Topological pathways to two-dimensional quantum turbulence

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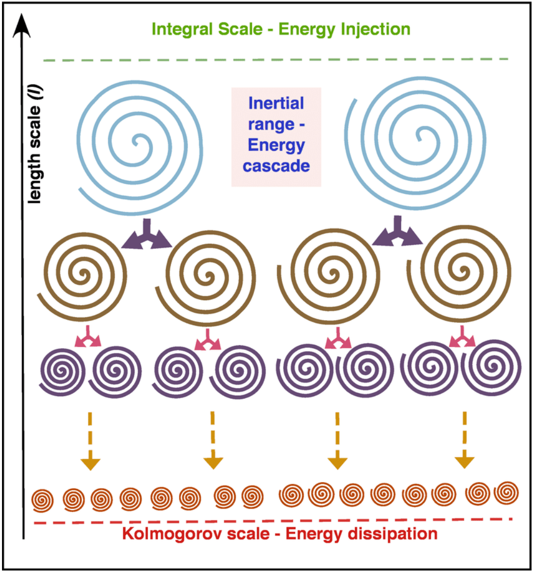
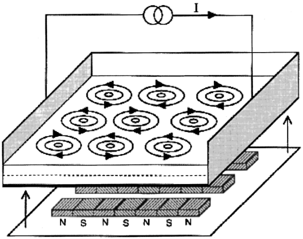
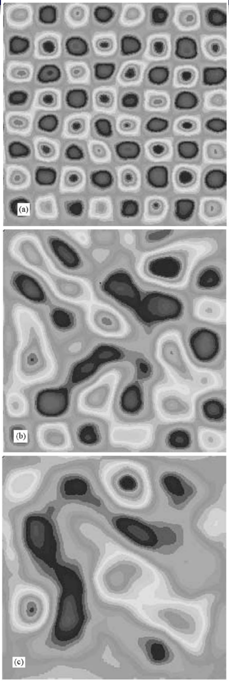
Breakdown of large eddies into progressively smaller ones



Richardson 1922:

«Big whirls have little whirls, that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.»

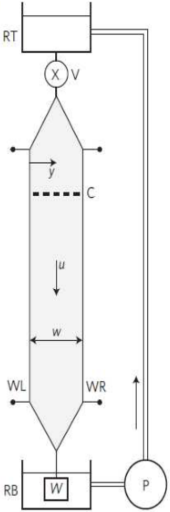




Thin layer of electrolyte

vortices driven by electromagnetical forces

P. Tabeling, Phys. Rep. (2002)

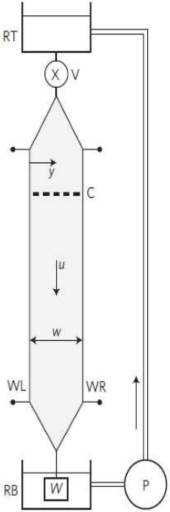


Tuan Tran et al., Nat. Phys. 2010

M. A. Rutgers, PRL 1998

Y. Couder, J.M. Chomaz, M. Rabaud, Physica D 1989

Soap films ... and more



Tuan Tran et al., Nat. Phys. 2010

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Great Red Spot

smaller vortices and storm clusters can be seen being engulfed by the GRS
but formation dynamics is disputed
Sánchez-Lavega et al.
Geophys. Res. Lett. 2024
1939: formation of white ovals which merged to form "Red Jr." in 2020

velocity field $\vec{v}(\vec{r}, t)$, vorticity: $\vec{\nabla} \times \vec{v} = \omega(\vec{r}, t)\, \vec{e}_z$
point vortex i, position $\vec{r}_i(t)$, circulation Γ_i: $\omega(\vec{r}, t) = \sum_i \Gamma_i\, \delta(\vec{r} - \vec{r}_i(t))$

$$H = -\frac{1}{2\pi} \sum_{i>j} \Gamma_i\, \Gamma_j \ln |\vec{r}_i - \vec{r}_j| \,, \quad \text{writing } \; \vec{r}_i(t) = (x_i, y_i) \; \text{ yields } \; \begin{cases} \Gamma_i\, \dot{x}_i = \partial H / \partial y_i \\ \Gamma_i\, \dot{y}_i = -\partial H / \partial x_i \end{cases}$$

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Onsager 1949

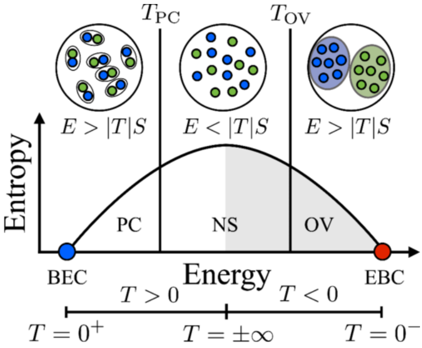
Phase space has a finite volume:
$\int d\Omega = (\int dxdy)^N = A^N$.

$\Phi(E) \stackrel{def}{=} \int_{H<E} d\Omega = \int_{-\infty}^{E} \Phi'(E) dE$.

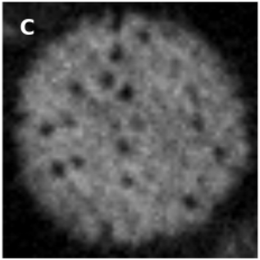
$\Phi'(E)$ is positive with $\Phi'(\pm\infty) = 0$
(since $\Phi(-\infty) = 0$, $\Phi(+\infty) = A^N$)

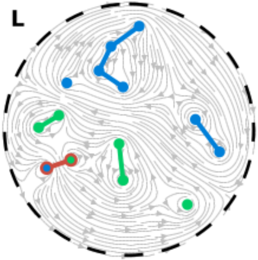
→ negative temperatures
(since $T^{-1} = (\partial S/\partial E) = \Phi''/\Phi'$)

« If $T < 0$ vortices of the same sign will tend to cluster, so as to use up excess energy at the least possible cost in terms of degrees of freedom.»



Simula, Davis, Helmerson PRL 2014

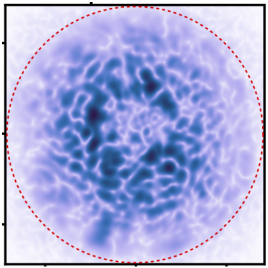


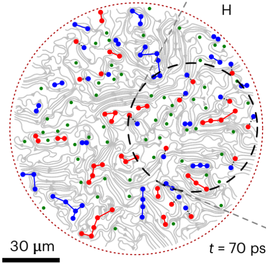


^{87}Rb atoms

group of Helmerson and Johnstone, Monash University (Science 2019)

see also group of Neely, Davis, and Rubinsztein-Dunlop, University of Queensland (Science 2019)

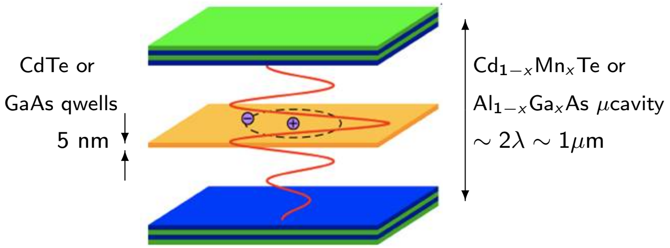




group of Sanvitto and Ballarini,
CNR-Nanotec Lecce
(Nat. Photon. 2023)

exciton-polaritons (hybrid light–matter quasiparticles)

Cavity polaritons



interacting bosons

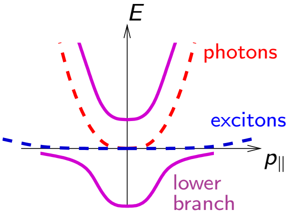
$m_{\mathrm{eff}} \lesssim 10^{-4}\, m_{\mathrm{e}}$

$T_{\mathrm{BEC}} \sim 10$ K

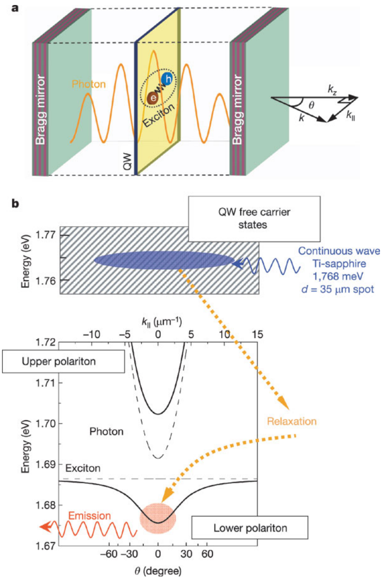
lifetime $\lesssim 50$ ps

optical detection

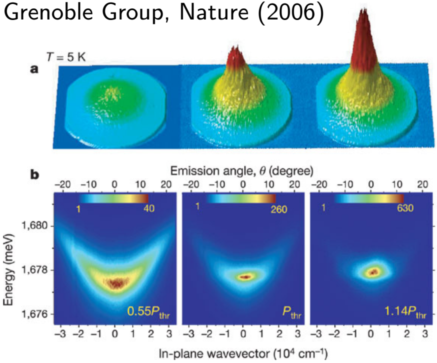
polarization degree of freedom

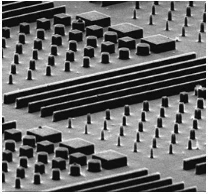


BEC of polaritons



Grenoble Group, Nature (2006)





Grenoble: institut Néel

Paris-Saclay: C2N

Paris: LKB

What is so special about superfluids ?

Superfluid order parameter: $\psi(\vec{r},t) = A\exp(i\,\chi)$

$\rightarrow$ ground state: stationnary and uniform: $A(\vec{r},t) = A_0$, $\chi(\vec{r},t) = -\mu_0 t/\hbar$

$\rightarrow$ Galilean boost, velocity $\vec{v}_0$: $\chi(\vec{r},t) = (-\mu t + m\,\vec{v}_0\cdot\vec{r})/\hbar \quad (\mu = \mu_0 + \frac{1}{2}mv_0^2)$

$\rightarrow$ non uniform velocity: $\chi(\vec{r},t)$ with $\quad \vec{v}(\vec{r},t) = \frac{\hbar}{m}\vec{\nabla}\chi$

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$\rightarrow$ non uniform velocity: $\chi(\vec{r},t)$ with $\quad \vec{v}(\vec{r},t) = \frac{\hbar}{m}\vec{\nabla}\chi$

This impose $\vec{\nabla}\times\vec{v} = 0$: irrotational flow.

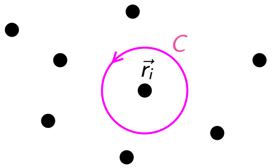
Except if there are points $\vec{r}_i$ where χ is ill defined, i.e. where $A(\vec{r}_i) = 0$.

Then $\omega(\vec{r}) = \sum_i \Gamma_i\,\delta(\vec{r}-\vec{r}_i)$.

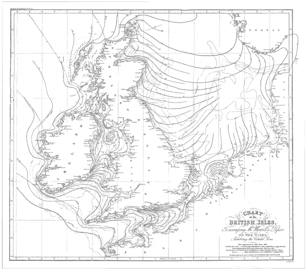
Onsager-Feynman quantization condition

$$\oint_C \vec{v}\cdot d\vec{r} = \Gamma_i = \frac{\hbar}{m}\oint_C d\chi = \frac{\hbar}{m}2\pi n_i$$

$n_i \in \mathbb{Z}$, typically $n_i = \pm 1$

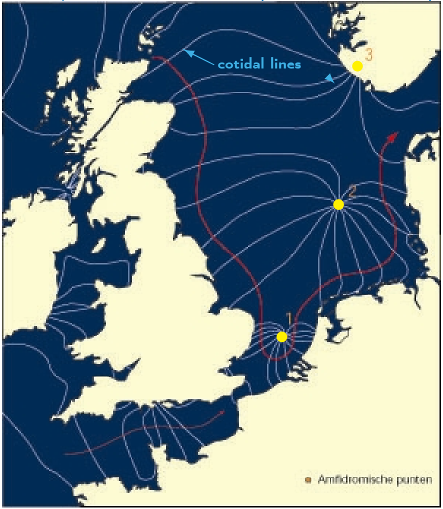


cotidal lines and amphidromic points. Whewell 1836



W. Whewell
Phil. Trans. R. Soc. 1836

Principal tidal constituent (M2: semi-diurnal)



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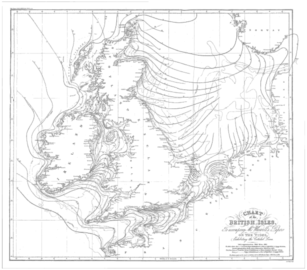

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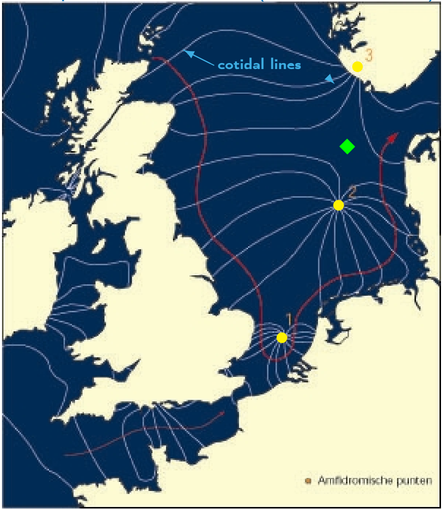

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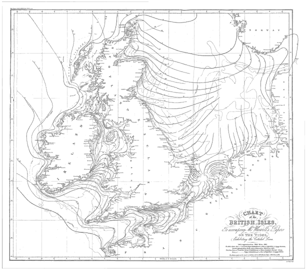
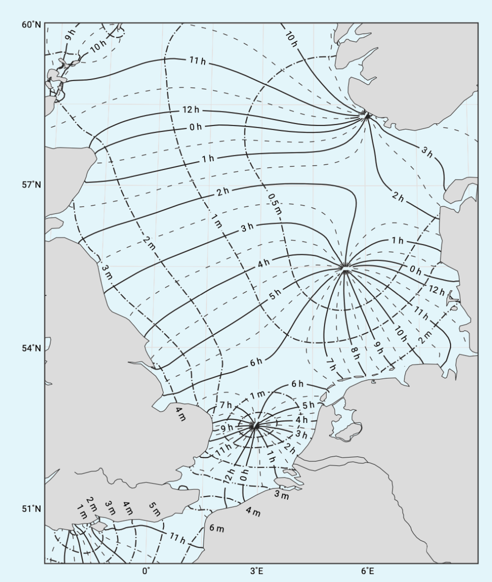


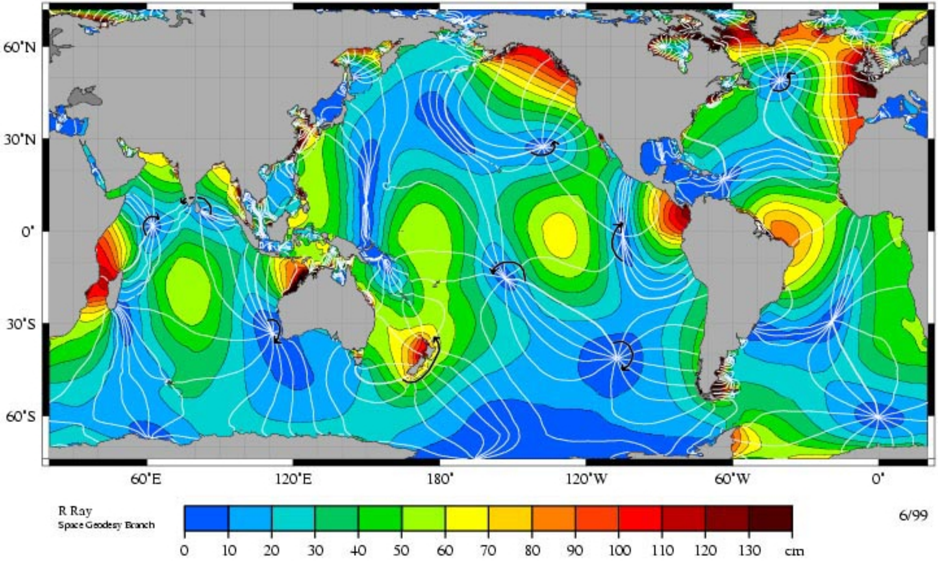
W. Whewell
Phil. Trans. R. Soc. 1836

$\psi = h(\vec{r}) \exp\{i(\chi(\vec{r}) - \omega t)\}$
where $\omega = \omega_{M2} = 2\pi / T_2$



co-range and cotidal lines

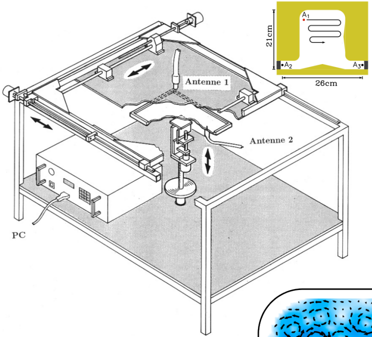
Amphidromic M2 points



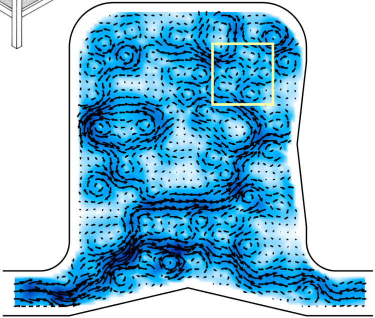
$T_{M2} = 12$ h 25 min, $\Delta t_{cotidal} = 1$ h 2 min, $\omega_{M2} \times \Delta t_{cotidal} = \pi/6$

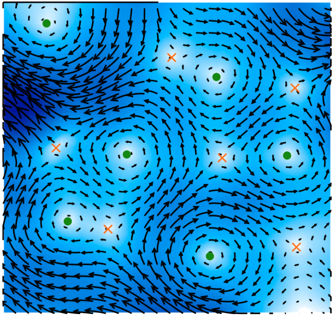
$\Longrightarrow$ quantized circulation of tidal current of the M2 component

Open electromagnetic resonator



U. Kuhl, Eur. Phys. J. (2007)



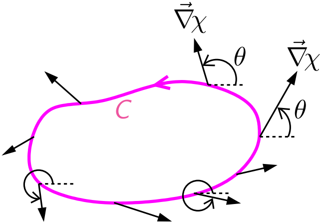


$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{i\chi(\vec{r})\}$

$\vec{v} = \frac{\hbar}{m}\vec{\nabla}\chi$

$I_{\rm V}$: vorticity, $I_{\rm P}$: Poincaré index



$I_{\rm V}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\chi \cdot {\rm d}\vec{\ell} = \oint_C \frac{{\rm d}\chi}{2\pi}$

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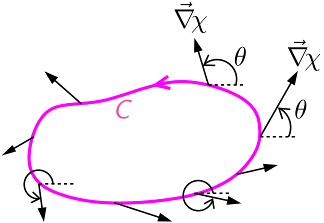

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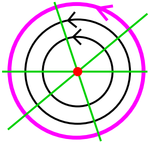
vortex



$I_{\rm V} \in \mathbb{Z}$

$I_{\rm P} = +1$

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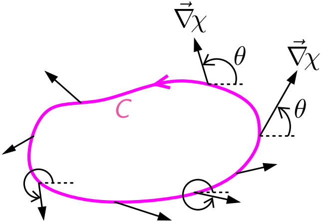

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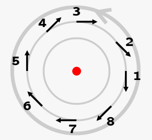
vortex



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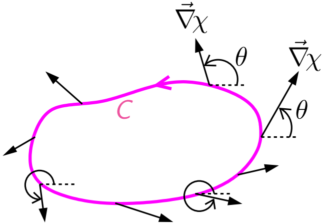

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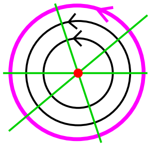
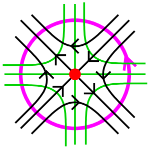
vortex		$I_{\rm V} \in \mathbb{Z}$ $I_{\rm P} = +1$
saddle		$I_{\rm V} = 0$ $I_{\rm P} = -1$

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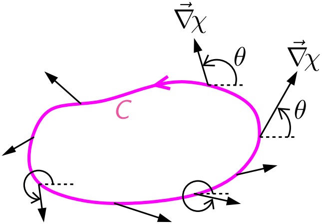

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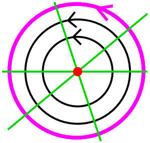
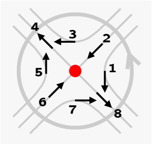
I_{V} : vorticity, I_{P} : Poincaré index

vortex		$I_{\mathrm{V}} \in \mathbb{Z}$ $I_{\mathrm{P}} = +1$
saddle		$I_{\mathrm{V}} = 0$ $I_{\mathrm{P}} = -1$

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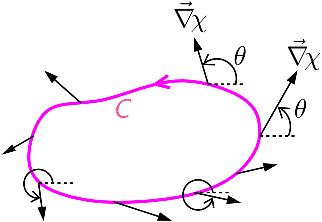

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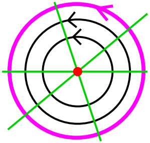
vortex



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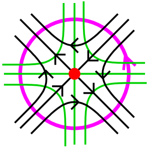
saddle

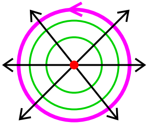


$I_{\rm V} = 0$

$I_{\rm P} = -1$

node

min/max of the phase



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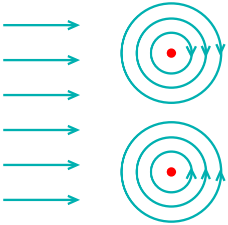
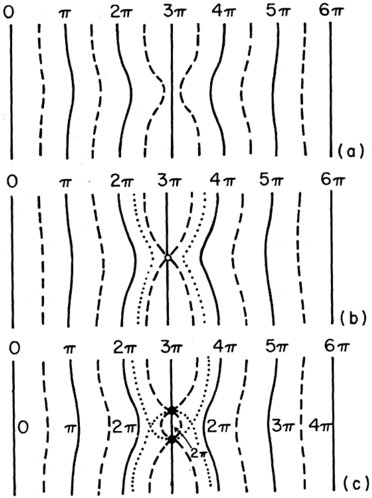
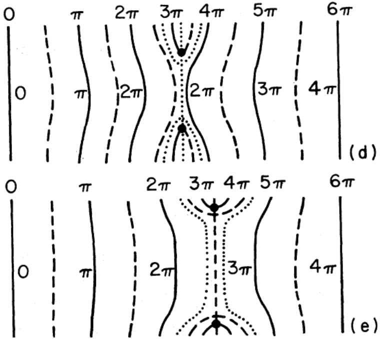


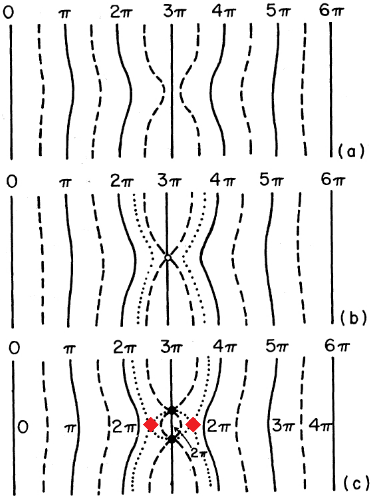

CLAC!



Anderson, RMP (1966)

Langer, Fisher, PRL (1967)





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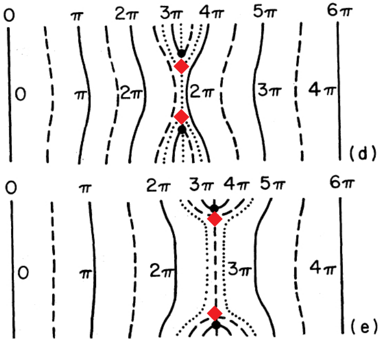
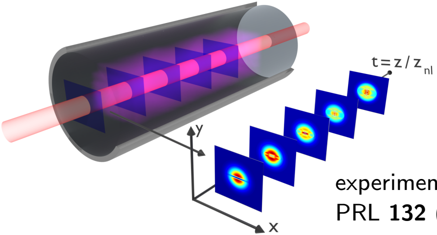
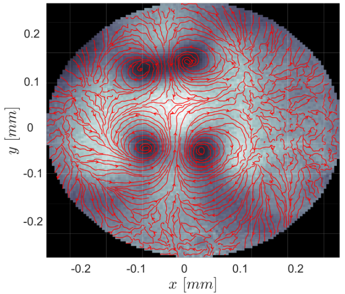
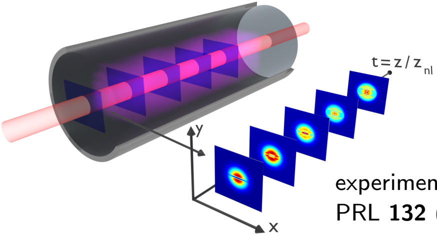




experiment in Nice
PRL **132** (2024)

output intensity and streamlines





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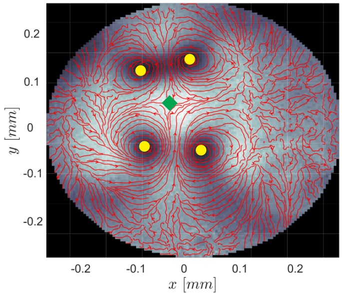
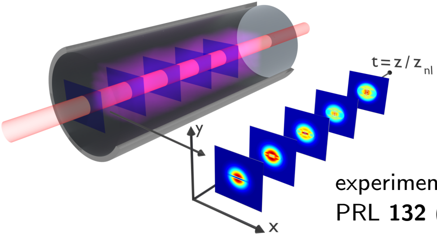


Experimental streamlines



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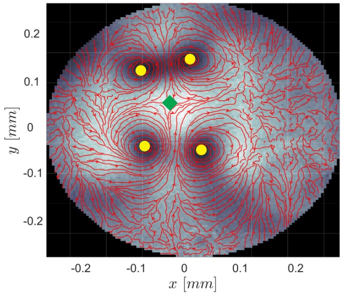
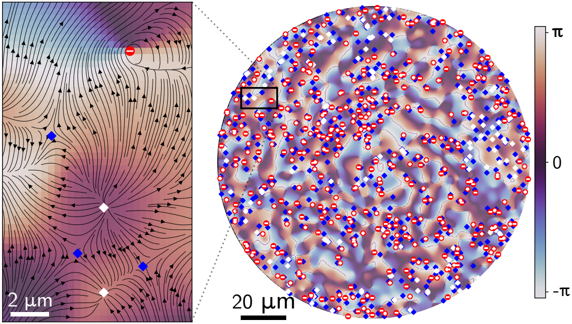




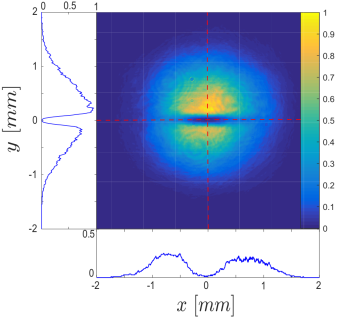
experiment in Lecce
arXiv:2411.11671

direct measure of the (large) number of vortices, saddles and nodes as a function of time

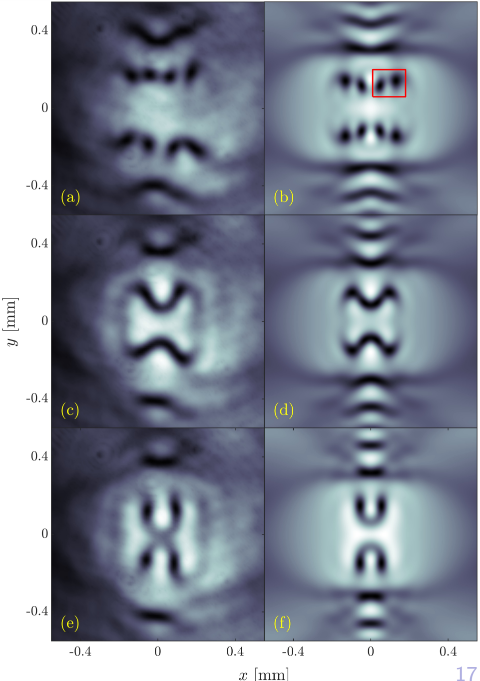
Reproducing the experimental results

$$\psi(\vec{r},0) = \sqrt{I_1}\exp\left(-\frac{r^2}{w_{\mathrm{G}}^2}\right) + \sqrt{I_2}\exp\left(-\frac{x^2}{w_x^2}-\frac{y^2}{w_y^2}\right)e^{i\Phi_2}$$

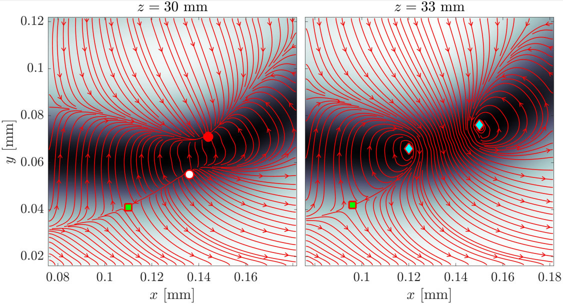
$I_1 = I_2 \qquad \Phi_2 \simeq \pi$



(a,b) : $\Phi_2 = 0.96\,\pi$
(c,d) : $\Phi_2 = \pi$
(e,f) : $\Phi_2 = 1.05\,\pi$



Mechanism of vortex formation/annihilation: fold-Hopf bifurcation



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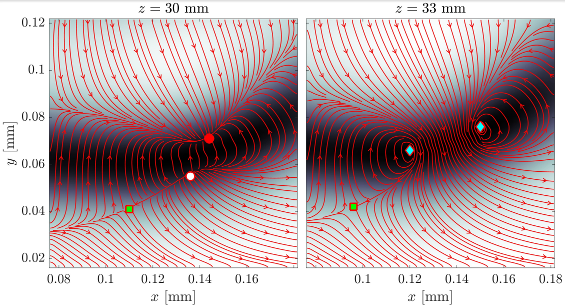
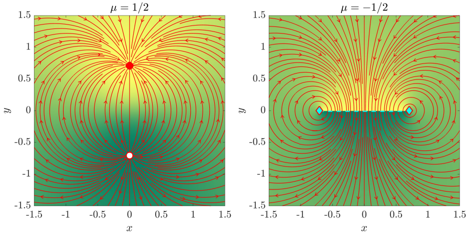
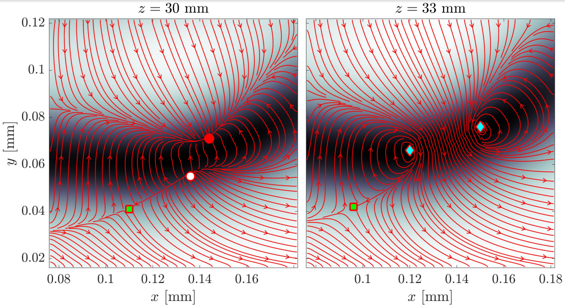


fold-Hopf bifurcation:

$$\begin{cases} v_x = -2\sigma xy \\ v_y = \mu + \sigma x^2 - y^2 \end{cases}$$

$\sigma = 1$, $\mu \in \mathbb{R}$





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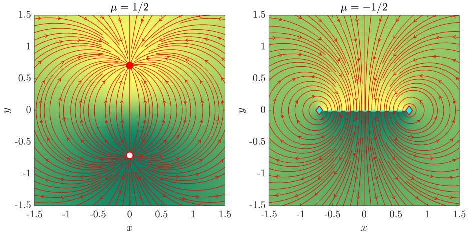
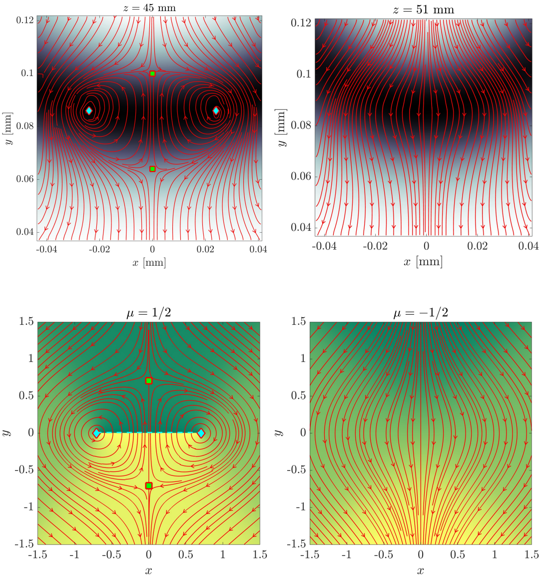


orbitally equivalent system: $\vec{v} = \vec{\nabla}\chi_{\mathrm{fH}}$, $\chi_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$.

✓ gradient system

✓ verifies Onsager-Feynman quantization condition

vortex formation/annihilation: Bristol mechanism ... also fold-Hopf



$$\vec{v} = \vec{\nabla}\chi_{\text{fH}} \qquad \text{where} \qquad \chi_{\text{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$$

$$\text{with} \qquad \sigma = -1 \quad \text{and} \quad \mu \in \mathbb{R}$$

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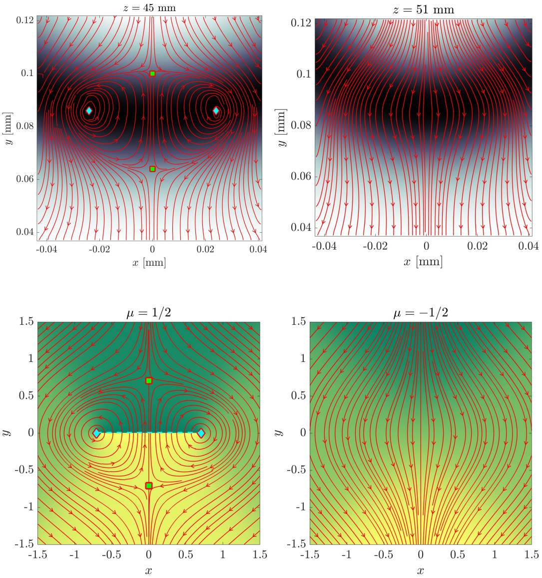
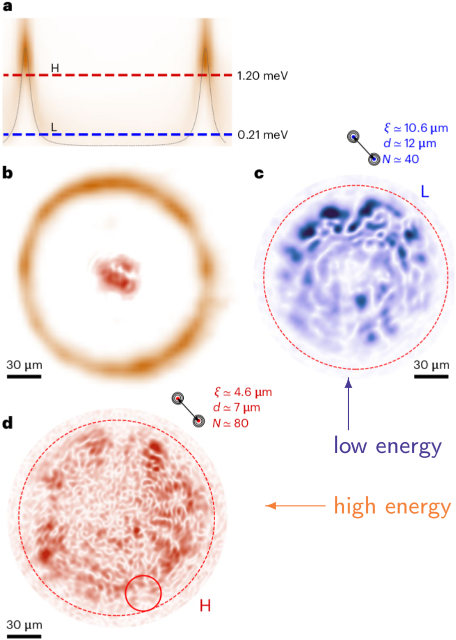


close to the bifurcation point

the phase of a model wave function (exact solution of Helmoltz equation) matches $\chi_{\mathrm{fH}}(\vec{r})$

$\vec{v} = \vec{\nabla}\chi_{\mathrm{fH}}$ where $\chi_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$

with $\sigma = -1$ and $\mu \in \mathbb{R}$



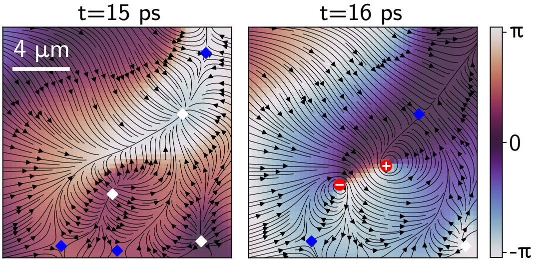
low energy (0.21 meV)

high energy (1.20 meV)

fold-Hopf bifurcations in Lecce experiment

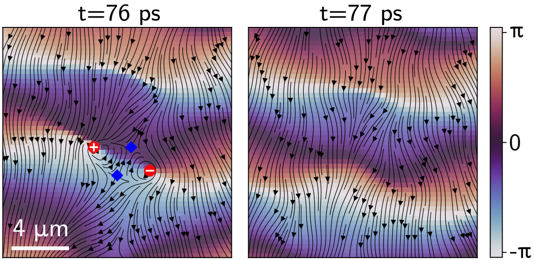
Node collision

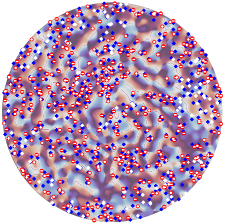
(fold-Hopf with $\sigma = 1$)



Bristol mechanism

(fold-Hopf with $\sigma = -1$)



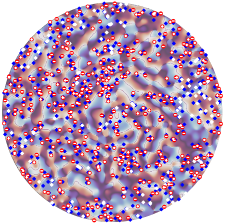


simplest possible processes

$$\text{node} + \text{node} \underset{b}{\overset{a}{\rightleftharpoons}} \text{vortex}_{(+)} + \text{vortex}_{(-)}$$

$$\varnothing \underset{d}{\overset{c}{\rightleftharpoons}} \text{node} + \text{saddle}$$

The (positive) quantities a, b, c, and d are the reaction rates.



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The (positive) quantities a, b, c, and d are the reaction rates.

Rate equations:

$$\frac{\mathrm{d}V_\pm}{\mathrm{d}t} = aN^2 - bV_+V_-,$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = c - dNS,$$
$$\frac{\mathrm{d}N}{\mathrm{d}t} = c - 2aN^2 + 2bV_+V_- - dNS,$$

where $N(t)$ denotes the number of nodes, $S(t)$ the number of saddles, and $V_+(t)$ [$V_-(t)$] the number of vortices with positive [negative] vorticity.

In the following $b = 0$

$$(b = 0) \qquad \frac{\mathrm{d}V_\pm}{\mathrm{d}t} = aN^2, \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = c - dNS, \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = c - 2aN^2 - dNS,$$

No imparted angular momentum:

$$V_+(t) - V_-(t) = C^{st} \ll typical\left(V_+(t) + V_-(t)\right) \quad \rightsquigarrow \quad V_+(t) = V_-(t) \equiv \frac{V(t)}{2}$$

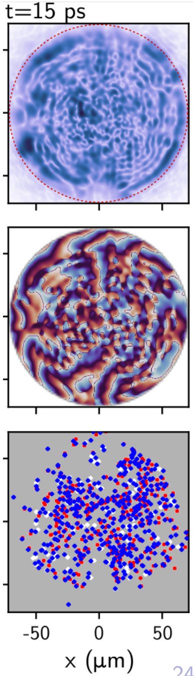
rescaled quantities

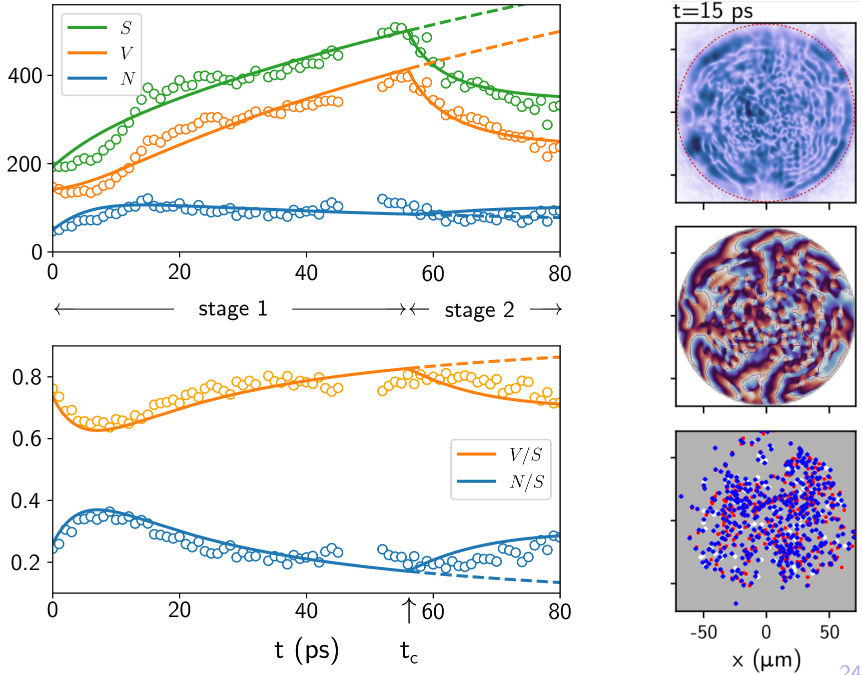
$\tau = t/t_0$, $n = N/N_0$, $v = V/N_0$, $s = S/N_0$

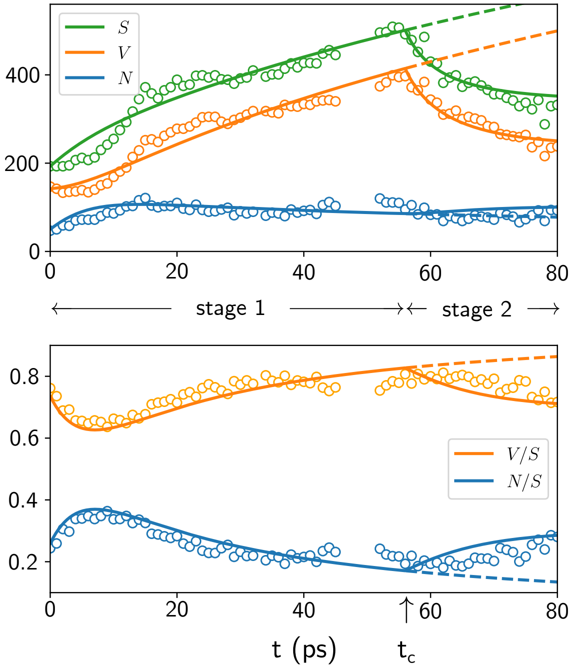
with $t_0 = 1/\sqrt{2ac}$ and $N_0 = \sqrt{c/2a}$

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}\tau} = n^2, \qquad \frac{\mathrm{d}s}{\mathrm{d}\tau} = 1 - \gamma ns, \quad \frac{\mathrm{d}n}{\mathrm{d}\tau} = 1 - n^2 - \gamma ns,}$$

where the single parameter $\gamma \equiv d/2a$ governs the qualitative features of the dynamical system.


t=15 ps
-50
0
50
x (μm)


S
V
N
400
200
0
0
20
40
60
80
stage 1
stage 2
0.8
0.6
0.4
0.2
V/S
N/S
t (ps)
t_c
t=15 ps
-50
0
50
x (μm)

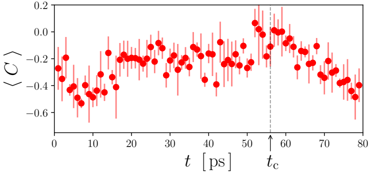


$$I_P = N(t) + V(t) - S(t)$$
$$= C^{st} \simeq 0$$

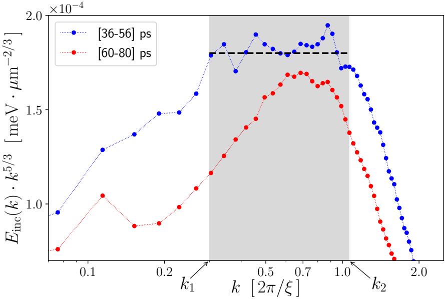
Quantitative evaluation of clustering:

$$C = \frac{1}{V}\sum_{i=1}^{V} c_i \qquad (c_i = \pm 1)$$

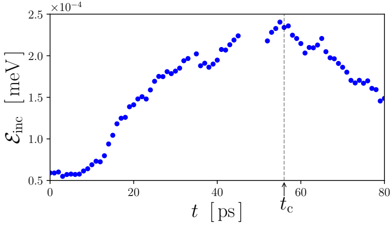
White, Barenghi, Proukakis, PRA (2012)



here: $E_{\mathrm{inc}} \propto k^{-5/3}$



$$\mathcal{E}_{\mathrm{inc}}(t) = \int_{k_1}^{k_2} E_{\mathrm{inc}}(k,t)\, dk$$

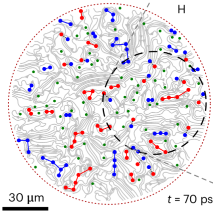


End of clustering

The vortices are no longer grouped in packs of the same vorticity.
Nodes don't seem to play a significant role.
$\rightsquigarrow$ Bristol mechanism:

$$\text{vortex}_{(+)} + \text{vortex}_{(-)} + \text{saddle} + \text{saddle} \underset{f}{\overset{e}{\rightleftharpoons}} \varnothing$$

for simplicity, take $f = 0$.



$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = n^2 - \varepsilon\, v^2 s^2, \quad \frac{\mathrm{d}s}{\mathrm{d}\tau} = 1 - \gamma ns - \varepsilon\, v^2 s^2 \quad \frac{\mathrm{d}n}{\mathrm{d}\tau} = 1 - n^2 - \gamma ns,$$

where $\varepsilon = \frac{1}{2} e N_0^3 t_0 = ec/(8a^2)$ is the rescaled rate of annihilation of saddles and vortices through the Bristol mechanism.

$S \approx V \rightsquigarrow$ Bristol is effectively a 4 vortices mechanism (also 3 and 2).

2D quantum vortices

carry **2** topological indices

Two mechanisms of vortex formation/annihilation:

- fulfill **topological** and **quantum** constrains
- Experimentally relevant

Involve critical points $\neq$ vortices:
candidate observables for studying the transition to turbulence and its decay

Conclusion and perspectives

2D quantum vortices

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- concepts of singular optics relevant for quantum fluids
- effective kinetic description $\approx$ Landau mean field
- Topological constrains $\rightarrow$ non trivial spatial correlations between (different types of) critical points?
- Thermodynamic counterpart in BKT transition ?

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Thank you for your attention

Refs: Congy, Azam, Kaiser, Pavloff, PRL **132**, 033804 (2024), Panico et al. arXiv:2411.11671

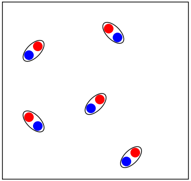
T_{BKT}

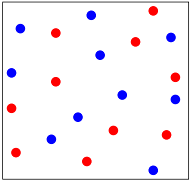
superfluid — normal → T

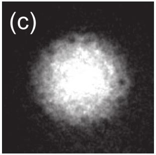
superfluid	normal
$\rho = \rho_s + \rho_n$	$\rho = \rho_n$
$\langle \psi^\dagger(\vec{r})\psi(\vec{0})\rangle \sim r^{-\alpha}$	$\langle \psi^\dagger(\vec{r})\psi(\vec{0})\rangle \sim \exp(-r/\ell)$

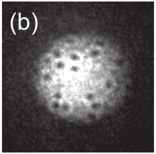
$\alpha_{crit} = \frac{1}{4}$

$$\alpha = \frac{1}{\rho_s \lambda_{th}^2} \leq \frac{1}{4}$$

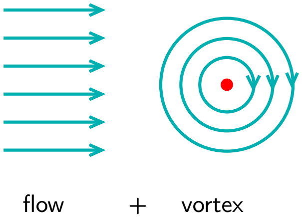




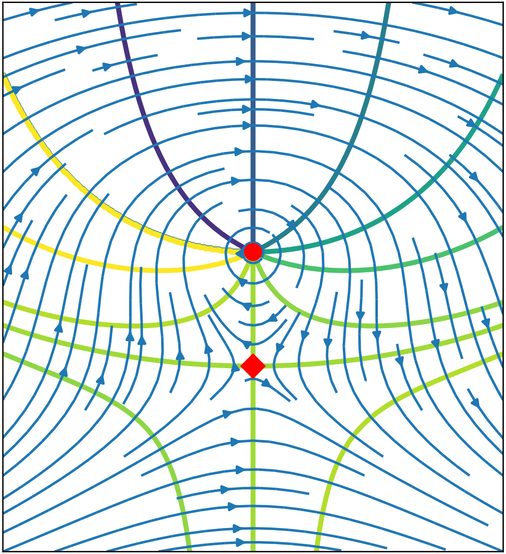




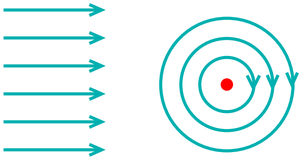
Shin group (Seoul)
PRL 2013



$$\psi = e^{ikx}\,(x - iy)$$

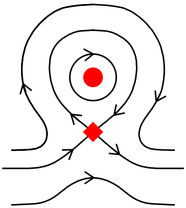


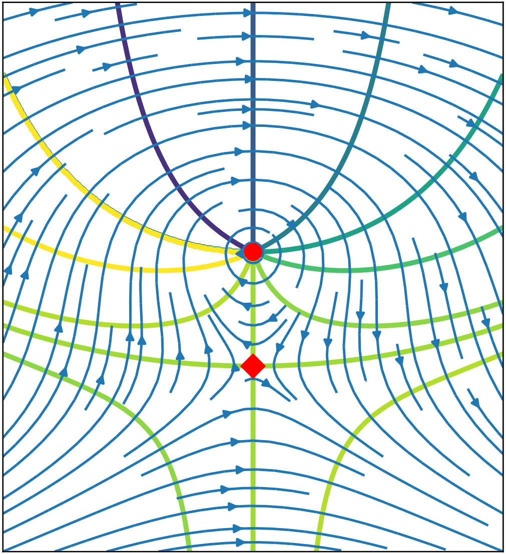
Saddles: model case

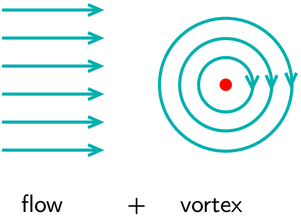


flow + vortex

$$\psi = e^{ikx}\,(x - iy)$$







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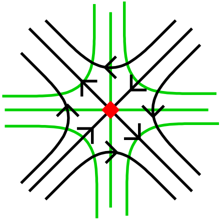
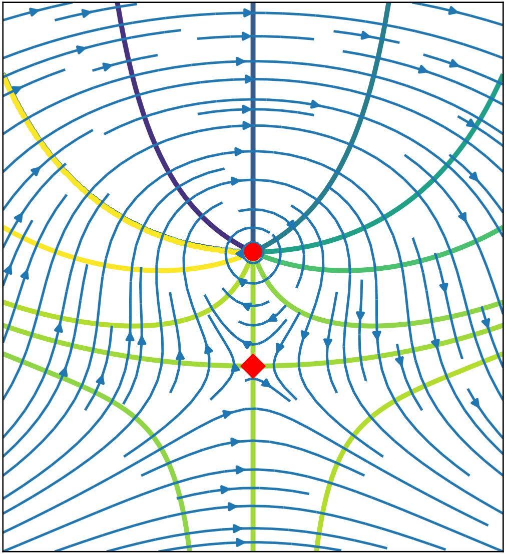




Other points with zero velocity: (elusive) nodes

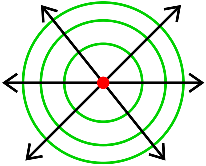
In a quantum fluid

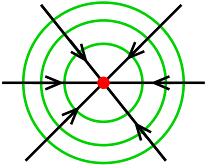
$$\Psi(\vec{r}) = A(\vec{r})\exp\{i\chi(\vec{r})\}$$

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}}\boldsymbol{\chi}$$

phase χ: velocity potential

min or max of χ:





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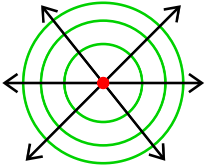
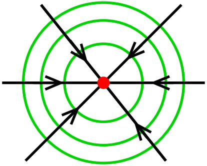




Nodes are forbidden:

- in an incompressible fluid:

$$\vec{\nabla} \cdot \vec{v} = 0 \Longrightarrow \vec{\nabla}^2 \chi = 0$$

Hence not seen, e.g., in a Coulomb gas model

- in a stationnary configuration:

No sink nor source !

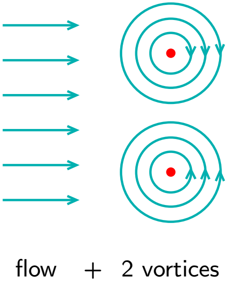
$$\vec{\nabla}^2 \psi + [f(A) + U(\vec{r}) - \mu]\psi = 0 \qquad (1)$$

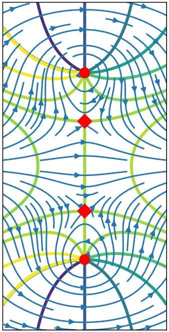
where $\psi = A \exp(i\chi)$:

$$\vec{\nabla}^2 \psi = \left\{ \vec{\nabla}^2 A - A|\vec{\nabla}\chi|^2 + i(A\vec{\nabla}^2\chi + 2\vec{\nabla}A \cdot \vec{\nabla}\chi) \right\} e^{iS}$$

at a stagnation point where $\vec{\nabla}\chi = 0$,

the imaginary part of (1) yields $\vec{\nabla}^2\chi = 0$





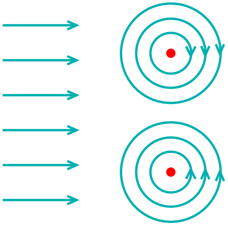
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

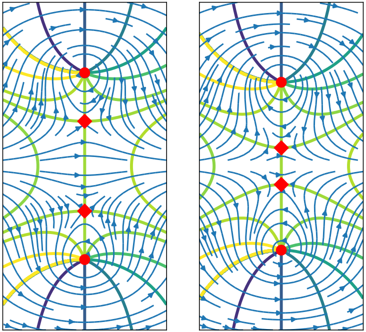
$k^2 b =$ 1.3

Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



flow + 2 vortices



solution of $(\Delta + k^2)\psi = 0$

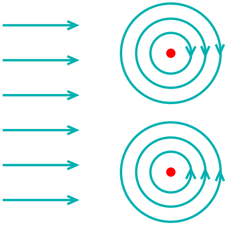
$$\psi = e^{ikx}\left[x - ik(y^2 - b)\right]$$

$k^2 b =$

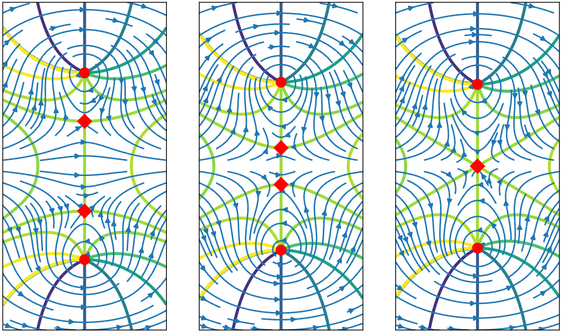
1.3	1.05	

Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



flow + 2 vortices



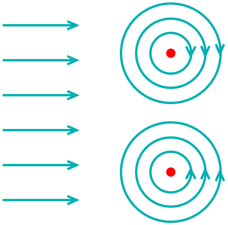
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

$k^2 b =$

1.3	1.05	1

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



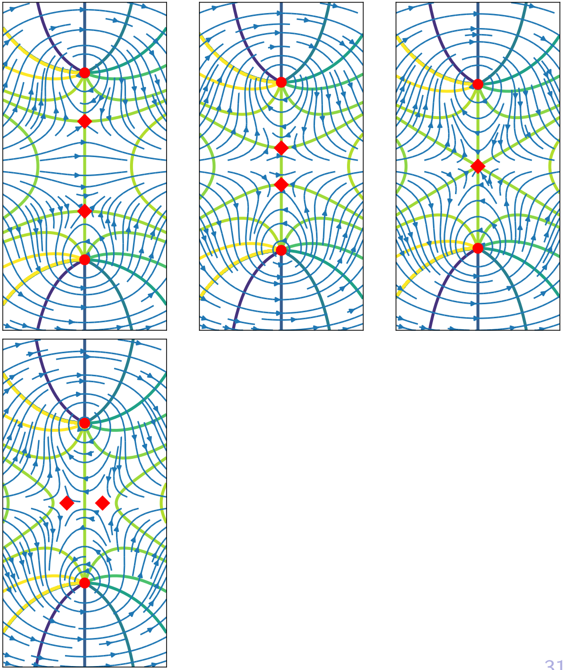
flow + 2 vortices

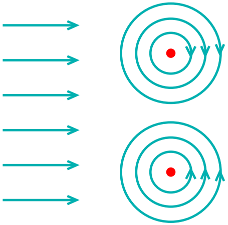
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{\mathrm{i}kx}\,[x - \mathrm{i}k(y^2 - b)]$$

$k^2 b =$

1.3	1.05	1
0.95		





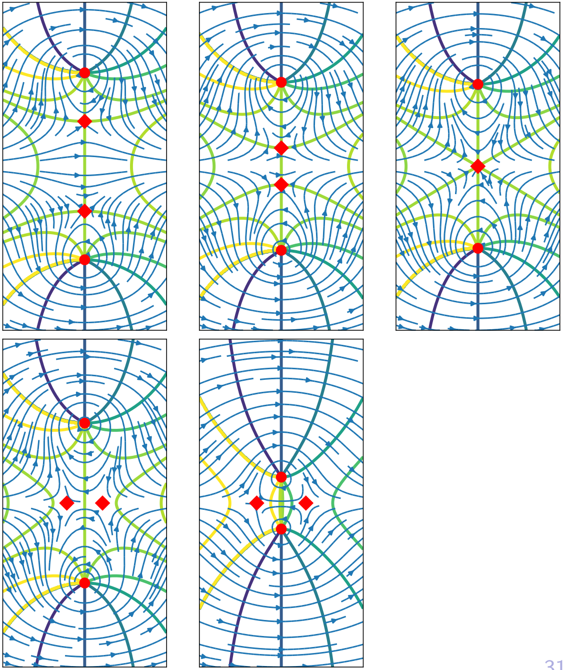
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

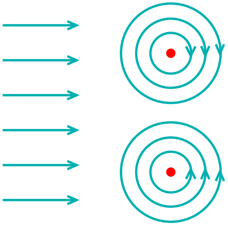
$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

$k^2 b =$

1.3	1.05	1
0.95	0.1	



Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



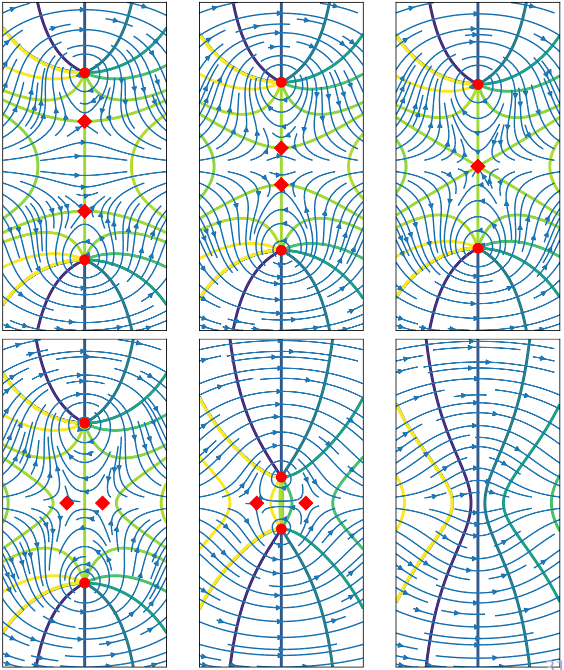
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

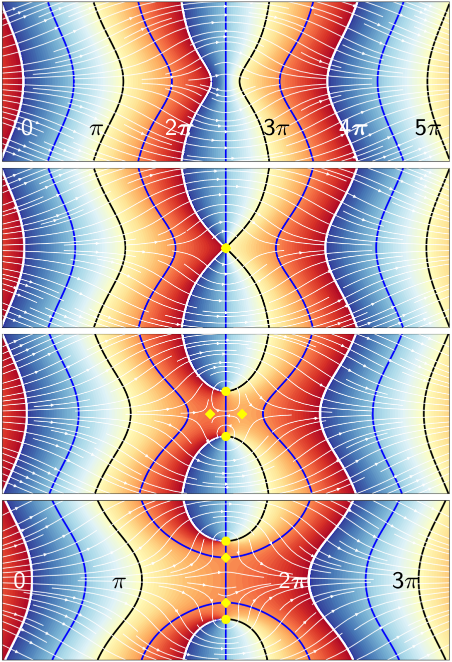
$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

$k^2 b =$

1.3	1.05	1
0.95	0.1	-0.1



Bristol $\leftrightarrow$ phase slippage

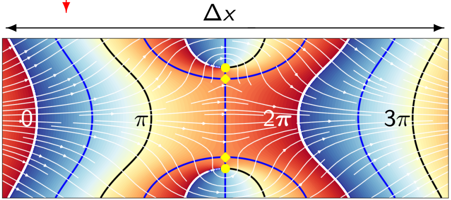


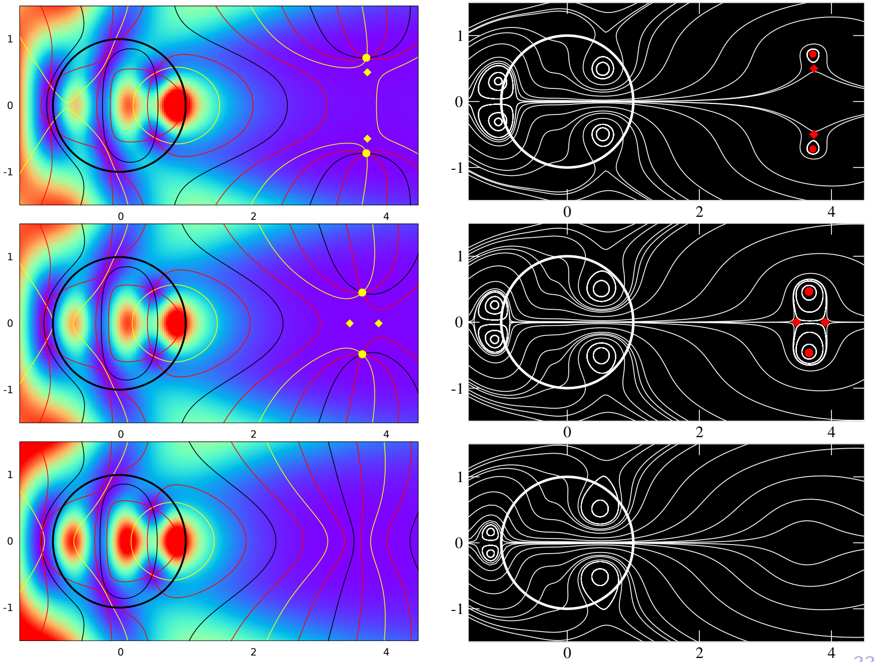
Initially:

$\Delta\chi = 5\pi$ along $y = 0$ and fixed Δx

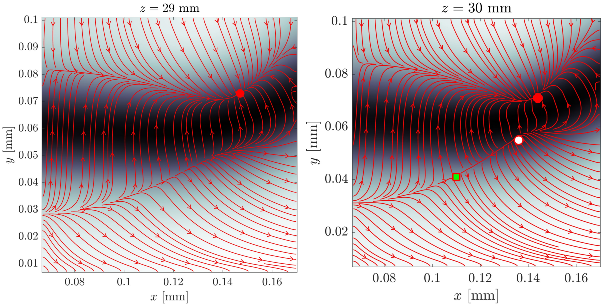
Eventually:

$\Delta\chi = 3\pi$ along $y = 0$, same Δx

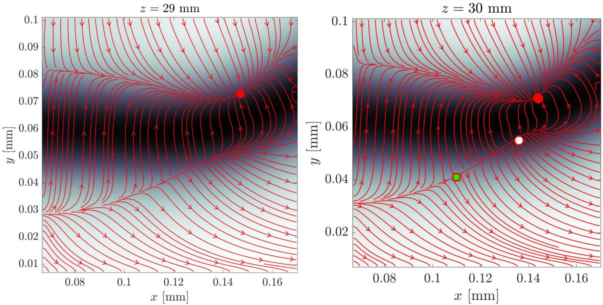



1
0
-1
0
2
4

Formation of saddles and nodes: saddle-node bifurcation



$\Phi_2 = 0.96\,\pi$

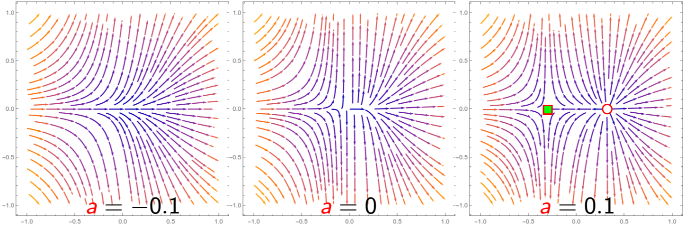


$\Phi_2 = 0.96\,\pi$

Saddle-node:

$$\begin{cases} v_x = x^2 - a \\ v_y = y \end{cases}$$

$a \in \mathbb{R}$



$$\vec{v} = \vec{\nabla}\left(\tfrac{1}{3}x^3 - ax + \tfrac{1}{2}y^2\right)$$

orbitally equivalently: $\vec{v} = \vec{\nabla}\chi$ where

$\chi(\vec{r}) = (almost)any\ fct\ of\ Z = \frac{1}{3}x^3 - ax + \frac{1}{2}y^2$