(Dispersive) shock wave dynamics

les Houches, march 2023

Morning glory roll cloud, Australia
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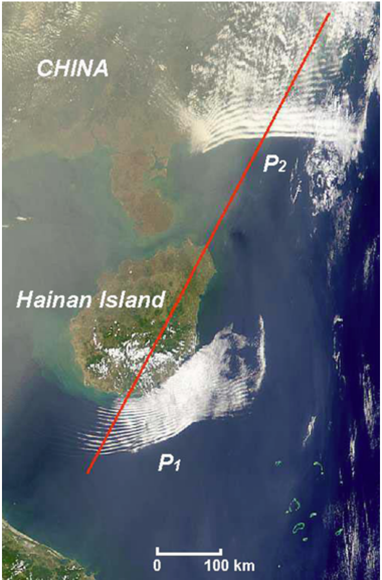
Waves generated by wind $\longrightarrow$
south China sea



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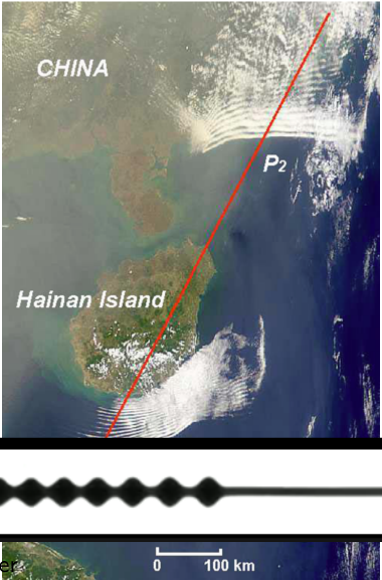
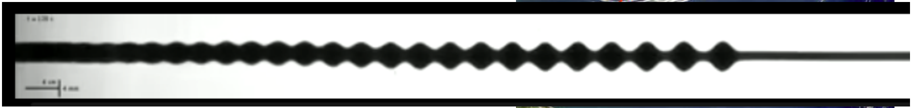
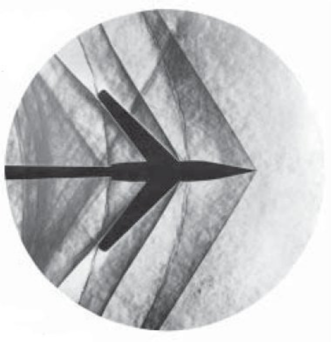


Morning glory roll cloud, Australia



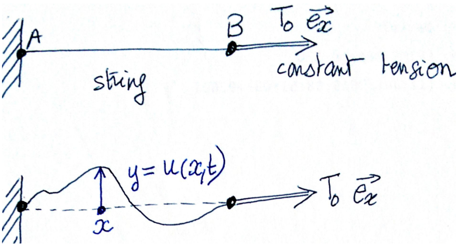
viscous fluid conduit, Mark Hoefer's group, Boulder dissipative shock



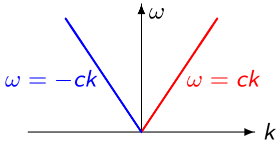
Schlieren photograph of shocks attached on a supersonic body

Dispersive shock

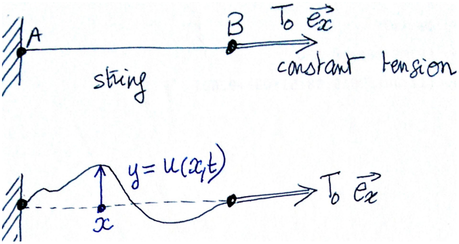
undular bore (mascaret) on river Dordogne vibrating string: $u_{tt} - c^2 u_{xx} = 0$ where $c^2 = T_0/\mu$



$$\underbrace{(u_t - c\,u_x)}_{\text{left mover}}\underbrace{(u_t + c\,u_x)}_{\text{right mover}} = 0$$



vibrating string: $u_{tt} - c^2 u_{xx} = 0$ where $c^2 = T_0/\mu$



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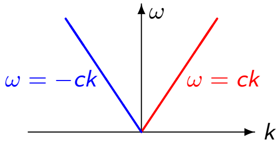
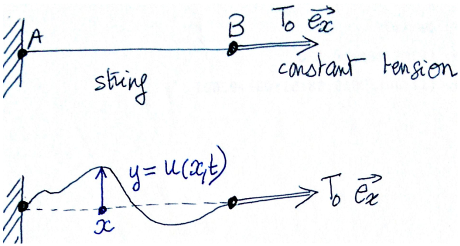


nonlinearize the right mover: $c \to c(u) = c_0 + c_1.u + \cdots$

upon re-parametrization:	$u_t + uu_x = 0$	Hopf equation
regularize the above:	$u_t + uu_x = \epsilon u_{xx}$	Burgers equation
other possible choice:	$u_t + uu_x = \epsilon u_{xxx}$	Korteweg de Vries eq.

Simple nonlinear PDEs

vibrating string: $u_{tt} - c^2 u_{xx} = 0$ where $c^2 = T_0/\mu$



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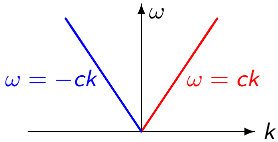

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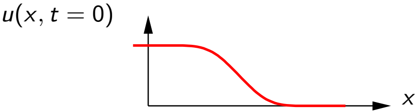
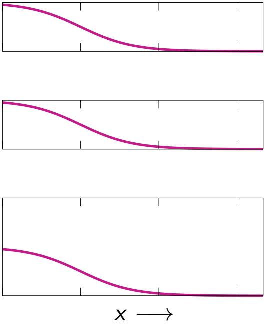
dispersion relation $u(x,t) = u_0 + A\exp[i(kx - \omega t)]$

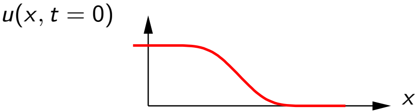
Burgers :	$\omega = u_0 k - i\epsilon k^2$	damping
KdV :	$\omega = u_0 k + \epsilon k^3$	dispersion



$$u_t + uu_x = \begin{cases} 0 & \text{Hopf} \\ \epsilon\, u_{xx} & \text{Burgers} \\ \epsilon\, u_{xxx} & \text{KdV} \end{cases}$$



dissipative or dispersive shock waves



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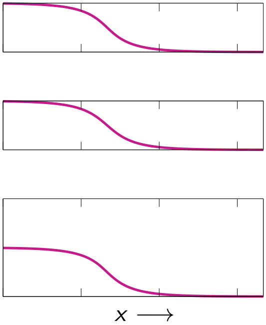

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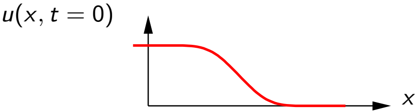

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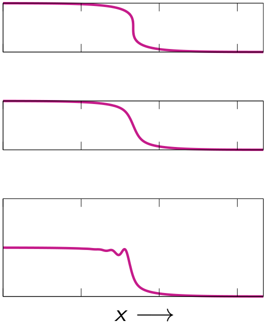

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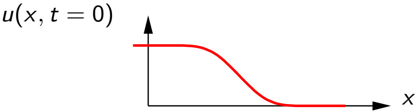

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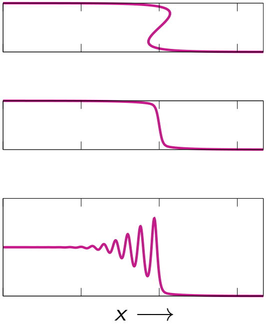
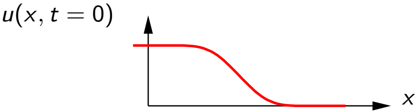


dissipative or dispersive shock waves



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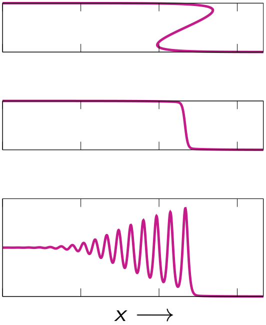
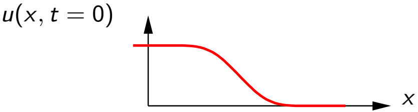


dissipative or dispersive shock waves



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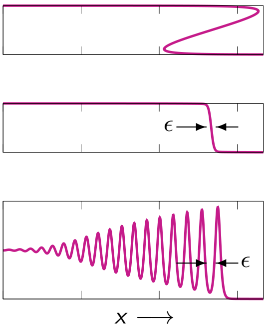
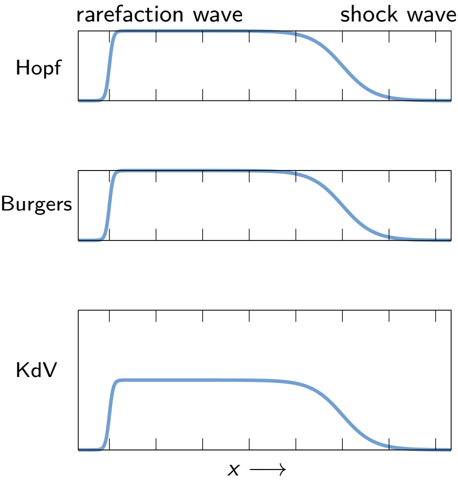


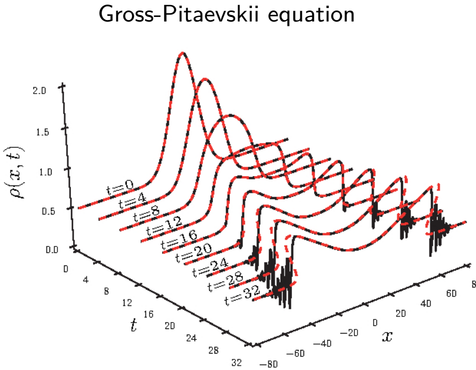
steepening
then breaking

Taylor profile

Gurevich Pitaevskii

uni-directional vs bi-directional motion





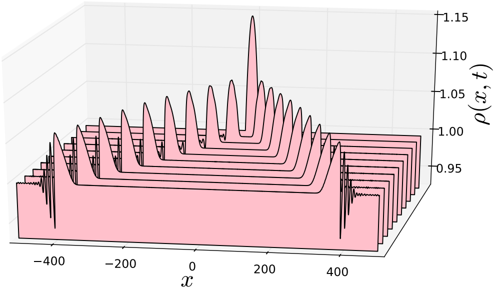
nonlinear problem:

$$i\psi_t = -\tfrac{1}{2}\psi_{xx} - \rho\,\psi$$

with $\rho = |\psi|^2$ and

$$\rho(|x| < x_0, 0) = \rho_0 + \rho_1 \left(1 - \frac{x^2}{x_0^2}\right)$$

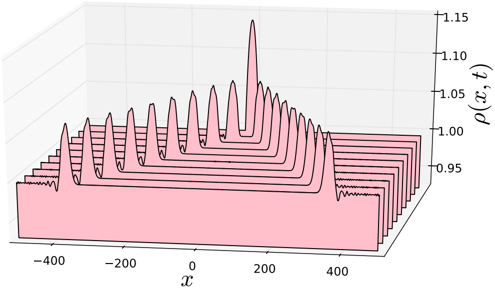
$\rho_0 = 1,\ \rho_1 = 0.15,\ x_0 = 20$

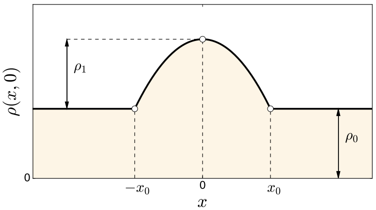


linear approximation:

$$\psi = \exp(-i\rho_0 t)(\sqrt{\rho_0} + \delta\psi)$$

$$i\delta\psi_t = -\tfrac{1}{2}\delta\psi_{xx} - \rho_0(\delta\psi + \delta\psi^*)$$





$$\rho(x,0) = \rho_0 + \rho_1 \left(1 - \frac{x^2}{x_0^2}\right)$$

wave breaking time:

$$c(\rho_0+\rho_1).t_{WB} = x_0 + c(\rho_0).t_{WB}$$

$$c(\rho_0+\rho_1) \simeq c_0.(1+\tfrac{1}{2}\rho_1/\rho_0) \text{ where } c_0 = c(\rho_0)$$

$$\boxed{c_0.t_{WB} \simeq 2\,x_0\,\frac{\rho_0}{\rho_1}}$$

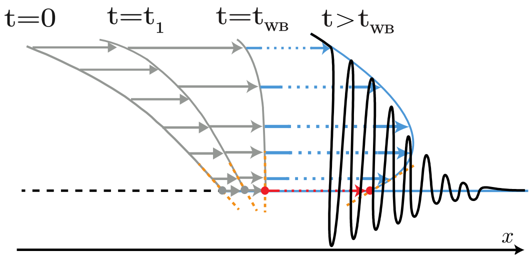
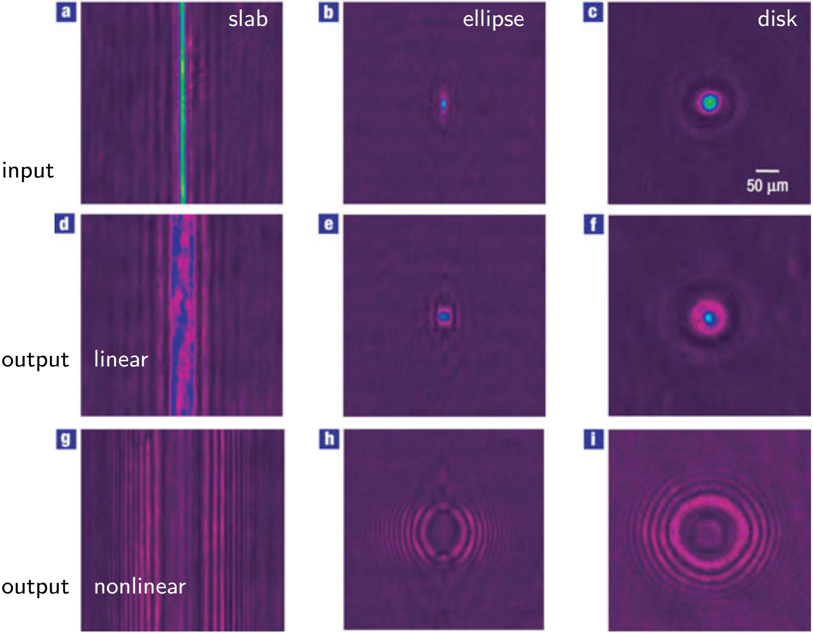
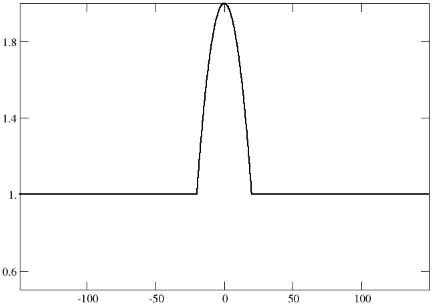


photo-refractive material: NL induced by a voltage bias across the crystal

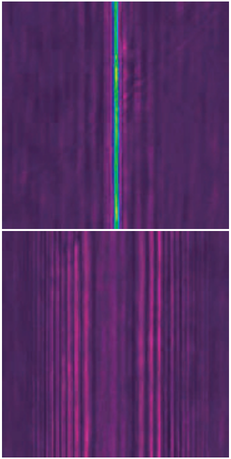


Nat. Phys. **3**, 46 (2007)



$0 \leq t \leq 60$

input



output

dispersionless hydrodynamics

$$\psi(x,t) = \sqrt{\rho}\,\exp\{i\,S\} \qquad S_x = u$$

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u u_x + \rho_x + \left(\frac{(\rho_x)^2}{8\rho} - \frac{\rho_{xx}}{4\rho}\right)_x = 0 \end{cases}$$

Riemann invariants

$$\lambda_\pm = \tfrac{1}{2} u \pm \sqrt{\rho}$$

$$\partial_t \lambda_\pm + V_\pm\,\partial_x \lambda_\pm = 0$$

with $V_\pm = \frac{1}{2}(3\lambda^\pm + \lambda^\mp) = u \pm \sqrt{\rho}$

- in a region where one the λ's is constant: simple wave, the method of characteristics works

- In a region where both λ's vary:

Hodograph transform

$x = x(\lambda^+, \lambda^-)$, $t = t(\lambda^+, \lambda^-)$

$$\frac{\partial x}{\partial \lambda_\pm} - V_\mp \frac{\partial t}{\partial \lambda_\pm} = 0$$

Euler-Poisson equation

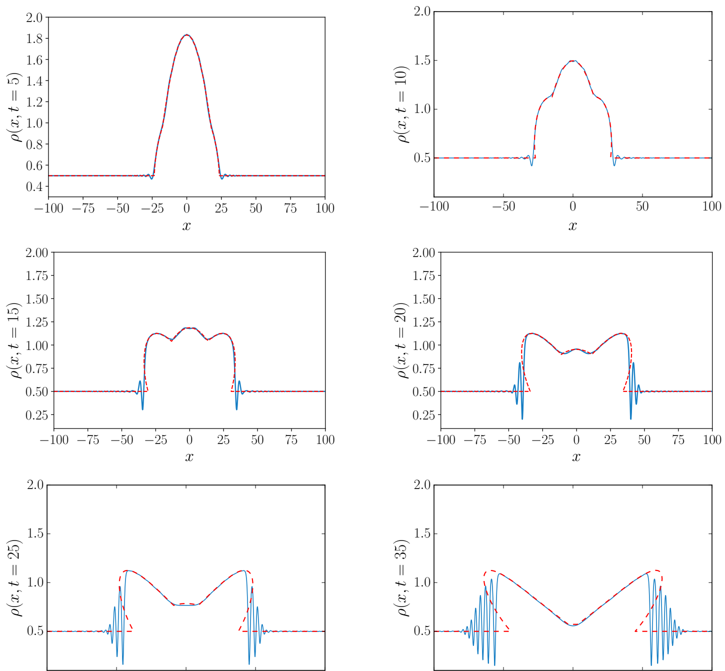
$$x - V_\pm t = \frac{\partial W}{\partial \lambda_\pm}$$

$$\frac{\partial^2 W}{\partial \lambda_+ \partial \lambda_-} = \frac{1}{2(\lambda_+ - \lambda_-)}\left(\frac{\partial W}{\partial \lambda_+} - \frac{\partial W}{\partial \lambda_-}\right)$$

solved by the so-called Riemann method

PRA **99** (2019), EPL **129** (2020), PRE **102** (2020)

NLS pre wave-breaking

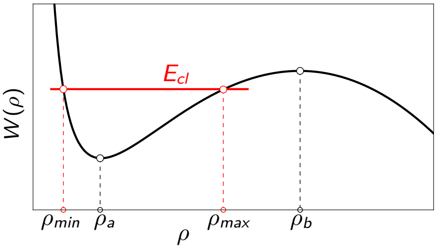


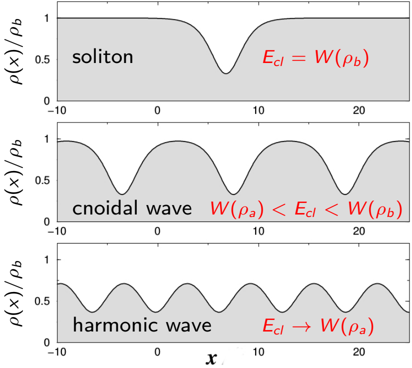
Stationary solutions of the NLS equation

$$-\tfrac{1}{2}A_{xx} + \left[\rho + \frac{J^2}{2\rho^2} - \mu\right] A = 0\,, \quad \text{where} \quad J = \rho(x)u(x) \quad \text{and} \quad A = \sqrt{\rho}$$

first integral:

$$\tfrac{1}{2}A_x^2 + W(\rho) = E_{cl}\,, \quad \text{where} \quad W(\rho) = -\frac{\rho^2}{2} + \mu\rho + \frac{J^2}{2\rho}\,.$$



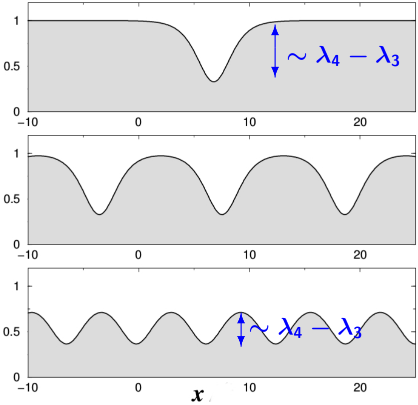


Single phase solutions of the NLS equation

$$\rho(x,t) = \tfrac{1}{4}(\lambda_4 - \lambda_3 - \lambda_2 + \lambda_1)^2 + (\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)\,\mathrm{sn}^2(k\,(x - Vt), m)$$

$$k = \sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}, \quad V = \tfrac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4),$$

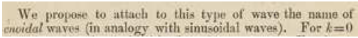
$$m = \tfrac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)} \in [0,1], \quad u(x,t) = V - \frac{C(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{\rho(x,t)}.$$



soliton limit: $\lambda_2 \to \lambda_3$

cnoidal wave: $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$

KdV
1895


We propose to attach to this type of wave the name of *cnoidal* waves (in analogy with sinusoidal waves). For $k=0$

sinusoidal limit: $\lambda_3 \to \lambda_4$

slow modulations $\lambda_i \to \lambda_i(x,t)$ with

$$\partial_t \lambda_i + \mathscr{V}_i(\{\lambda_j\})\, \partial_x \lambda_i = 0$$

$$\mathscr{V}_1(\{\lambda_j\}) = \frac{1}{2}\sum_{i=1}^{4} \lambda_i - \frac{(\lambda_4-\lambda_1)(\lambda_3-\lambda_1)K(m)}{(\lambda_4-\lambda_1)K(m)-(\lambda_3-\lambda_1)E(m)} \qquad m = \frac{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}$$

Gurevich-Pitaevskii problem

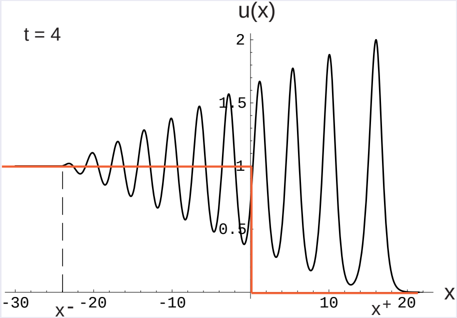
Gurevich & Pitaevskii (1973)

simple case: decay of an initial discontinuity → dispersive shock wave

no characteristic length : self-similar solution depending on $\zeta = x/t$ and matching to the right and left boundaries with a non dispersive flow.

$$u_t + u u_x + u_{xxx} = 0$$

$$u(x, t=0) = \begin{cases} 1 \text{ if } x < 0\,, \\ 0 \text{ if } x > 0\,. \end{cases}$$



Whitham modulation equations

Forest & Lee (1986), Pavlov (1987)

slow modulations $\lambda_i \to \lambda_i(x,t)$ with

$$\partial_t \lambda_i + \mathscr{V}_i(\{\lambda_j\})\, \partial_x \lambda_i = 0$$

$$\mathscr{V}_1(\{\lambda_j\}) = \tfrac{1}{2}\sum_{i=1}^{4} \lambda_i - \frac{(\lambda_4-\lambda_1)(\lambda_3-\lambda_1)K(m)}{(\lambda_4-\lambda_1)K(m)-(\lambda_3-\lambda_1)E(m)} \qquad m = \frac{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}{(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)}$$

Gurevich-Pitaevskii problem

Gurevich & Pitaevskii (1973)

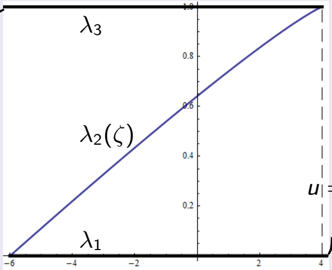
simple case: decay of an initial discontinuity $\to$ dispersive shock wave

no characteristic length : self-similar solution depending on $\zeta = x/t$ and matching to the right and left boundaries with a non dispersive flow.

$u = \lambda_3 = 1$

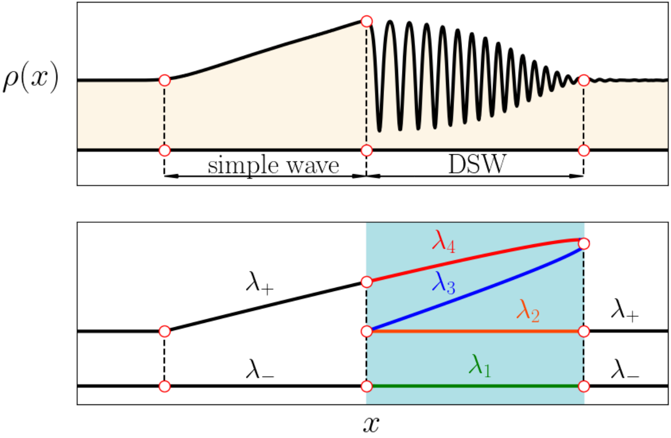
$$(\mathscr{V}_i - \zeta)\,\frac{\mathrm{d}\lambda_i}{\mathrm{d}\zeta} = 0$$

$$u(x, t=0) = \begin{cases} 1 \text{ if } x < 0\,, \\ 0 \text{ if } x > 0\,. \end{cases}$$



$u = \lambda_1 = 0$

x^- x^+

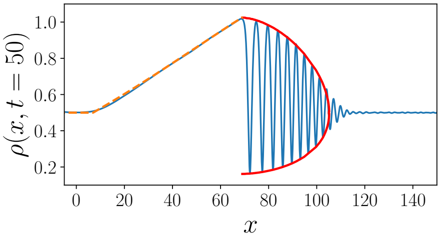


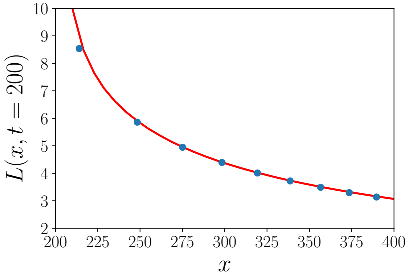
• in the simple wave region $\rho(x,t) = \frac{1}{4}(\lambda_+ - \lambda_-)^2$ remember $\lambda_\pm = \frac{1}{2}u \pm \sqrt{\rho}$

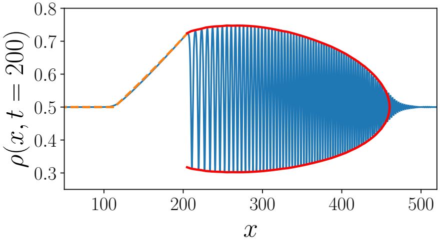
• in the DSW region: two (x,t)-dependent λ's.
Hodograph method then Euler-Poisson equation $\to$ $\lambda_3(x,t)$ and $\lambda_4(x,t)$

$$\rho(x,t) = \frac{1}{4}(\lambda_4 - \lambda_3 - \lambda_2 + \lambda_1)^2 + (\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)\,\mathrm{sn}^2(k\,(x - Vt), m)$$

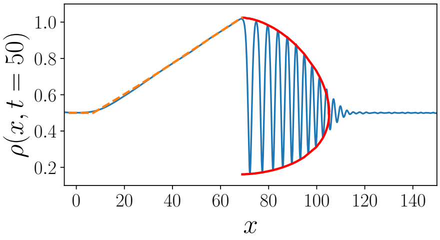
Realistic setting

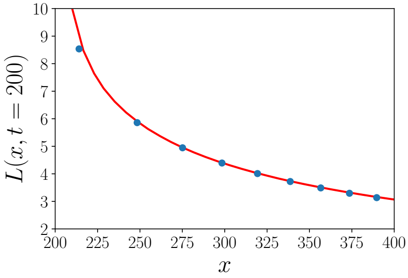


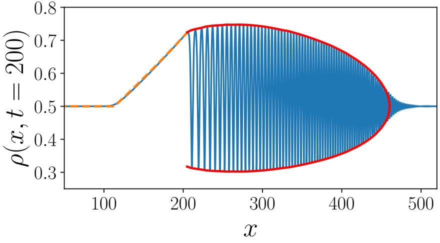


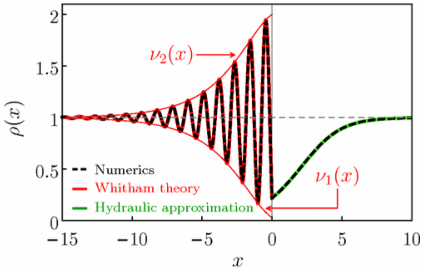


Realistic setting

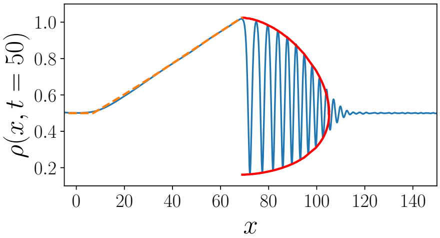


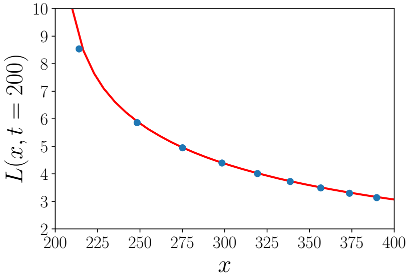


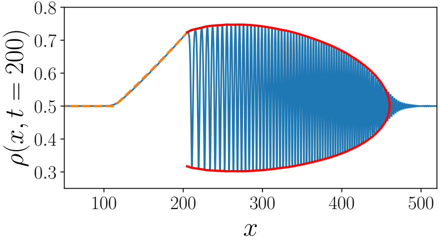


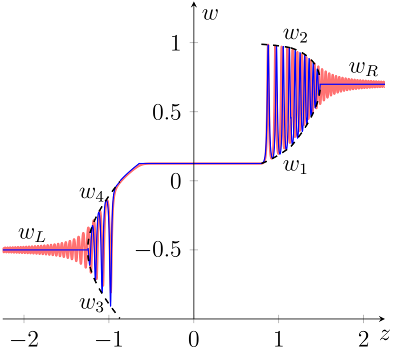


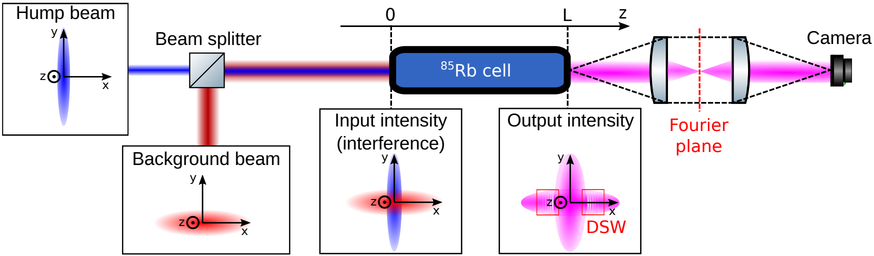
Realistic setting











$$\vec{E}(z,\vec{r}_\perp) = \mathcal{A}(z,\vec{r}_\perp)\exp\{i(k_0 z - \omega t)\}\vec{e}_z$$

complex amplitude × carrier wave

paraxial approximation:

$$i\partial_z \mathcal{A} = -\frac{1}{2n_0k_0}\vec{\nabla}_\perp^2\mathcal{A} + \frac{k_0 n_2|\mathcal{A}|^2}{1+|\mathcal{A}|^2/I_{\rm sat}}\mathcal{A} - \frac{i}{\Lambda_{\rm abs}}\mathcal{A}$$

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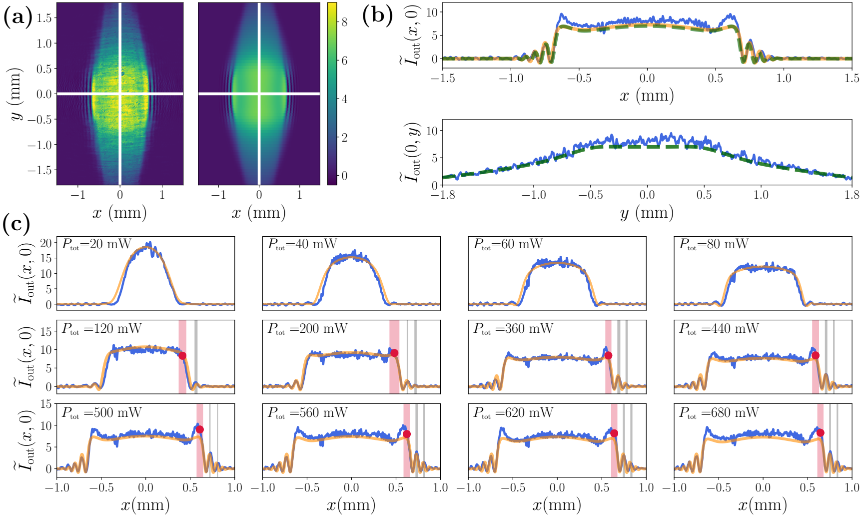
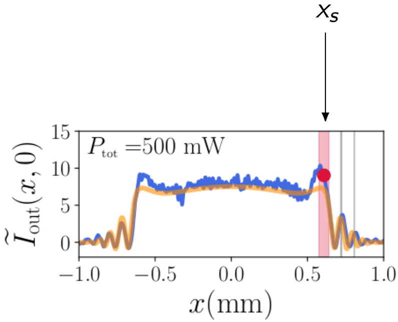
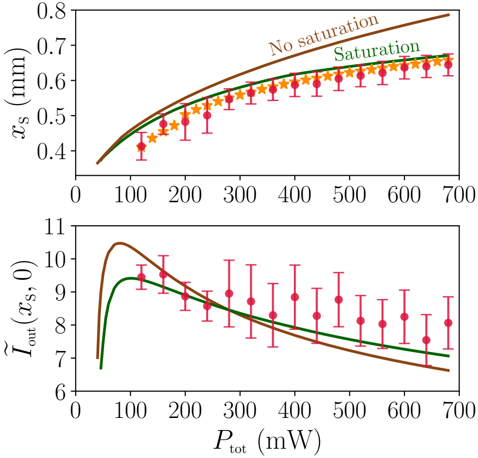
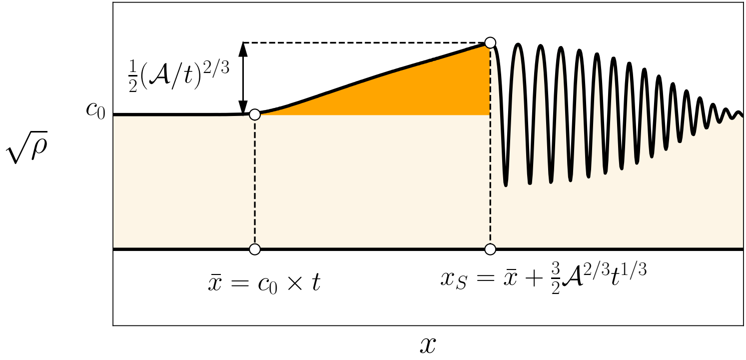


plots with background removed


x_s
P_{tot} =500 mW
$\tilde{I}_{out}(x,0)$
x(mm)


No saturation
Saturation
x_S (mm)
$\tilde{I}_{out}(x_S,0)$
P_{tot} (mW)

$t \gg t_{WB}$: asymptotic weak shock theory



New (asymptotically) conserved quantity

$$\mathcal{A} = \sqrt{2}\int_{\bar{x}}^{x_S} (\sqrt{\rho} - c_0)^{1/2} dx \simeq 2\, x_0\, \sqrt{c_0}\, F(\rho_0/\rho_1)$$

where

$$F(\alpha) = \int_0^{\pi/2} \cos\theta \left(\sqrt{1 + \frac{\cos^2\theta}{\alpha}} - 1 \right)^{1/2} d\theta$$

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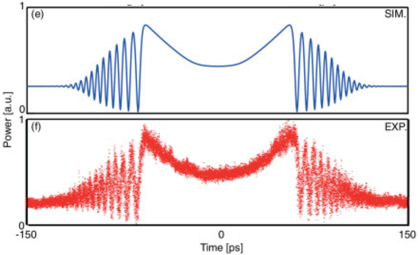
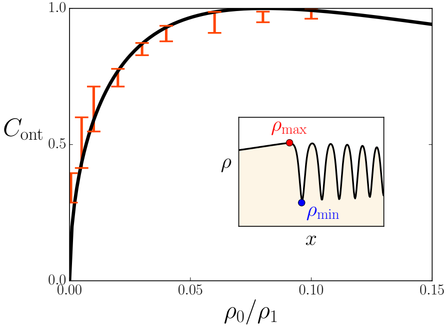
Fiber optics : $$-i\partial_z A = -\frac{\beta_2}{2}\partial_t^2 A + \gamma|A|^2 A + \frac{i\alpha}{2}A$$



$t_0 = 18.3$ ps, $L = 3$ km, $P_1 = 5.9$ W
$\gamma = 3W^{-1}.km^{-1}$, $\beta_2 = 2.5 \times 10^{-26}\, s^2/m$
$P_{ref} = 1W$
$\rho_1 = P_1/P_{ref} = 5.9$, $t = \gamma P_{ref} L = 9$,
$x_0 = t_0\sqrt{\gamma P_{ref}/\beta_2} = 6.3$

$$C_{ont} = \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}}$$



C_{ont} is a function of a single scaling parameter : $\xi = \frac{x_0}{c_0 t} F(\rho_0/\rho_1)$

$$C_{ont} = 4\,\frac{(2\xi)^{2/3}}{4 + (2\xi)^{4/3}} \qquad C_{ont} = 1 \text{ for } \xi = \sqrt{2}$$

Conclusion

rich analogy between nonlinear optics and hydrodynamics

- modulational instability
- observation of dispersive shock waves
- analogy with superfluid motion
- in the presence of disorder : competition between SF and Anderson localization
- possible formation of “sonic” horizon

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Mathematical description: Riemann + Whitham

- describes the early, pre-breaking, dispersiveless spreading
- analytic result for the wave-breaking time
- describes the later, post-breaking, dispersive shock
- analytic result for the asymptotic weak shock parameters

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Thank you for your attention