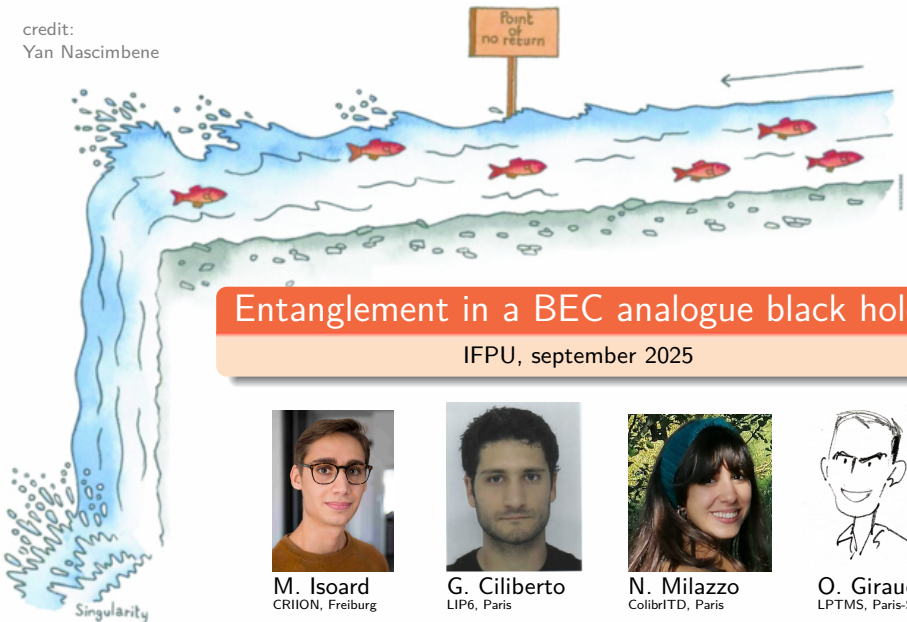


credit:
Yan Nascimbene



Entanglement in a BEC analogue black hole

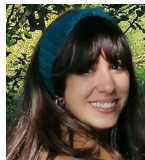
IFPU, september 2025



M. Isoard
CRIION, Freiburg



G. Ciliberto
LIP6, Paris



N. Milazzo
ColibriTD, Paris



O. Giraud
LPTMS, Paris-Saclay



J. C. Maxwell (1870)

The recognition of the formal analogy between two systems leads to a knowledge of both, more profound than could be obtained by studying each system separately.

J. Bouveresse (1999)

Prodiges et vertiges de l'analogie

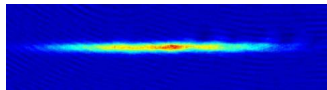
(Wonders and miracles of analogy)

Chez les déconstructionnistes, on fait de la théorie à peu près comme on fait de la poésie ou de la musique.

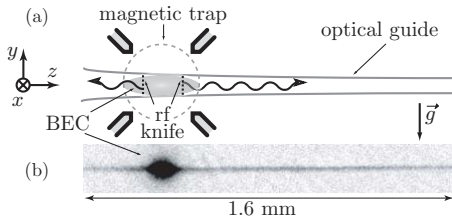
Among deconstruction philosophers theory is performed in much the same way as poetry or music.



	general relativity	BEC analogues
Hawking radiation ?	YES $\neq 0\rangle$ for incoming and outgoing modes	YES for the same reason relies on kinematic features
Thermality ?	YES black hole thermodynamics	roughly YES (but disputed) if $\lambda \rightarrow \infty$ is sound-like
Entanglement	YES $ BH\rangle = 2$ modes squeezed state for Schwarzschild observer	YES rich structure because 3 modes



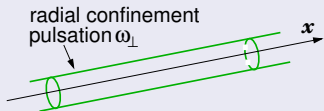
quasi-1D condensate
 longitudinal size $\sim 10^2 \mu\text{m}$
 transverse size $\sim 1 \mu\text{m}$



Guerin *et al.*, Phys. Rev. Lett. (2006)

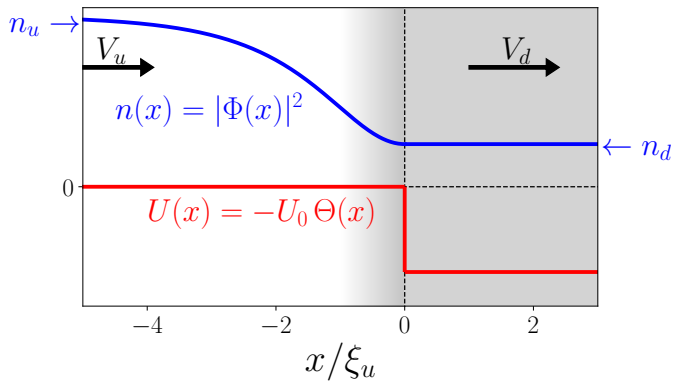
low T , quantum,
low c , 1D.

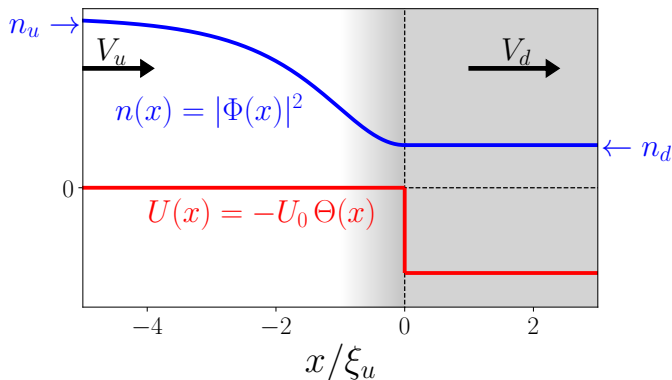
tight harmonic radial confinement :



$$V_{\perp}(\vec{r}_{\perp}) = \frac{1}{2} m \omega_{\perp}^2 r_{\perp}^2 .$$

→ **1D model** : $\Psi(x, t)$

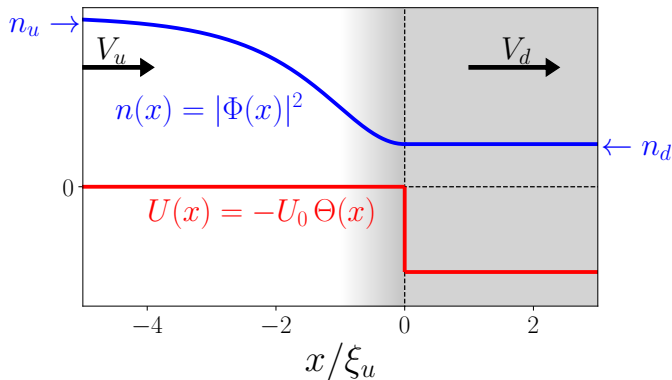




$$-\frac{\hbar^2}{2m} \psi_{xx} + \left(U(x) + g|\Phi|^2 \right) \psi = \mu \psi$$

$$\frac{V_d}{V_u} = \frac{n_u}{n_d} = \left(\frac{c_u}{V_u} \right)^2 = \frac{V_d}{c_d} = \left(\frac{c_u}{c_d} \right)^2$$

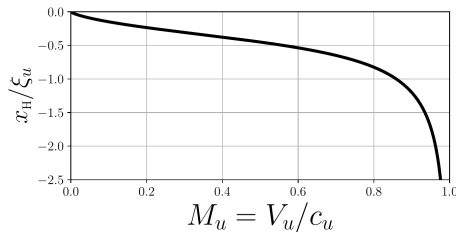
$$M_{u/d} = \frac{V_{u/d}}{c_{u/d}} : \quad M_d = M_u^{-2}$$

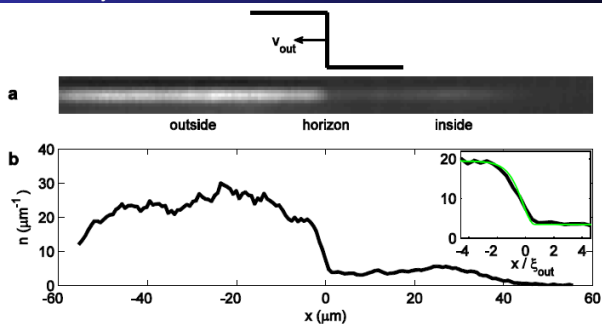


$$-\frac{\hbar^2}{2m} \psi_{xx} + (U(x) + g|\Phi|^2) \psi = \mu \psi$$

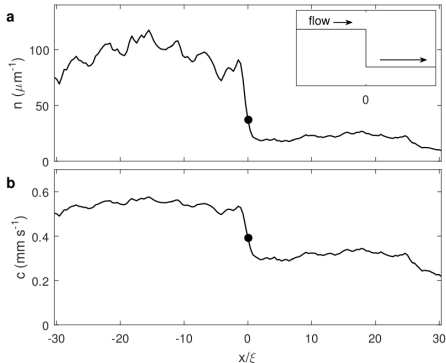
$$\frac{V_d}{V_u} = \frac{n_u}{n_d} = \left(\frac{c_u}{V_u} \right)^2 = \frac{V_d}{c_d} = \left(\frac{c_u}{c_d} \right)^2$$

$$M_{u/d} = \frac{V_{u/d}}{c_{u/d}} : \quad M_d = M_u^{-2}$$





2016



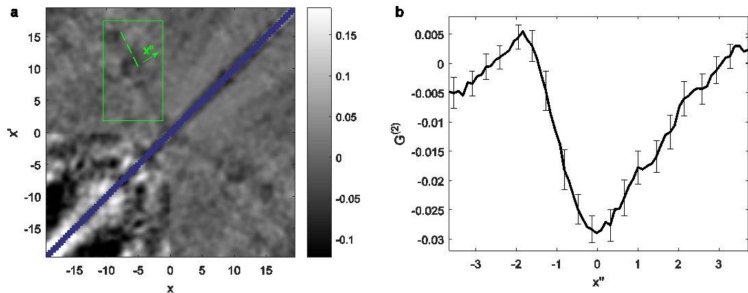
2019

$$M_d = V_d/c_d$$

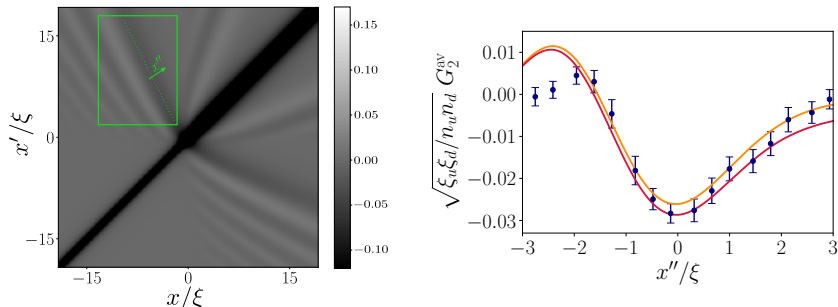
$$M_d|_{\text{exp}} = 2.9 = M_d|_{\text{theo}}$$

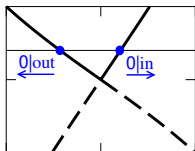
$$T_H/(gn_u)|_{\text{exp}} = 0.124 \quad (0.35 \text{ nK})$$

$$T_H/(gn_u)|_{\text{theo}} = 0.106$$

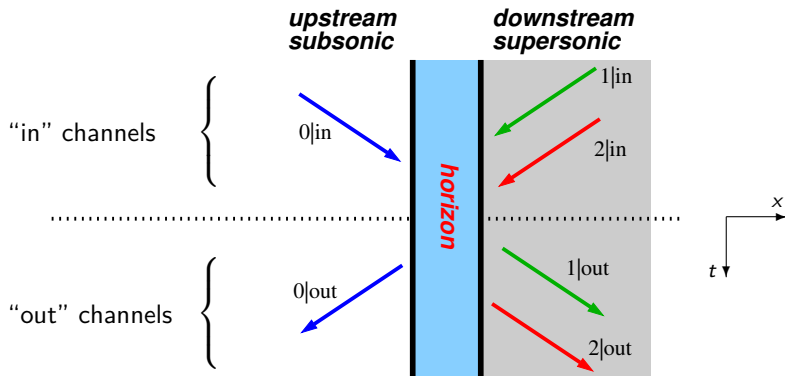
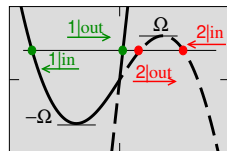


$$T_H|_{\text{exp}} = 0.35 \text{ nK} \quad T_H/(gn_u)|_{\text{exp}} = 0.124 \quad T_H/(gn_u)|_{\text{theo}} = 0.106 \quad \left(\frac{v_d}{c_d} = 2.9 \right)$$



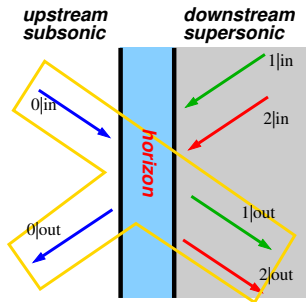


Propagation channels

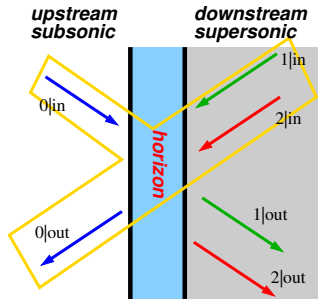


quantization modes: either 3 ingoing modes or 3 outgoing ones

example of ingoing mode: $\hat{b}_0$

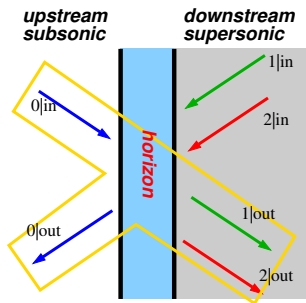


example of outgoing mode: $\hat{c}_0$

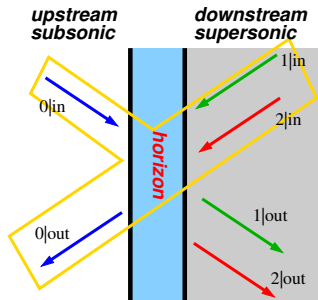


quantization modes: either 3 ingoing modes or 3 outgoing ones

example of ingoing mode: $\hat{b}_0$



example of outgoing mode: $\hat{c}_0$



$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix}_{out} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}_{in}$$

$$\begin{cases} \hat{c}_0 = \text{Hawking mode} \\ \hat{c}_1 = \text{Comoving=Spectator=Witness=Companion} \\ \hat{c}_2 = \text{Partner} \end{cases}$$

Bogoliubov modes are related through:

$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix}_{out} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}_{in}$$

Theoretical tool: 6×6 covariance matrices $\sigma^{(b)}$ and $\sigma^{(c)}$:

$$\sigma_{ij} = \frac{1}{2} \langle \hat{\xi}_i \hat{\xi}_j + \hat{\xi}_j \hat{\xi}_i \rangle - \langle \hat{\xi}_i \rangle \langle \hat{\xi}_j \rangle \quad \text{with} \quad \xi = \sqrt{2}(\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)$$

$$\hat{q}_i = \frac{1}{\sqrt{2}}(\hat{b}_i + \hat{b}_i^\dagger) \quad \text{for } \sigma^{(b)} \quad \text{and} \quad \hat{q}_i = \frac{1}{\sqrt{2}}(\hat{c}_i + \hat{c}_i^\dagger) \quad \text{for } \sigma^{(c)}$$

at $T = 0$, $\rho = |0\rangle_b \langle 0|$, $\sigma^{(b)} = \mathbb{1}_6$ and $\sigma^{(c)}$ is 6×6 : $\sigma^{(c)} = \mathcal{S} \sigma^{(b)} \mathcal{S}^T$
 where $\mathcal{S}$ is a symplectic matrix: $\mathcal{S}^T \Omega \mathcal{S} = \Omega$ with $\Omega = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}$.

By another (local unitary) symplectic transform one can write

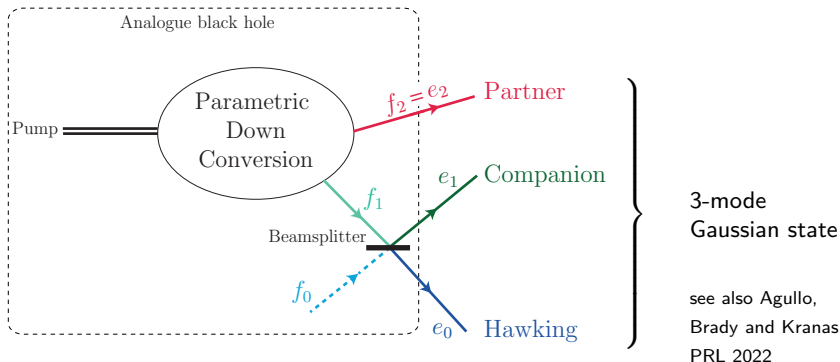
$$\mathcal{S} \sigma^{(c)} \mathcal{S}^T = \mathbb{1}_2 \oplus \sigma_{sq},$$

where σ_{sq} is the 4×4 covariance matrix of a two mode squeezed state

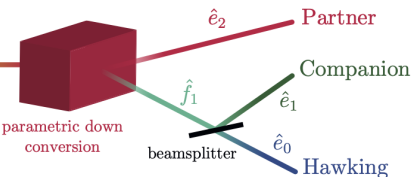
Alternative approach to that of Busch and Parentani PRD (2014), Finazzi and Carusotto PRA (2014) and de Nova, Zapata and Sols NJP (2015)

$$|0\rangle_b = \exp \left\{ r_2 \left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2 \right) \right\} |0\rangle_c, \text{ where } \sinh^2 r_2 = |S_{22}|^2 - 1 = |S_{02}|^2 + |S_{12}|^2$$

$$\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} = \begin{pmatrix} -\sin \theta & \cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_0 \\ \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}, \quad \cos \theta = \frac{|S_{02}|}{\sqrt{|S_{22}|^2 - 1}} \quad \text{and} \quad \begin{aligned} \hat{e}_0 &= (|S_{02}|/|S_{02}|) \hat{c}_0 \\ \hat{e}_1 &= (|S_{12}|/|S_{12}|) \hat{c}_1 \\ \hat{e}_2 &= (S_{22}/|S_{22}|) \hat{c}_2 \end{aligned}$$



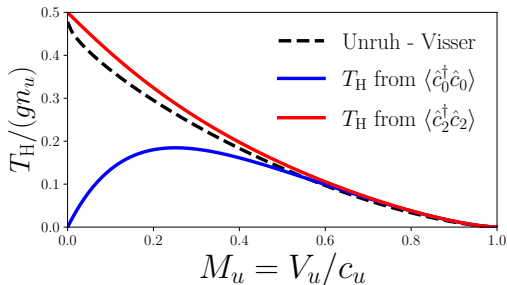
$$|0\rangle_b = \exp \left\{ r_2 \left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2 \right) \right\} |0\rangle_c$$



Hawking temperature

$$\langle \hat{f}_2^\dagger \hat{f}_2 \rangle = \langle \hat{f}_1^\dagger \hat{f}_1 \rangle = \frac{1}{\exp(\hbar\omega / T_{\text{eff}}(\omega)) - 1}$$

$$T_H = \lim_{\omega \rightarrow 0} T_{\text{eff}}(\omega) = \lim_{\omega \rightarrow 0} \hbar\omega |S_{22}(\omega)|^2$$



Gray body factor

$$\Gamma_0 = \lim_{\omega \rightarrow 0} \underbrace{\cos^2 \theta}_{\text{transmission}} = \frac{4M_u}{(1 + M_u)^2}$$

← identical to Anderson, Fabbri, Balbinot PRD 2015 and Linder, Schützhold, Unruh PRD 2016

non monotonous (qualitative: Yes/No)

- Cauchy Schwarz criterion
- PPT criterion (Generalized Peres-Horodecki parameter)

de Nova, Sols, Zapata PRA 2014

Busch, Carusotto, Parentani PRA 2014

Busch Parentani, PRD 2014

Finazzi, Carusotto, PRA 2014

Steinhauer PRD 2015

Boiron, Fabbri, Larré, Pavloff, Westbrook, PRL 2015

de Nova, Sols, Zapata, NJP 2015

Coutant, Weinfurter, PRD 2018

entanglement monotones (quantitative)

- entanglement entropy (pure state) generalizes to entanglement of formation
- (squared) logarithmic negativity generalizes to **Gaussian contangle**^a which verifies the monogamy inequalities

$$G^{(i|jk)} \geq G^{(i|j)} + G^{(i|k)}$$

^aAdesso, Illuminati, NJP 2006

Giovanazzi, PRL 2011

Horstmann, Schützhold, Reznik, Fagnocchi, Cirac, NJP 2011

Bruschi, Friis, Fuentes, Weinfurter, NJP 2013

Jacquet, König, SciPost 2020

Nambu, Osawa, PRD 2021

$\rho_{(i)}$: single mode reduced density matrix.

Purity = $\text{tr}[\rho_{(i)}^2]$

local mixedness $a_i = 2\langle \hat{c}_i^\dagger \hat{c}_i \rangle + 1 = (\text{purity})^{-1}$

bipartite entanglement (say between 0 and 2)

– trace out mode 1

– partial transpose $\sigma^{\text{PT}} = \Lambda \sigma \Lambda$, with $\Lambda = \sigma_z \oplus \mathbb{1}_2$

– compute the symplectic eigenvalues $\nu_{(\pm)}^{\text{PT}}$

– separability: $\rho^{\text{PT}} \geq 0 \iff$ all $\nu^{\text{PT}} \geq 1$

– $\nu_+^{\text{PT}} > 1$, hence criterion for non-separability: $\nu_-^{\text{PT}} < 1$

$$\nu_-^{\text{PT}} = \frac{a_0 + a_2}{2} - \sqrt{\left(\frac{a_0 - a_2}{2}\right)^2 + 4|\langle \hat{c}_0 \hat{c}_2 \rangle|^2}$$

The corresponding value of the “PPT measure” is $\Lambda^{(0|2)} = 1 - \nu_-^{\text{PT}}$

recipe for continuous variables

Martin & Vennin, PRA 2016

- (1) define a pseudo-spin operator $\hat{\mathbf{S}}^{(i)}(\omega)$ for each mode ($i = 0, 1, 2, \omega \geq 0$)
- (2) use this pseudo-spin to write a two-mode CHSH operator $\hat{\mathcal{B}}^{(i|j)}$
- (3) show that the expectation value $\langle \hat{\mathcal{B}}^{(i|j)} \rangle$ may violate a Bell inequality.

(1) Gour-Khanna-Mann-Revzen pseudo-spins

Phys. Lett. A 2004

$|n\rangle$: eigenvalue of $\hat{c}_i^\dagger(\omega)\hat{c}_i(\omega)$ associated with eigenvalue n .

$|q\rangle$: eigenvector of $\hat{q}_i(\omega) = \frac{1}{\sqrt{2}} (\hat{c}_i(\omega) + \hat{c}_i^\dagger(\omega))$ associated with eigenvalue q .

$$\hat{S}_z^{(i)} = \sum_n (|2n\rangle\langle 2n| - |2n+1\rangle\langle 2n+1|) = \int dq |q\rangle\langle -q|$$

$$\hat{S}_y^{(i)} = \dots \quad \hat{S}_x^{(i)} = \dots$$

$$[\hat{S}_x^{(i)}, \hat{S}_y^{(i)}] = 2i\hat{S}_z^{(i)}$$

$\hat{\mathbf{S}} \cdot \mathbf{n}$ squares to unity for any normalized vector $\mathbf{n} \leadsto$ eigenvalues ± 1 .

Beware: infinitely degeneracy! all $|2n\rangle$ ($|2n+1\rangle$) are eigenvectors of $\hat{S}_z^{(i)}$ with eigenvalue $+1$ (-1)

(2) CHSH operator

choose 4 measurement directions (normalized vectors $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{b}'$) and define

$$\hat{\mathcal{B}}^{(ij)}(\omega) = (\mathbf{a} + \mathbf{a}') \cdot \hat{\mathbf{S}}^{(i)} \otimes \mathbf{b} \cdot \hat{\mathbf{S}}^{(j)} + (\mathbf{a} - \mathbf{a}') \cdot \hat{\mathbf{S}}^{(i)} \otimes \mathbf{b}' \cdot \hat{\mathbf{S}}^{(j)} \quad (1)$$

Local realism

(λ : hidden variable)

the outcome of the measurement of $\mathbf{a} \cdot \hat{\mathbf{S}}^{(i)} \otimes \mathbf{b} \cdot \hat{\mathbf{S}}^{(j)}$ is the product

- of the measurement $M^{(i)}(\lambda, \mathbf{a}) = \pm 1$ of $\mathbf{a} \cdot \hat{\mathbf{S}}^{(i)}$
- with the measurement $M^{(j)}(\lambda, \mathbf{b}) = \pm 1$ of $\mathbf{b} \cdot \hat{\mathbf{S}}^{(j)}$

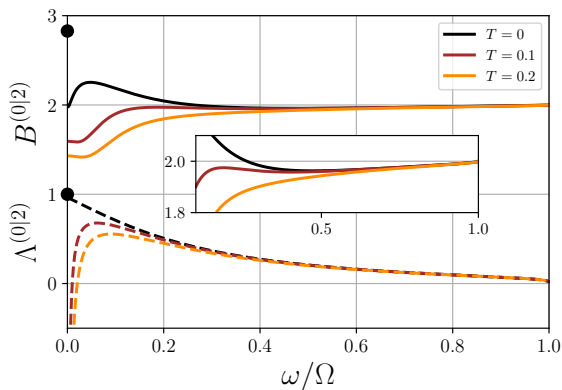
There are only two possible cases: either $M^{(i)}(\lambda, \mathbf{a})$ and $M^{(j)}(\lambda, \mathbf{a}')$ are equal (their difference cancels and they sum to ± 2) either they are opposite (their sum cancels and their difference is ± 2).

It follows from (1) that such a measurement of $\hat{\mathcal{B}}^{(ij)}$ yields a result ± 2 .

$$\text{local realism} \leadsto \text{Bell inequality : } -2 \leq \langle \hat{\mathcal{B}}^{(ij)}(\omega) \rangle \leq 2$$

(3) Violation of Bell inequality

$$B^{(i|2)}(\omega) \equiv \max_{a,a',b,b'} \langle \hat{\mathcal{B}}^{(i|2)}(\omega) \rangle \quad (i = 0, 1)$$

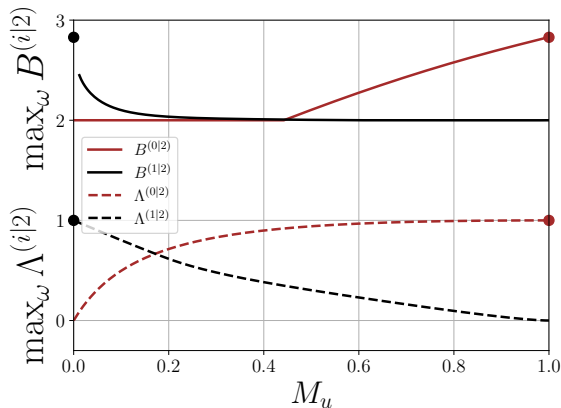


0 = Hawking
1 = Companion
2 = Partner

$$M_u = 0.59$$

(3) Violation of Bell inequality

$$B^{(i|2)}(\omega) \equiv \max_{a,a',b,b'} \langle \hat{\mathcal{B}}^{(i|2)}(\omega) \rangle \quad (i = 0, 1)$$



0 = Hawking
1 = Companion
2 = Partner

$T = 0$

assumption: Gaussian state

$$\Lambda^{(0|2)} = 1 - \frac{a_0 + a_2}{2} + \sqrt{\left(\frac{a_0 - a_2}{2}\right)^2 + 4|\langle \hat{c}_0 \hat{c}_2 \rangle|^2}$$

$$B^{(0|2)} = 2 \sqrt{\frac{4}{\pi^2} \arctan^2 \left(\frac{2|\langle \hat{c}_0 \hat{c}_2 \rangle|}{\sqrt{A_{02}}} \right) + \frac{1}{A_{02}^2}}.$$

where

$$a_i = 2\langle \hat{c}_i^\dagger \hat{c}_i \rangle + 1 \quad A_{02} = a_0 a_2 - 4|\langle \hat{c}_0 \hat{c}_2 \rangle|^2,$$

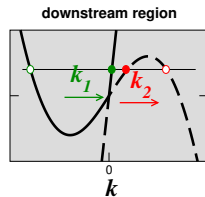
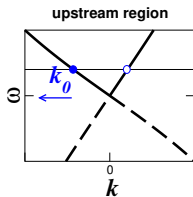
$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix}_{out} = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ S_{1,0} & S_{1,1} & S_{1,2} \\ S_{2,0} & S_{2,1} & S_{2,2} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}_{in}$$

Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega) \hat{c}_2(\omega) \rangle = S_{0,2}(\omega) S_{2,2}^*(\omega)$$

$$\propto \int_{-L_u}^0 dx \int_0^{L_d} dx' e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x, x')$$

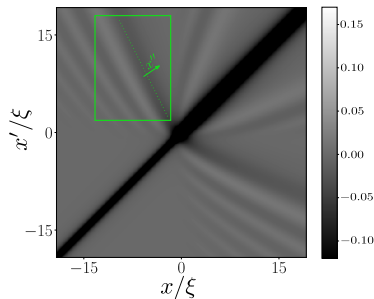
G_2 : density-density correlation function



Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega) \hat{c}_2(\omega) \rangle = S_{0,2}(\omega) S_{2,2}^*(\omega)$$

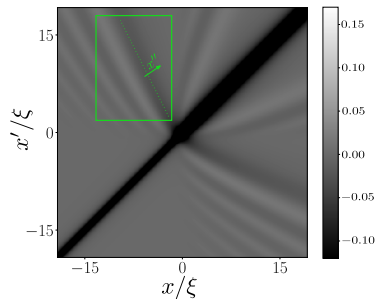
$$\propto \int_{-L_u}^0 dx \int_0^{L_d} dx' e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x, x')$$



Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega) \hat{c}_2(\omega) \rangle = S_{0,2}(\omega) S_{2,2}^*(\omega)$$

$$\propto \int_{-L_u}^0 dx \int_0^{L_d} dx' e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x, x')$$



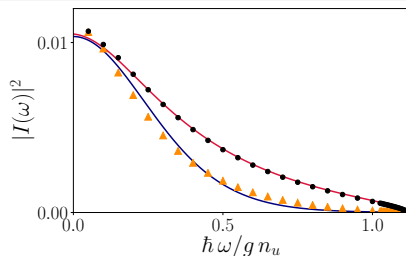
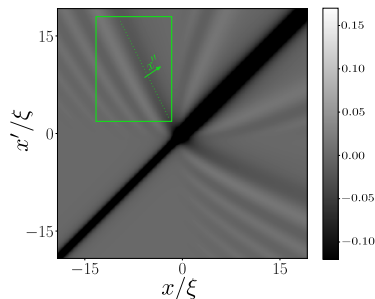
de Nova, Sols, Zapata, NJP 2015:

$$\frac{L_u}{|\partial\omega/\partial q(k_0)|} = \frac{L_d}{\partial\omega/\partial q(k_2)}$$

Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega) \hat{c}_2(\omega) \rangle = S_{0,2}(\omega) S_{2,2}^*(\omega)$$

$$\propto \int_{-L_u}^0 dx \int_0^{L_d} dx' e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x, x')$$



blue curve: fully thermal radiation

- $|S_{0,2}(\omega)|^2 = n_{th}$ at $T_H|_{exp}$
- discard mode 1
- $|\langle \hat{c}_0 \hat{c}_2 \rangle|^2 \simeq n_{th}(1 + n_{th})$

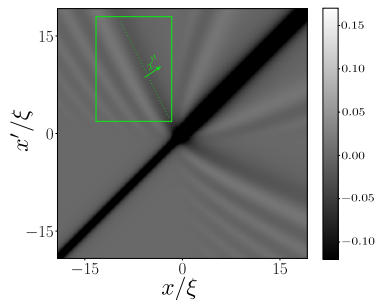
de Nova, Sols, Zapata, NJP 2015:

$$\frac{L_u}{|\partial\omega/\partial q(k_0)|} = \frac{L_d}{\partial\omega/\partial q(k_2)}$$

Steinhauer, PRD 2015:

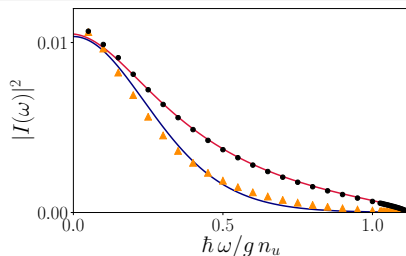
$$\langle \hat{c}_0(\omega) \hat{c}_2(\omega) \rangle = S_{0,2}(\omega) S_{2,2}^*(\omega)$$

$$\propto \int_{-L_u}^0 dx \int_0^{L_d} dx' e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x, x')$$



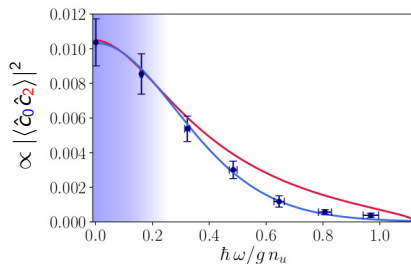
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$$\frac{L_u}{|\partial\omega/\partial q(k_0)|} = \frac{L_d}{\partial\omega/\partial q(k_2)}$$



blue curve: fully thermal radiation

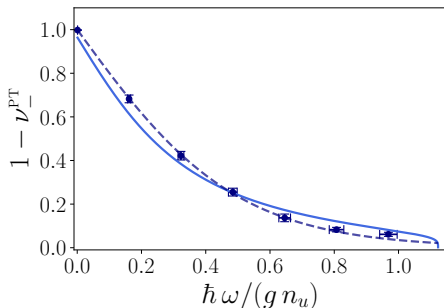
- $|S_{0,2}(\omega)|^2 = n_{th}$ at $T_H|_{exp}$
- discard mode 1
- $|\langle \hat{c}_0 \hat{c}_2 \rangle|^2 \simeq n_{th}(1 + n_{th})$



Assume:

- that the state is Gaussian and that $T = 0$
- that the occupation of the companion is negligible (then $\langle \hat{c}_0^\dagger \hat{c}_0 \rangle = \langle \hat{c}_2^\dagger \hat{c}_2 \rangle$)
- that $\langle \hat{c}_0 \hat{c}_2 \rangle$ is accurately extracted from the density-density correlation signal

$$\nu_-^{\text{PT}} \simeq \sqrt{1 + 4|\langle \hat{c}_0 \hat{c}_2 \rangle|^2} - 2|\langle \hat{c}_0 \hat{c}_2 \rangle|$$



points from de Nova *et al.*
Nature 2019

solid line: waterfall model

dashed line:

$|\langle \hat{c}_0 \hat{c}_2 \rangle|^2 \simeq n_{th}(1 + n_{th})$ at $T_H|_{exp}$

$$\hat{S}_y^0 \otimes \hat{S}_y^1 \otimes \hat{S}_z^2 |0_{\omega=0}\rangle^{\text{in}} = +|0_{\omega=0}\rangle^{\text{in}}$$

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$|x_{\pm}\rangle$ eigenvector of $\hat{S}_x$ with eigenvalue ± 1

GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|x_+ x_- x_- \rangle + |x_- x_+ x_+ \rangle)$$

$$\hat{S}_y^0 \otimes \hat{S}_y^1 \otimes \hat{S}_z^2 |0_{\omega=0}\rangle^{\text{in}} = +|0_{\omega=0}\rangle^{\text{in}}$$

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$|x_{\pm}^n\rangle$ eigenvector of $\hat{S}_x$ with eigenvalue ± 1 $n \in \mathbb{N}$

degenerate GHZ state

$$|0_{\omega=0}\rangle^{\text{in}} = \sum_{n,l,m} C_{n,l,m} (|x_+^n x_-^l x_-^m\rangle + |x_-^n x_+^l x_+^m\rangle)$$

$$\hat{S}_y^0 \otimes \hat{S}_y^1 \otimes \hat{S}_z^2 |0_{\omega=0}\rangle^{\text{in}} = +|0_{\omega=0}\rangle^{\text{in}}$$

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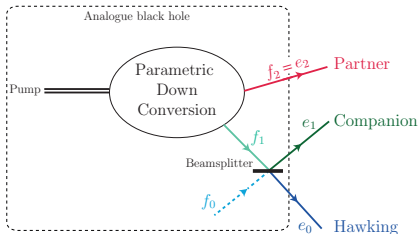
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effect of degeneracy

GHZ state whose
entanglement resists
partial tracing !

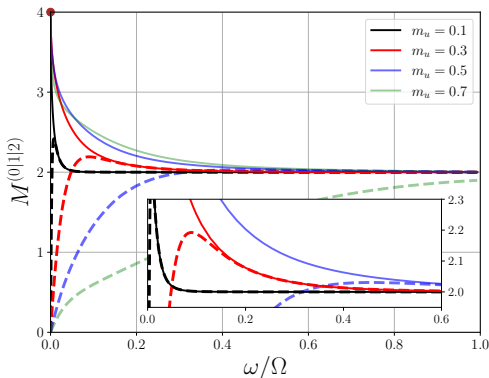


Mermin parameter

$$M^{(0|1|2)}(\omega) \equiv \max_{a,a',b,b',c,c'} |\langle \hat{\mathcal{M}}^{(0|1|2)}(\omega) \rangle|$$

where

$$\begin{aligned} \hat{\mathcal{M}}^{(0|1|2)}(\omega) = & -a \cdot \hat{S}^0 \otimes b \cdot \hat{S}^1 \otimes c \cdot \hat{S}^2 + a \cdot \hat{S}^0 \otimes b' \cdot \hat{S}^1 \otimes c' \cdot \hat{S}^2 \\ & + a' \cdot \hat{S}^0 \otimes b \cdot \hat{S}^1 \otimes c' \cdot \hat{S}^2 + a' \cdot \hat{S}^0 \otimes b' \cdot \hat{S}^1 \otimes c \cdot \hat{S}^2 \end{aligned}$$



solid: $T = 0$

dashed: $T = 0.1 g n_u$

BECs offer interesting prospects to observe analogous Hawking radiation

general perspective : **quantum effects** with **nonlinear matter waves**

BEC experiments (Steinhauer) provides indisputable evidences

➡ of the occurrence of a sonic horizon.

➡ of the associated acoustic Hawking radiation.

The thermality of analogous Hawking radiation is still a matter of debate.

👉 the quantum nature of the Hawking process



→ can be directly addressed by present time experimental setups



→ no indisputable result yet



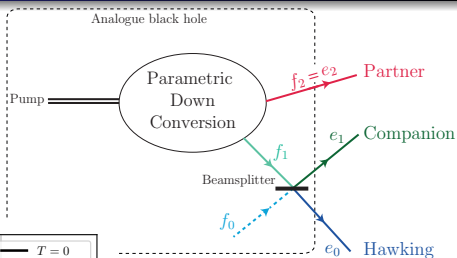
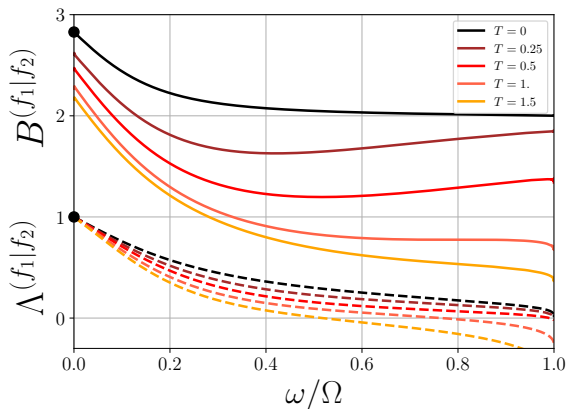
Gondret *et al.* PRL 2025 : proof of entanglement in pairs produced
via dynamical Casimir effect

Thank you for your attention

$$B^{(f_1|f_2)} = 2\sqrt{1 + \frac{4}{\pi^2} \arctan^2[\sinh(2r_2)]}$$

$$\Lambda^{(f_1|f_2)} = 1 - \exp(-2r_2)$$

$$r_2(\omega) = \langle \hat{c}_2^\dagger \hat{c}_2 \rangle$$



$$|0\rangle_b = \exp \left\{ r_2 \left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2 \right) \right\} |0\rangle_c$$

$$\text{Monogamy inequality : } G^{(i|jk)} \geq G^{(i|j)} + G^{(i|k)}$$

☞ $(i|jk)$ pure Gaussian state: squared logarithmic negativity $(\ln \|\rho^{\text{PT}}\|)^2$

$$G^{(i|jk)} = 4 r_i^2 = 4 \operatorname{arsinh}^2 \left(\sqrt{\langle \hat{c}_i^\dagger \hat{c}_i \rangle} \right)$$

☞ $\rho^{(i|j)}$ mixed (k has been traced out) $G^{(i|j)}$: Gaussian contangle

$$G^{(0|1)} = 0 \quad G^{(j|2)} = \operatorname{arsinh}^2 \left(\frac{2|\langle \hat{c}_j \hat{c}_2 \rangle|}{1 + \langle \hat{c}_k^\dagger \hat{c}_k \rangle} \right)$$

Monogamy inequality : $G^{(i|jk)} \geq G^{(i|j)} + G^{(i|k)}$

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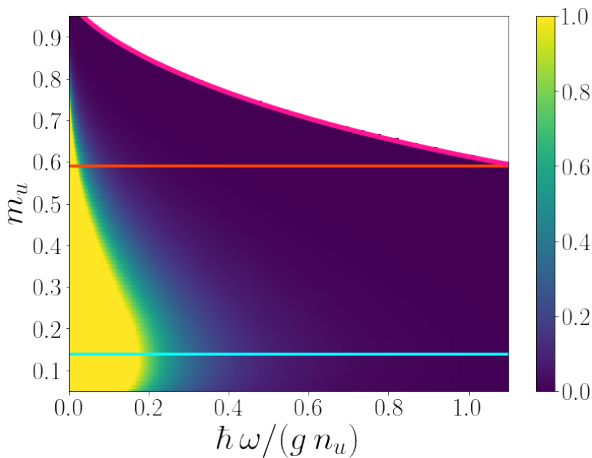
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residual contangle:

$$G_{\text{res}} = \min \left(G^{(i|jk)} - G^{(i|j)} - G^{(i|k)} \right)$$



residual contangle:

$$G_{res} = \min (G^{(i|jk)} - G^{(i|j)} - G^{(i|k)})$$

