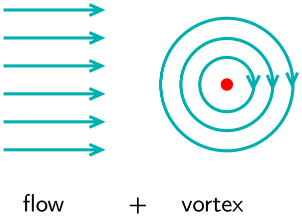


$\vec{E} = \psi(x, y, z) \exp\{i(k_0 z - \omega_0 t)\}\vec{e}_x$ linearly polarized carrier wave

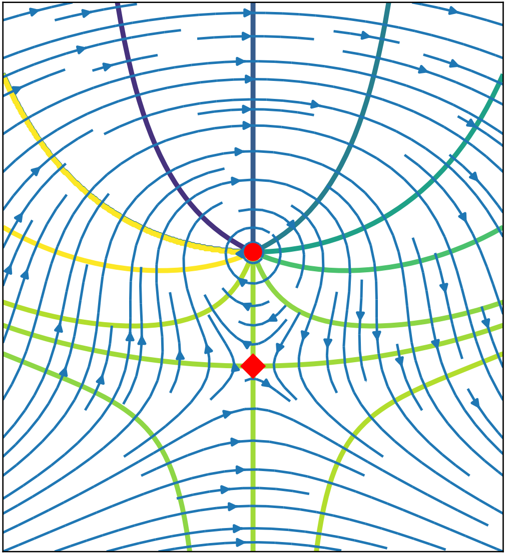
Paraxial approximation:

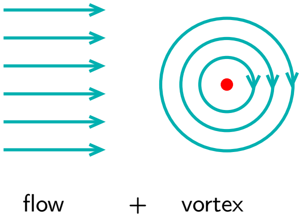
$$i\,\partial_z \psi = -\frac{1}{2 n_0 k_0}(\partial_x^2 + \partial_y^2)\psi + k_0 n_2 |\psi|^2 \psi - \frac{i}{2\,\Lambda_{\rm abs}}\,\psi,$$

$|\psi|^2$ in W.mm^{-2}. n_2 nonlinear Kerr coefficient. $\Lambda_{\rm abs} = -z_{\rm max}/\ln(\mathcal{T})$, $z_{\rm max} = 7$ cm is the total length of propagation through the nonlinear medium and $\mathcal{T} \simeq 0.2$ denotes the coefficient of energy transmission.

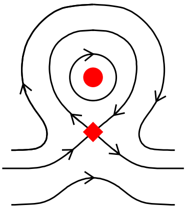


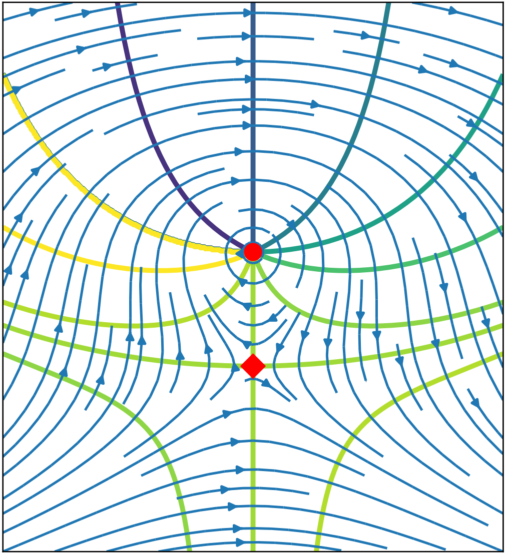
$$\psi = e^{\mathrm{i}kx}\,(x - \mathrm{i}y)$$

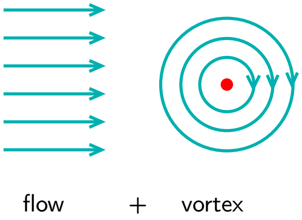




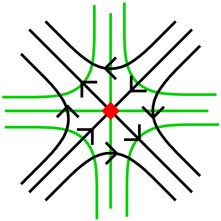
$$\psi = e^{ikx}\,(x - iy)$$

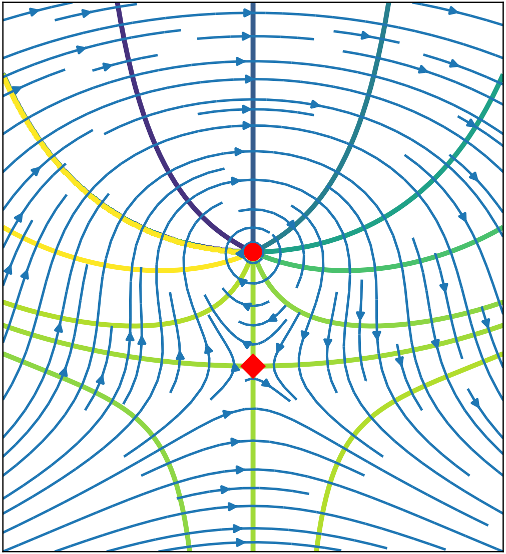






$$\psi = e^{ikx}\,(x - iy)$$





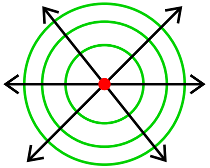
In a quantum fluid

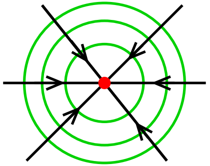
$$\Psi(\vec{r}) = A(\vec{r}) \exp\{iS(\vec{r})\}$$

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}} \boldsymbol{S}$$

phase S: velocity potential

min or max of S:





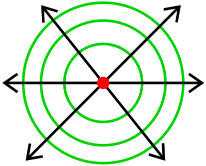
In a quantum fluid

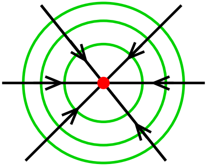
$$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$$

$$\vec{v} = \vec{\nabla} S$$

phase S: velocity potential

min or max of S:





Nodes are forbidden:

- in a 2D incompressible fluid:

$$\vec{\nabla}\cdot\vec{v} = 0 \Longrightarrow \vec{\nabla}^2 S = 0$$

Hence not seen, e.g., in a Coulomb gas model

- in a stationnary configuration:

No sink nor source !

$$\vec{\nabla}^2\psi + [f(A) + U(\vec{r}) - \mu]\psi = 0 \qquad (1)$$

where $\psi = A\exp(iS)$:

$$\vec{\nabla}^2\psi = \left\{\vec{\nabla}^2 A - A|\vec{\nabla}S|^2 + i(A\vec{\nabla}^2 S + 2\vec{\nabla}A\cdot\vec{\nabla}S)\right\} e^{iS}$$

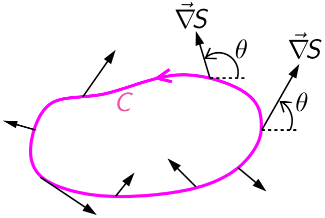
at a point where $\vec{\nabla}S = 0$,

the imaginary part of (1) yields $\vec{\nabla}^2 S = 0$

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$

$\vec{v} = \vec{\nabla}S$



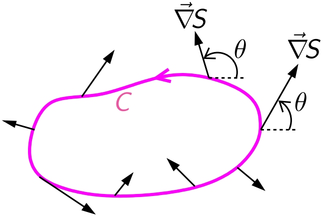
$$I_{\mathrm{V}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_{\mathrm{P}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r}) \exp\{iS(\vec{r})\}$

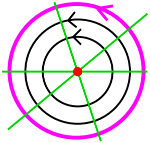
$\vec{v} = \vec{\nabla} S$



$$I_{\mathrm{V}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_{\mathrm{P}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

vortex



$I_{\mathrm{V}} \in \mathbb{Z}$

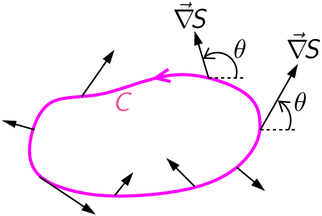
$I_{\mathrm{P}} = +1$

Two topological indices

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$

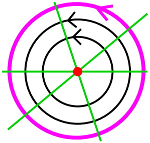
$\vec{v} = \vec{\nabla} S$



$$I_{\mathrm{V}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla} S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_{\mathrm{P}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla} \theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

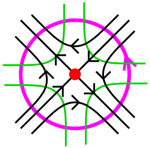
vortex



$I_{\mathrm{V}} \in \mathbb{Z}$

$I_{\mathrm{P}} = +1$

saddle



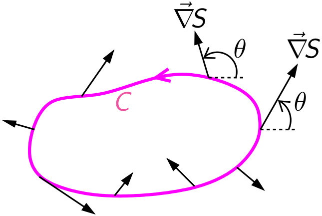
$I_{\mathrm{V}} = 0$

$I_{\mathrm{P}} = -1$

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$

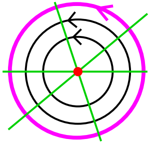
$\boldsymbol{\vec{v} = \vec{\nabla} S}$



$$I_{\mathrm{V}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_{\mathrm{P}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

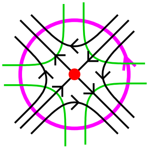
vortex



$I_{\mathrm{V}} \in \mathbb{Z}$

$I_{\mathrm{P}} = +1$

saddle

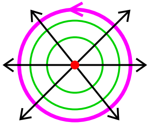


$I_{\mathrm{V}} = 0$

$I_{\mathrm{P}} = -1$

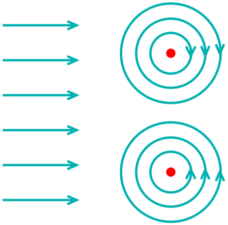
node

min/max of the phase

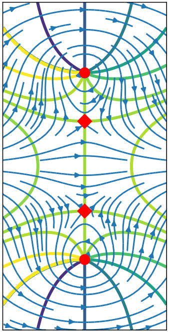


$I_{\mathrm{V}} = 0$

$I_{\mathrm{P}} = +1$



flow + 2 vortices

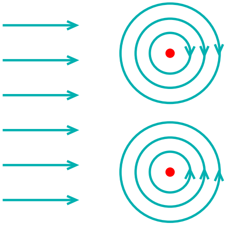


solution of $(\Delta + k^2)\psi = 0$

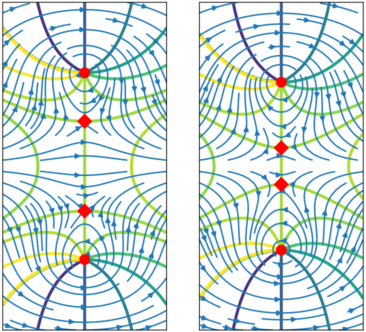
$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

$k^2 b =$ 1.3

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



flow + 2 vortices



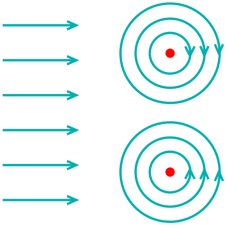
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx}\left[x - ik(y^2 - b)\right]$$

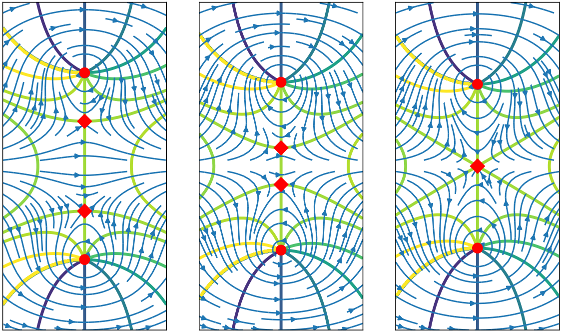
$k^2 b =$

1.3	1.05

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



flow + 2 vortices



solution of $(\Delta + k^2)\psi = 0$

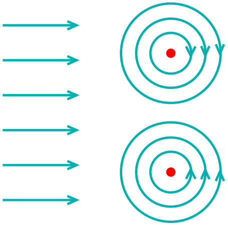
$$\psi = e^{ikx}\,[x - ik(y^2 - b)]$$

$k^2 b =$

1.3	1.05	1

Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



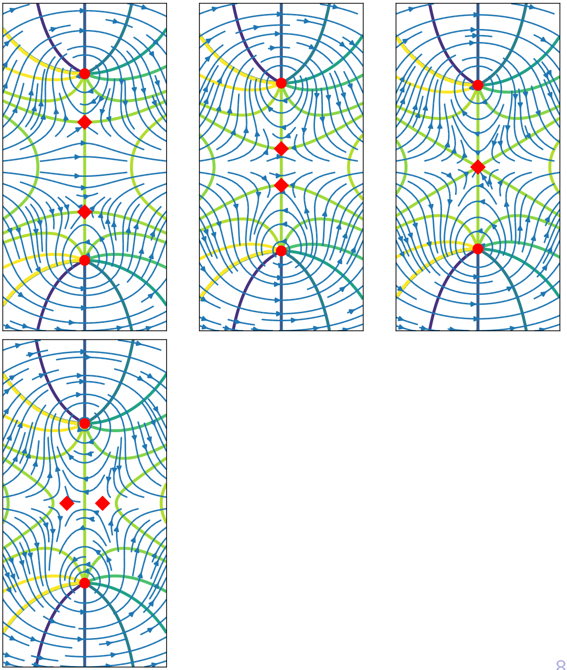
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx}\,[x - ik(y^2 - b)]$$

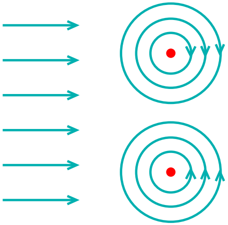
$k^2 b =$

1.3	1.05	1
0.95		



Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



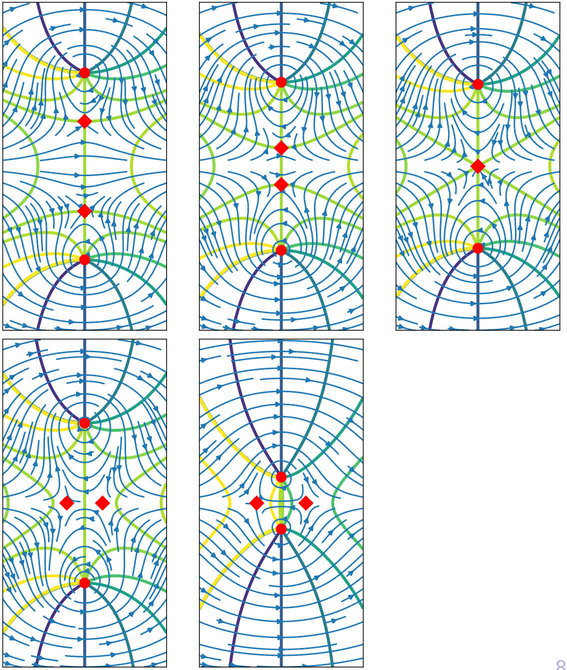
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx}\,[x - ik(y^2 - b)]$$

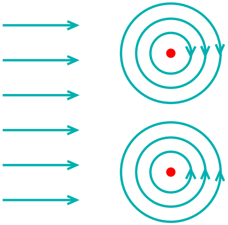
$k^2 b =$

1.3	1.05	1
0.95	0.1	



Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



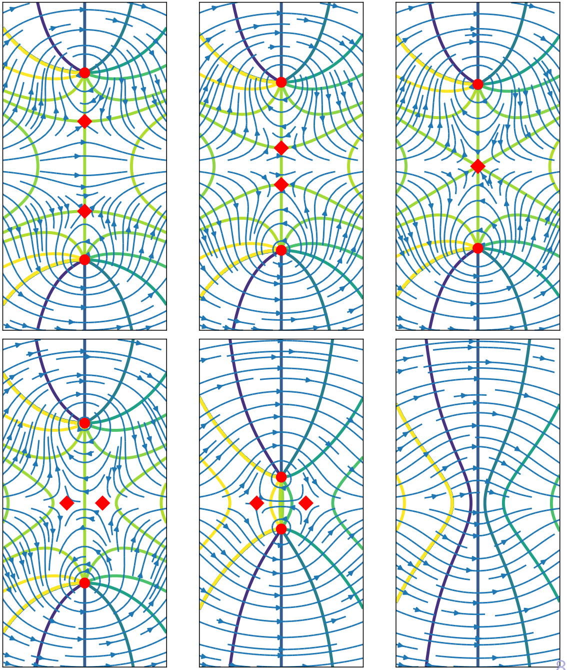
flow + 2 vortices

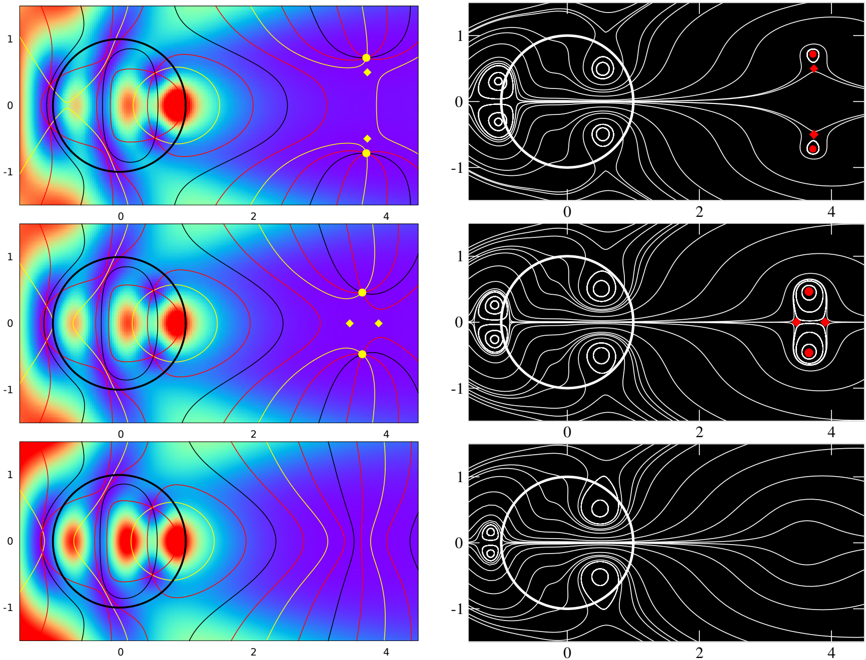
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx}\left[x - ik(y^2 - b)\right]$$

$k^2 b =$

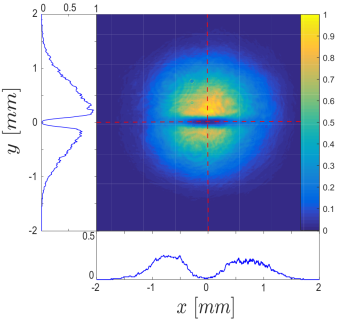
1.3	1.05	1
0.95	0.1	-0.1



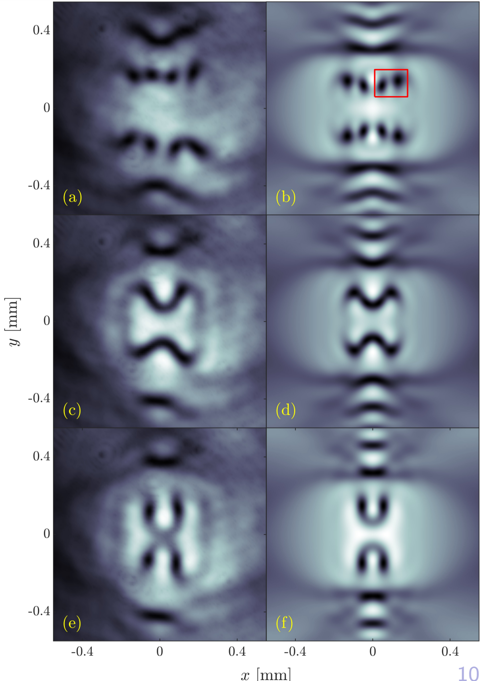

1
0
-1
0
2
4
1
0
-1
0
2
4
1
0
-1
0
2
4
1
0
-1
0
2
4
1
0
-1
0
2
4
1
0
-1
0
2
4

$$\psi(\vec{r},0) = \sqrt{I_1}\exp\left(-\frac{r^2}{w_{\mathrm{G}}^2}\right) + \sqrt{I_2}\exp\left(-\frac{x^2}{w_x^2} - \frac{y^2}{w_y^2}\right) e^{i\Phi_2}$$

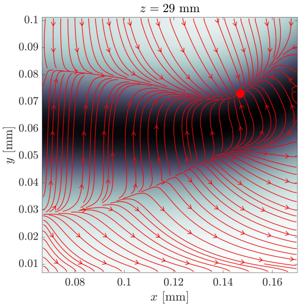
$I_1 = I_2$ $\Phi_2 \simeq \pi$

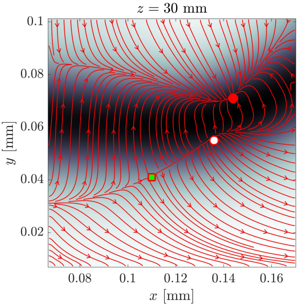


(a,b) : $\Phi_2 = 0.96\,\pi$
(c,d) : $\Phi_2 = \pi$
(e,f) : $\Phi_2 = 1.05\,\pi$



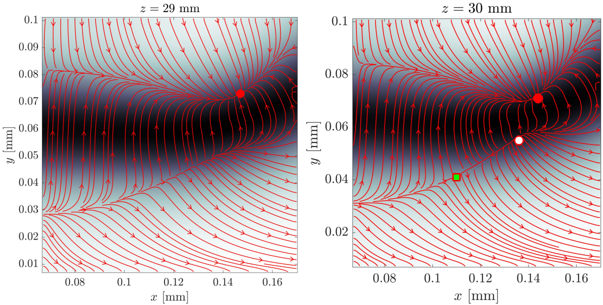
Formation of saddles and nodes: saddle-node bifurcation





$\Phi_2 = 0.96\,\pi$

Formation of saddles and nodes: saddle-node bifurcation

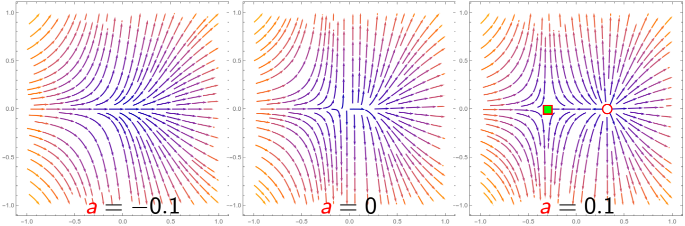


$\Phi_2 = 0.96\,\pi$

Saddle-node:

$$\begin{cases} v_x = x^2 - a \\ v_y = y \end{cases}$$

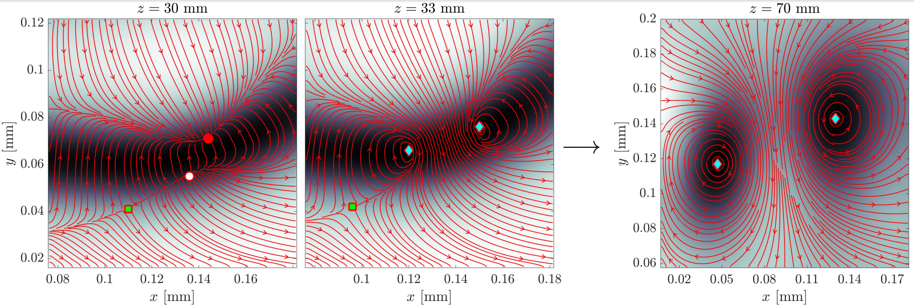
$a \in \mathbb{R}$

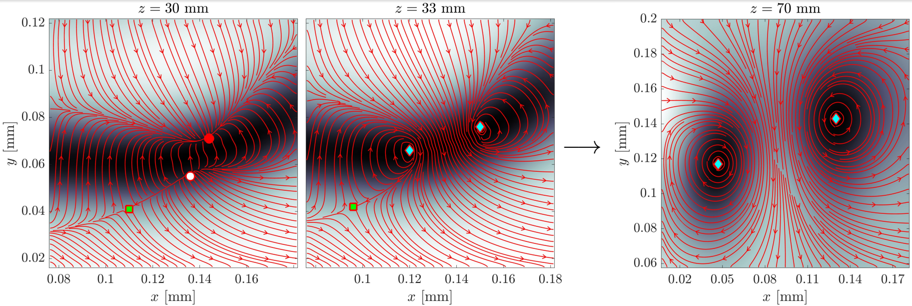


$$\vec{v} = \vec{\nabla}\left(\tfrac{1}{3}x^3 - ax + \tfrac{1}{2}y^2\right)$$

orbitally equivalently: $\vec{v} = \vec{\nabla} S$ where
$S(\vec{r}) = (almost)any\ fct\ of\ Z = \frac{1}{3}x^3 - ax + \frac{1}{2}y^2$

Mechanism of vortex formation: fold-Hopf bifurcation

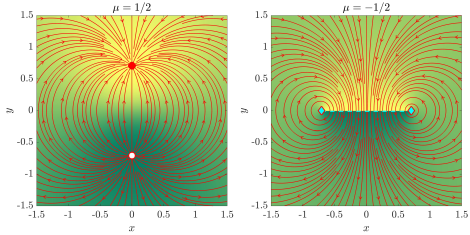




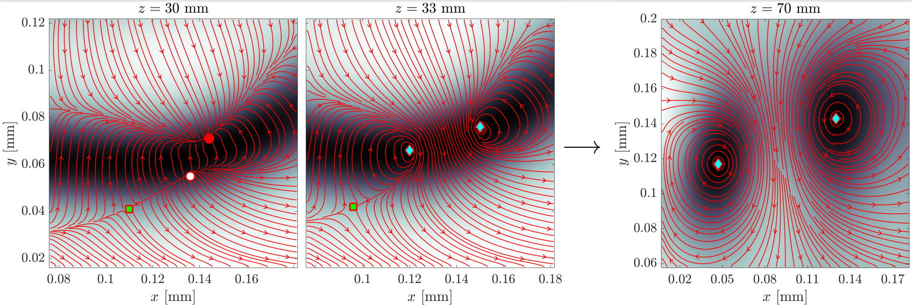
fold-Hopf bifurcation:

$$\begin{cases} v_x = -2\sigma xy \\ v_y = \mu + \sigma x^2 - y^2 \end{cases}$$

$\sigma = 1, \ \mu \in \mathbb{R}$



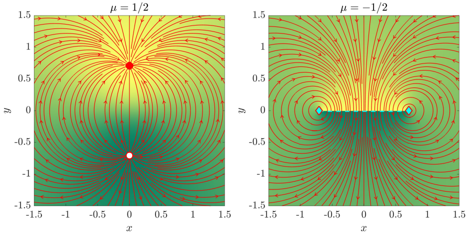
Mechanism of vortex formation: fold-Hopf bifurcation



fold-Hopf bifurcation:

$$\begin{cases} v_x = -2\sigma xy \\ v_y = \mu + \sigma x^2 - y^2 \end{cases}$$

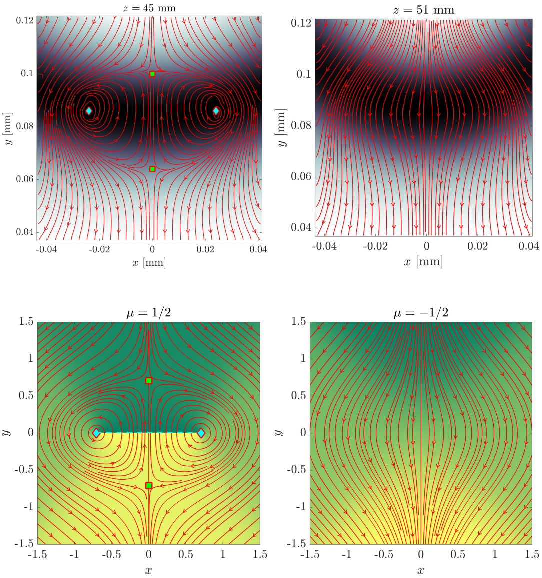
$\sigma = 1$, $\mu \in \mathbb{R}$



orbitally equivalent system: $\vec{v} = \vec{\nabla} S_{\mathrm{fH}}$, $S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$.

✓ gradient system

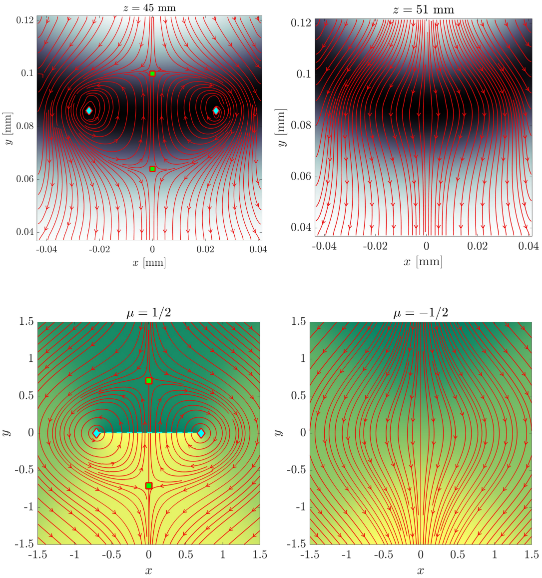
✓ verifies Onsager-Feynman quantization condition



$\Phi_2 = 1.05\,\pi$

$$\vec{v} = \vec{\nabla} S_{\mathrm{fH}} \quad \text{where} \quad S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$$

$$\text{with} \quad \sigma = -1 \quad \text{and} \quad \mu \in \mathbb{R}$$



$\Phi_2 = 1.05\,\pi$

close to the bifurcation point

the phase of the model wave function (solution of Helmoltz equation) matches $S_{\mathrm{fH}}(\vec{r})$

$\vec{v} = \vec{\nabla} S_{\mathrm{fH}}$ where $S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$

with $\sigma = -1$ and $\mu \in \mathbb{R}$

Conclusion and perspectives

Two mechanisms of vortex formation:

- Experimentally relevant
- fulfill **topological** and **quantum** constrains

Involve critical points $\neq$ vortices: candidate observables for studying the transition to turbulence

➪ Can one (predict/describe) the kinetics of the # of vortices, saddles and nodes?

$$\text{conservation of } I_{\mathrm{P}} \implies (\#\ \text{vortices}) + (\#\ \text{nodes}) - (\#\ \text{saddles}) = \text{cst}$$

➥ Topological constrains $\rightarrow$ non trivial correlations between different types of critical points?

☞ Can one think of other nonlinear phenomena that are topologically constrained?

Two mechanisms of vortex formation:

- Experimentally relevant
- fulfill **topological** and **quantum** constrains

Involve critical points $\neq$ vortices: candidate observables for studying the transition to turbulence

➪ Can one (predict/describe) the kinetics of the # of vortices, saddles and nodes?

conservation of $I_{\mathrm{P}} \implies$ (# vortices) $+$ (# nodes) $-$ (# saddles) $=$ cst

➥ Topological constrains → non trivial correlations between different types of critical points?

☞ Can one think of other nonlinear phenomena that are topologically constrained?

Thank you for your attention

ref: Congy, Azam, Kaiser, Pavloff arXiv:2308.02305