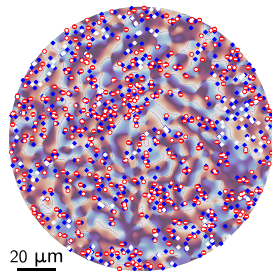
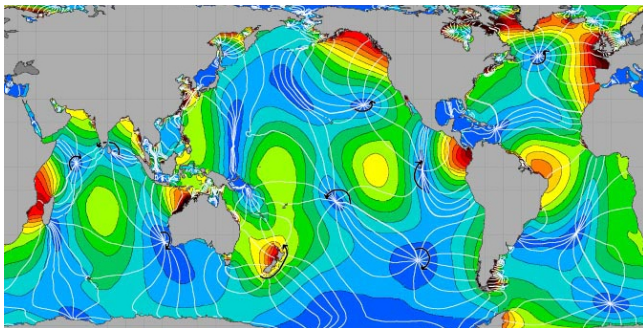


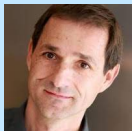
Topological pathways to two-dimensional quantum turbulence

Nicolas Pavloff, LPTMS, Université Paris-Saclay





P. Azam
DGA Toulouse



R. Kaiser
Inst. Phys. Nice



T. Congy
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G. Ciliberto
LIP6, Paris



R. Panico
Univ. Bonn



G. Martone
Nanotec Lecce



D. Ballarini
Nanotec Lecce



A. Lanotte
Nanotec Lecce

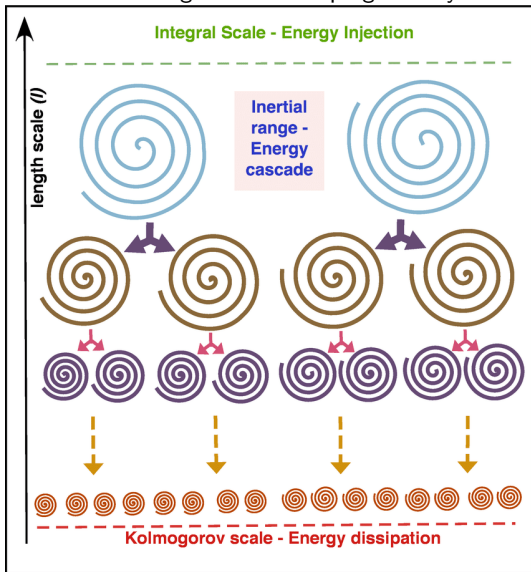
Refs:

Congy, Azam, Kaiser, Pavloff, Phys. Rev. Lett. **132**, 033804 (2024),

Panico et al., Phys. Rev. Res. **7**, L022063 (2025).

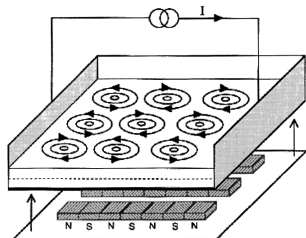
3D: Richardson-Kolmogorov cascade

Breakdown of large eddies into progressively smaller ones

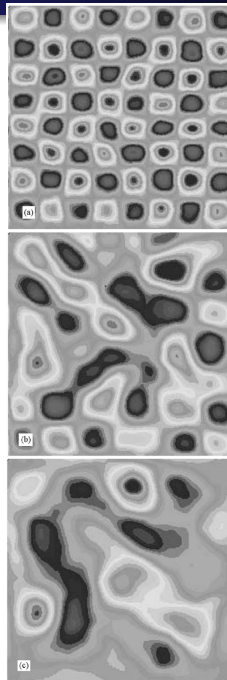


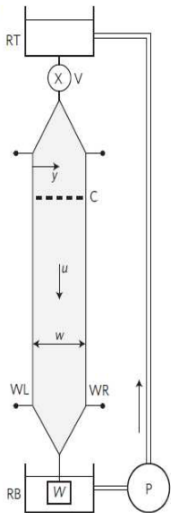
Richardson 1922:

«Big whirls have little
whirls, that feed on their
velocity,
and little whirls have
lesser whirls and so on to
viscosity.»



Thin layer of electrolyte
vortices driven by electromagnetical forces
P. Tabeling, Phys. Rep. (2002)

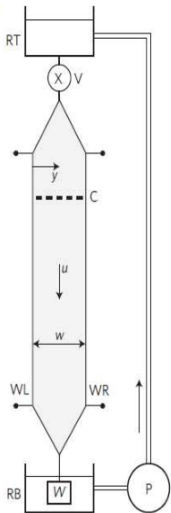




Tuan Tran et al., Nat. Phys. 2010

M. A. Rutgers, PRL 1998

Y. Couder, J.M. Chomaz, M. Rabaud, Physica D 1989



Great Red Spot



smaller vortices and storm clusters can be seen being engulfed by the GRS but formation dynamics is disputed
Sánchez-Lavega et al.
Geophys. Res. Lett. 2024
1939: formation of white ovals which merged to form "Red Jr." in 2020

Tuan Tran et al., Nat. Phys. 2010

M. A. Rutgers, PRL 1998

Y. Couder, J.M. Chomaz, M. Rabaud, Physica D 1989

velocity field $\vec{v}(\vec{r}, t)$, $\vec{r} = x\vec{e}_x + y\vec{e}_y \rightsquigarrow$ vorticity: $\vec{\nabla} \times \vec{v} = \omega(\vec{r}, t) \vec{e}_z$
point vortex i , position $\vec{r}_i(t)$, circulation Γ_i : $\omega(\vec{r}, t) = \sum_i \Gamma_i \delta(\vec{r} - \vec{r}_i(t))$

$$H = -\frac{1}{2\pi} \sum_{i>j} \Gamma_i \Gamma_j \ln |\vec{r}_i - \vec{r}_j|, \quad \text{writing } \vec{r}_i(t) = (x_i, y_i) \text{ yields } \begin{cases} \Gamma_i \dot{x}_i = \partial H / \partial y_i \\ \Gamma_i \dot{y}_i = -\partial H / \partial x_i \end{cases}$$

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Onsager 1949

Phase space has a finite volume:

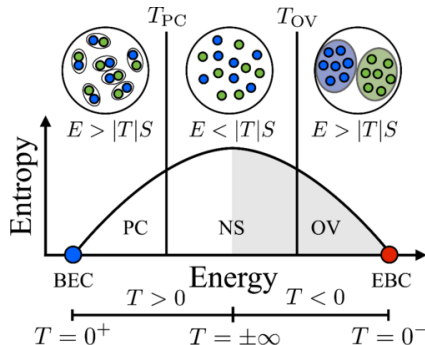
$$\int d\Omega = (\int dx dy)^N = A^N.$$

$$\Phi(E) \stackrel{\text{def}}{=} \int_{H < E} d\Omega = \int_{-\infty}^E \Phi'(E) dE.$$

$\Phi'(E)$ is positive with $\Phi'(\pm\infty) = 0$
(since $\Phi(-\infty) = 0$, $\Phi(+\infty) = A^N$)

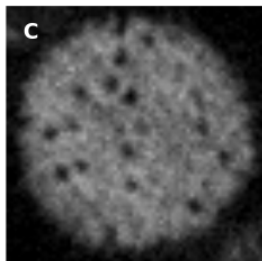
→ **negative temperatures**
(since $T^{-1} = (\partial S / \partial E) = \Phi'' / \Phi'$)

« If $T < 0$ vortices of the same sign will tend to cluster, so as to use up excess energy at the least possible cost in terms of degrees of freedom. »

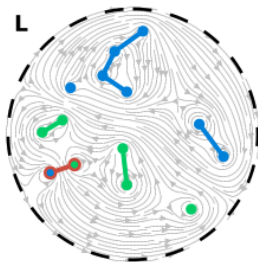


Simula, Davis, Helmersson PRL 2014

Vortex clustering in two dimensional superfluids

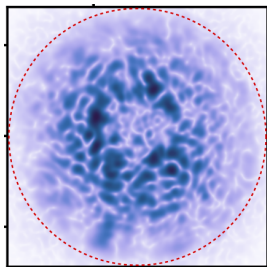


^{87}Rb atoms

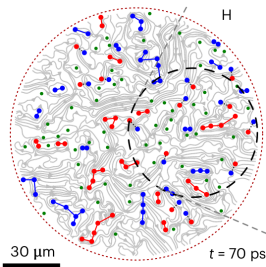


group of Helmerson and
Johnstone, Monash
University (Science 2019)

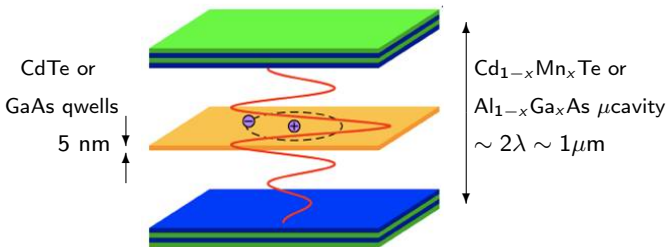
see also group of Neely,
Davis, and
Rubinsztein-Dunlop,
University of Queensland
(Science 2019)



exciton-polaritons (hybrid light-matter quasiparticles)



group of Sanvitto and
Ballarini,
CNR-Nanotec Lecce
(Nat. Photon. 2023)



interacting bosons

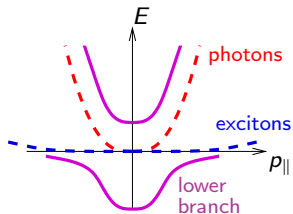
$$m_{\text{eff}} \lesssim 10^{-4} m_e$$

$$T_{\text{BEC}} \sim 10 \text{ K}$$

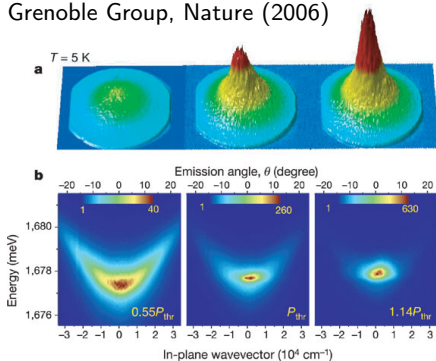
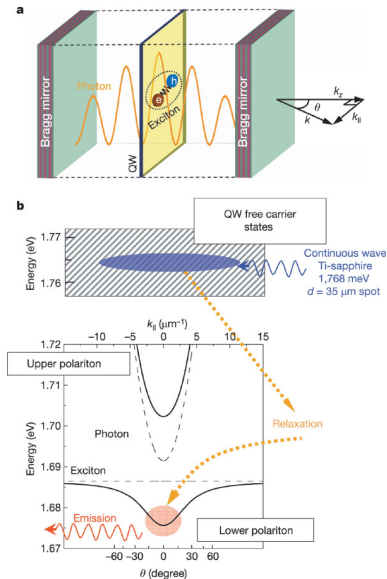
$$\text{lifetime} \lesssim 50 \text{ ps}$$

optical detection

polarization degree of freedom



Grenoble Group, Nature (2006)



Grenoble: institut Néel
Paris-Saclay: C2N
Paris: LKB

What is so special about superfluids ?

Superfluid order parameter: $\psi(\vec{r}, t) = A \exp(i \chi)$

→ ground state: stationary and uniform: $A(\vec{r}, t) = A_0$, $\chi(\vec{r}, t) = -\mu_0 t / \hbar$

→ Galilean boost, velocity $\vec{v}_0$: $\chi(\vec{r}, t) = (-\mu t + m \vec{v}_0 \cdot \vec{r}) / \hbar$ ($\mu = \mu_0 + \frac{1}{2} m v_0^2$)

→ non uniform velocity: $\chi(\vec{r}, t)$ with $\vec{v}(\vec{r}, t) = \frac{\hbar}{m} \vec{\nabla} \chi$

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→ non uniform velocity: $\chi(\vec{r}, t)$ with $\vec{v}(\vec{r}, t) = \frac{\hbar}{m} \vec{\nabla} \chi$

This imposes $\vec{\nabla} \times \vec{v} = 0$: irrotational flow.

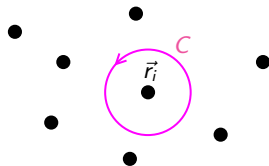
Except if there are points $\vec{r}_i$ where χ is ill defined, i.e. where $A(\vec{r}_i) = 0$.

Then $\vec{\nabla} \times \vec{v} = \omega(\vec{r}, t) \vec{e}_z$ with $\omega(\vec{r}, t) = \sum_i \Gamma_i \delta(\vec{r} - \vec{r}_i(t))$.

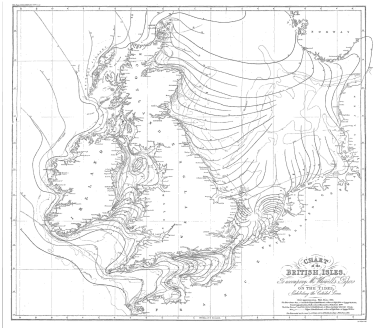
Onsager-Feynman quantization condition

$$\oint_C \vec{v} \cdot d\vec{r} = \Gamma_i = \frac{\hbar}{m} \oint_C d\chi = \frac{\hbar}{m} 2\pi n_i$$

$$n_i \in \mathbb{Z}, \quad \text{typically } n_i = \pm 1$$



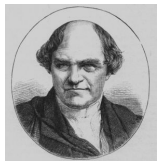
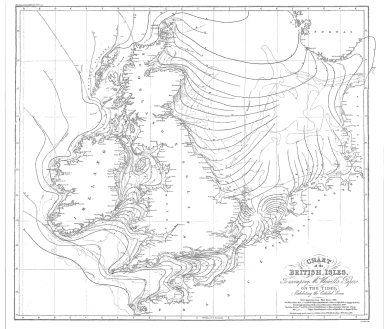
cotidal lines and amphidromic points. Whewell 1836



W. Whewell
Phil. Trans. R. Soc. 1836

Principal tidal constituent (M2: semi-diurnal)



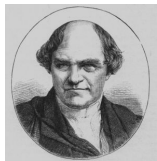
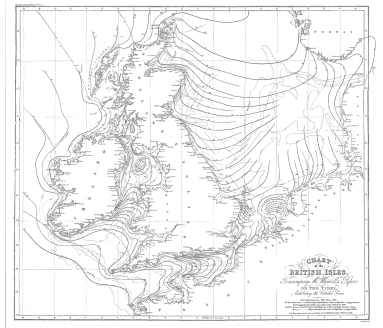


W. Whewell
Phil. Trans. R. Soc. 1836

Principal tidal constituent (M2: semi-diurnal)



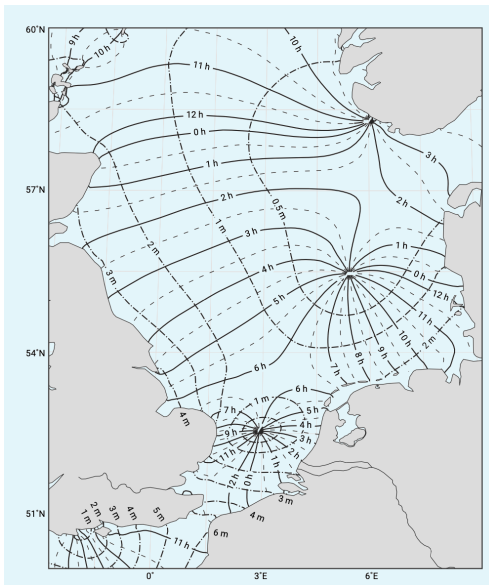
cotidal lines and amphidromic points. Whewell 1836



W. Whewell
Phil. Trans. R. Soc. 1836

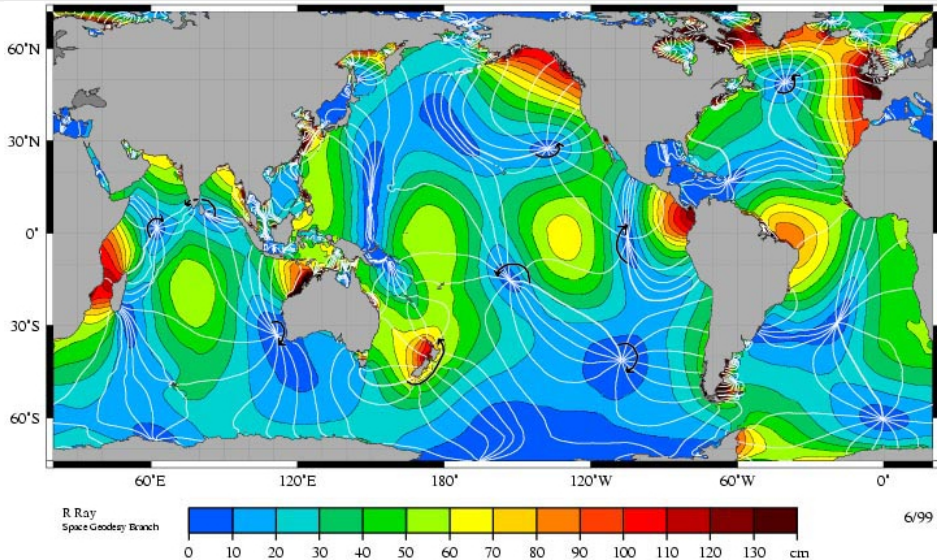
$$\psi = h(\vec{r}) \exp\{i(\chi(\vec{r}) - \omega t)\}$$

where $\omega = \omega_{M2} = 2\pi/T_2$



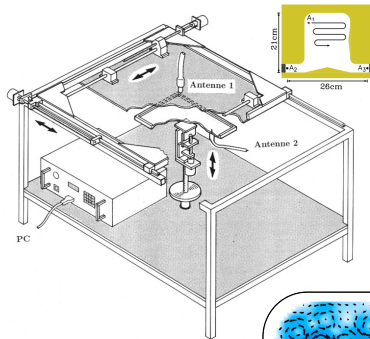
co-range and cotidal lines

Amphidromic M2 points

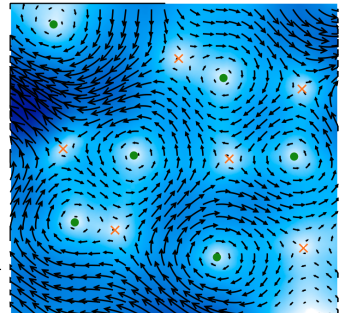
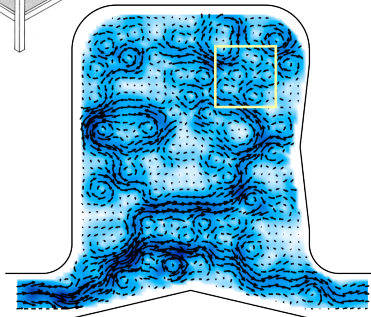


$$T_{M2} = 12 \text{ h } 25 \text{ min}, \Delta t_{\text{cotidal}} = 1 \text{ h } 2 \text{ min}, \omega_{M2} \times \Delta t_{\text{cotidal}} = \pi/6$$

⇒ quantized circulation of tidal current of the M2 component



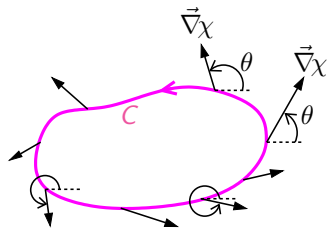
U. Kuhl, Eur. Phys. J. (2007)



$$\vec{r} = x\vec{e}_x + y\vec{e}_y$$

$$\Psi(\vec{r}) = A(\vec{r}) \exp\{i\chi(\vec{r})\}$$

$$\vec{v} = \frac{\hbar}{m} \vec{\nabla} \chi$$



$$I_V(C) = \frac{1}{2\pi} \oint_C \vec{\nabla} \chi \cdot d\vec{\ell} = \oint_C \frac{d\chi}{2\pi}$$

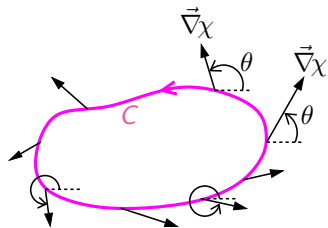
$$I_P(C) = \frac{1}{2\pi} \oint_C \vec{\nabla} \theta \cdot d\vec{\ell} = \oint_C \frac{d\theta}{2\pi}$$

I_V : vorticity, I_P : Poincaré index

$$\vec{r} = x\vec{e}_x + y\vec{e}_y$$

$$\Psi(\vec{r}) = A(\vec{r}) \exp\{i\chi(\vec{r})\}$$

$$\vec{v} = \frac{\hbar}{m} \vec{\nabla} \chi$$

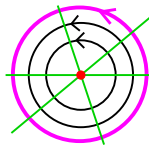


$$l_V(C) = \frac{1}{2\pi} \oint_C \vec{\nabla} \chi \cdot d\vec{\ell} = \oint_C \frac{d\chi}{2\pi}$$

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l_V : vorticity, l_P : Poincaré index

vortex



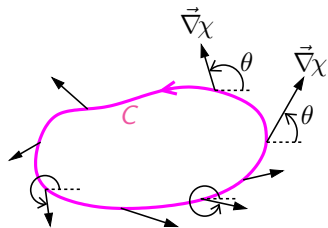
$$l_V \in \mathbb{Z}$$

$$l_P = +1$$

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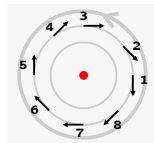


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vortex



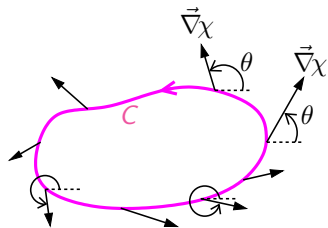
$$l_V < 0$$

$$l_P = +1$$

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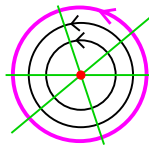


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l_V : vorticity, l_P : Poincaré index

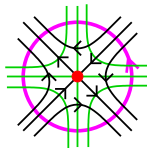
vortex



$$l_V \in \mathbb{Z}$$

$$l_P = +1$$

saddle



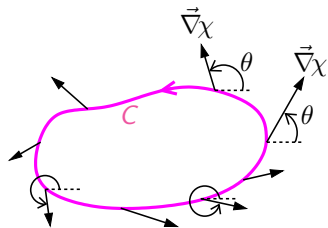
$$l_V = 0$$

$$l_P = -1$$

$$\vec{r} = x\vec{e}_x + y\vec{e}_y$$

$$\Psi(\vec{r}) = A(\vec{r}) \exp\{i\chi(\vec{r})\}$$

$$\vec{v} = \frac{\hbar}{m} \vec{\nabla} \chi$$

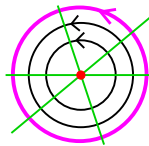


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l_V : vorticity, l_P : Poincaré index

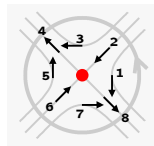
vortex



$$l_V \in \mathbb{Z}$$

$$l_P = +1$$

saddle



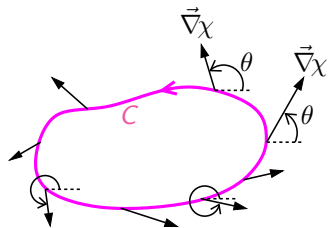
$$l_V = 0$$

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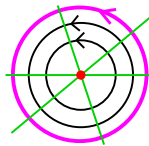


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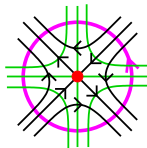
vortex



$$l_V \in \mathbb{Z}$$

$$l_P = +1$$

saddle

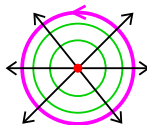


$$l_V = 0$$

$$l_P = -1$$

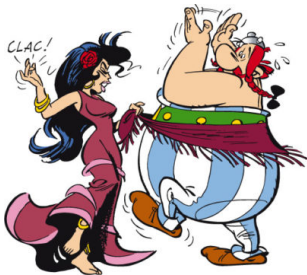
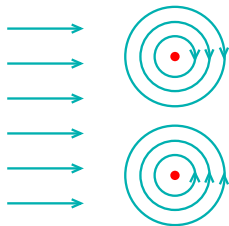
node

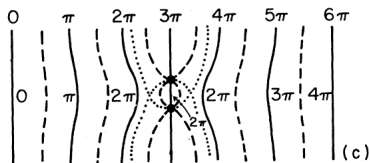
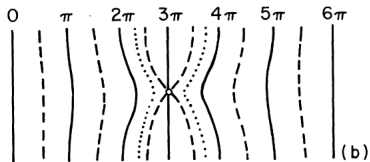
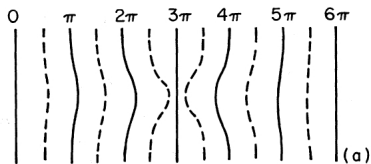
min/max of
the phase



$$l_V = 0$$

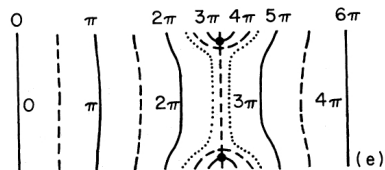
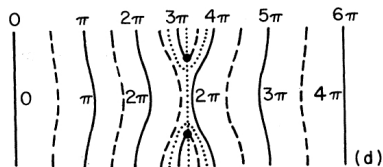
$$l_P = +1$$

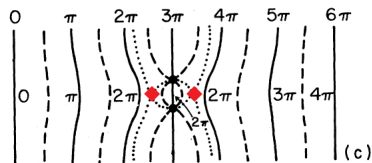
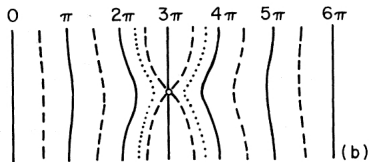
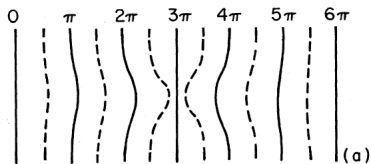




Anderson, RMP (1966)

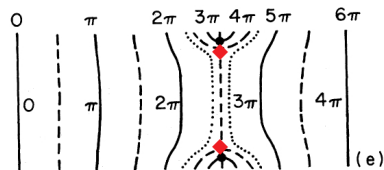
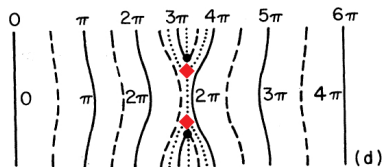
Langer, Fisher, PRL (1967)

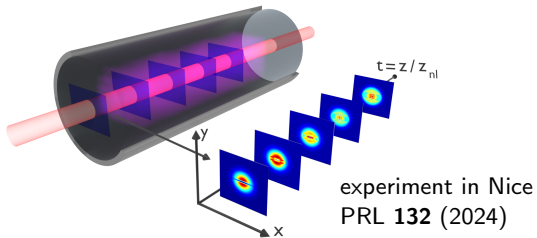




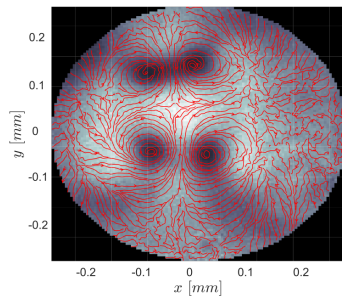
Anderson, RMP (1966)

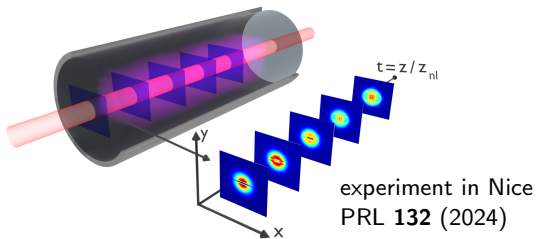
Langer, Fisher, PRL (1967)



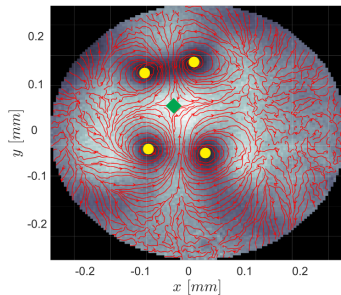


output intensity and streamlines





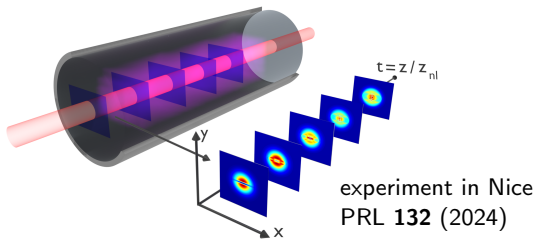
output intensity and streamlines



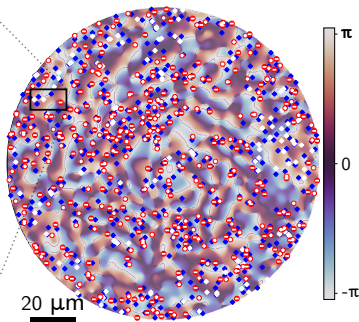
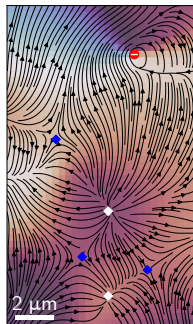
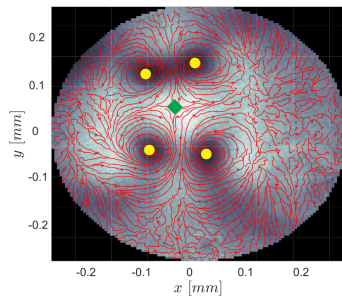
$$\vec{E}(z, \vec{r}_\perp) = \mathcal{A}(z, \vec{r}_\perp) \exp\{i(k_0 z - \omega t)\} \vec{e}_z \quad \text{complex amplitude} \times \text{carrier wave}$$

paraxial approximation:

$$i\partial_z \mathcal{A} = -\frac{1}{2n_0 k_0} \vec{\nabla}_\perp^2 \mathcal{A} + \frac{k_0 n_2 |\mathcal{A}|^2}{1 + |\mathcal{A}|^2 / I_{\text{sat}}} \mathcal{A} - \frac{i}{\Lambda_{\text{abs}}} \mathcal{A}$$



output intensity and streamlines

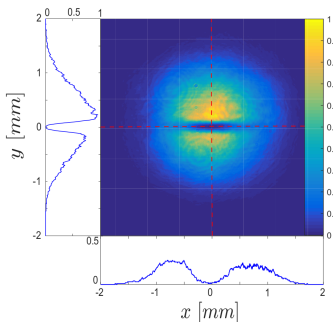
experiment in Lecce
PRR 7 (2025)

direct measure of the (large)
number of vortices, saddles and
nodes as a function of time

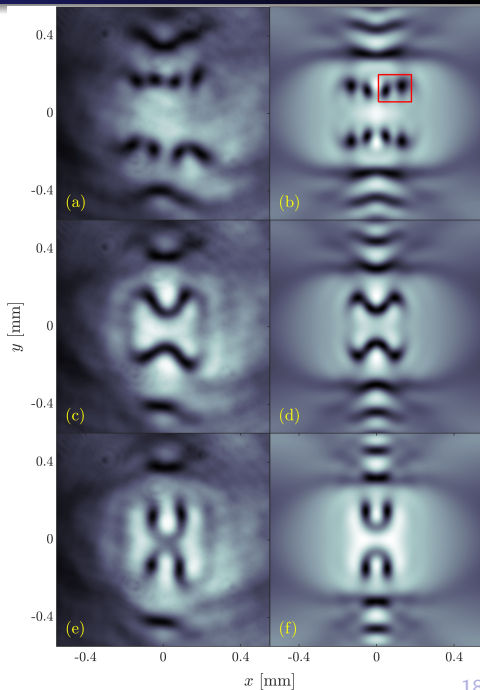
Reproducing the experimental results

$$\psi(\vec{r}, 0) = \sqrt{I_1} \exp\left(-\frac{r^2}{w_G^2}\right) + \sqrt{I_2} \exp\left(-\frac{x^2}{w_x^2} - \frac{y^2}{w_y^2}\right) e^{i\Phi_2}$$

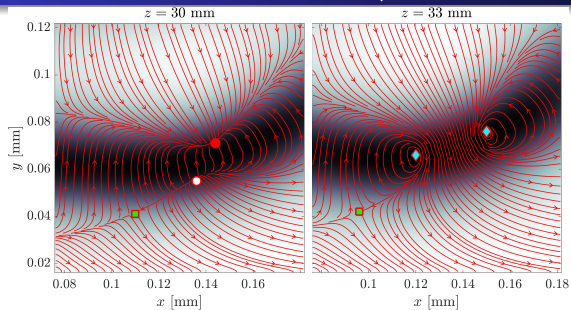
$$I_1 = I_2 \quad \Phi_2 \simeq \pi$$



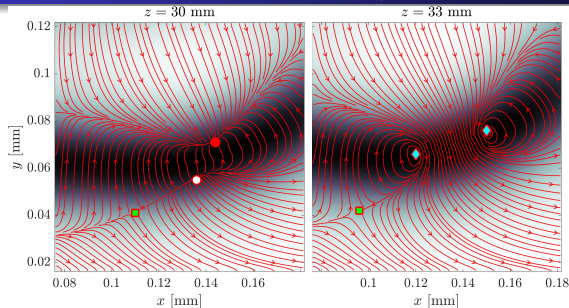
- (a,b) : $\Phi_2 = 0.96 \pi$
 (c,d) : $\Phi_2 = \pi$
 (e,f) : $\Phi_2 = 1.05 \pi$



Mechanism of vortex formation/annihilation: fold-Hopf bifurcation



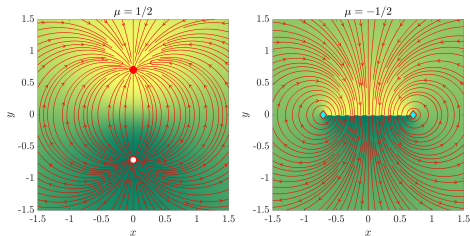
Mechanism of vortex formation/annihilation: fold-Hopf bifurcation



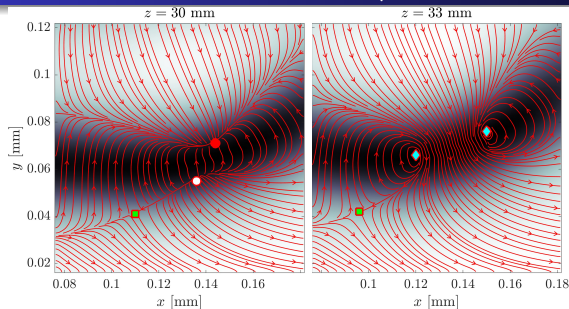
fold-Hopf bifurcation:

$$\begin{cases} v_x = -2\sigma xy \\ v_y = \mu + \sigma x^2 - y^2 \end{cases}$$

$$\sigma = 1, \mu \in \mathbb{R}$$



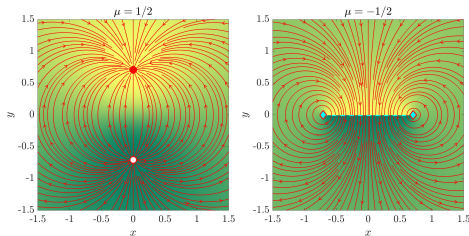
Mechanism of vortex formation/fold-Hopf bifurcation



fold-Hopf bifurcation:

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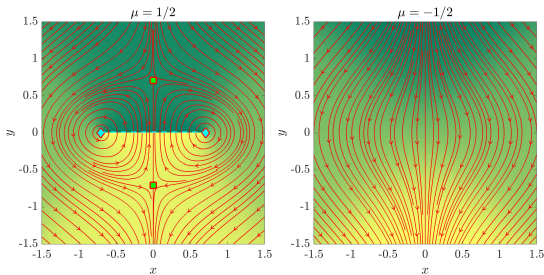
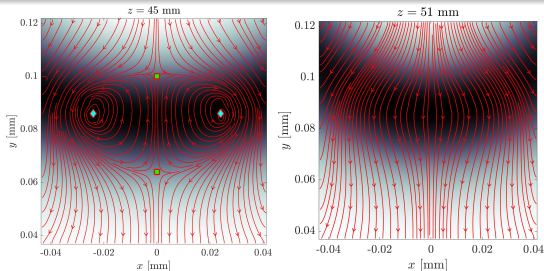
$\sigma = 1, \mu \in \mathbb{R}$



orbitally equivalent system: $\vec{v} = \vec{\nabla} \chi_{\text{fH}}, \chi_{\text{fH}}(\vec{r}) \equiv \arg [x^2 + \sigma(y^2 + \mu) + i\sigma y]$.

- ✓ gradient system
- ✓ verifies Onsager-Feynman quantization condition

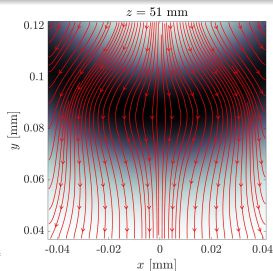
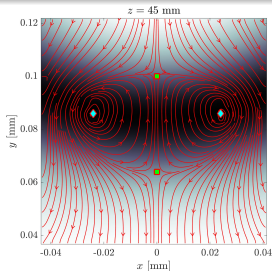
vortex formation/annihilation: Bristol mechanism ... also fold-Hopf



$$\vec{v} = \vec{\nabla} \chi_{\text{fH}} \quad \text{where} \quad \chi_{\text{fH}}(\vec{r}) \equiv \arg [x^2 + \sigma(y^2 + \mu) + i\sigma y]$$

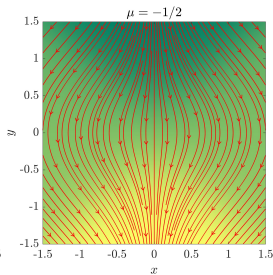
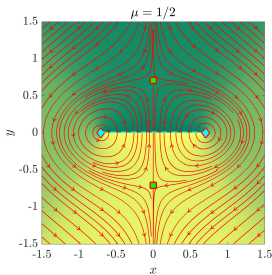
with $\sigma = -1$ and $\mu \in \mathbb{R}$

vortex formation/annihilation: Bristol mechanism ... also fold-Hopf



close to the bifurcation point

the phase of a model wave function (exact solution of Helmholtz equation) matches $\chi_{fH}(\vec{r})$

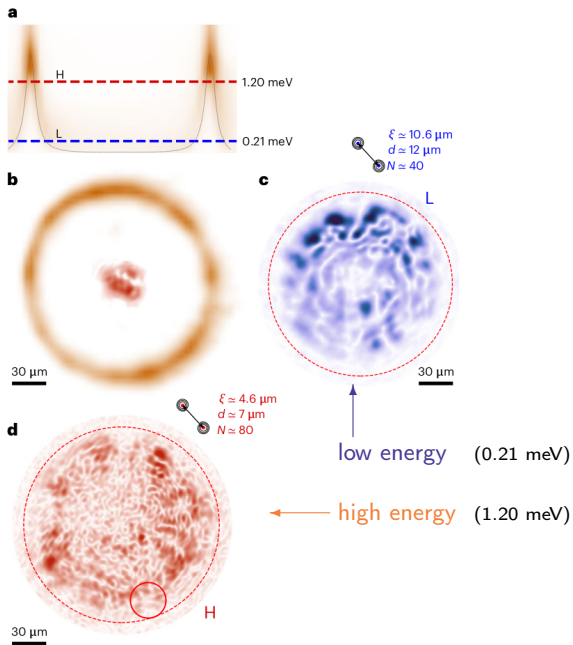


$$\vec{v} = \vec{\nabla} \chi_{fH}$$

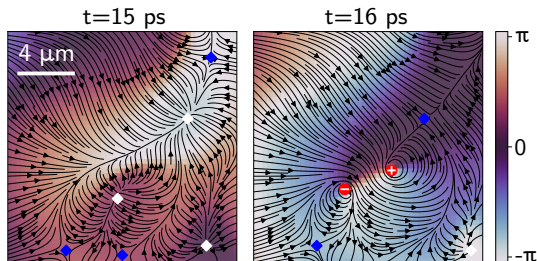
where
with

$$\chi_{fH}(\vec{r}) \equiv \arg [x^2 + \sigma(y^2 + \mu) + i\sigma y]$$

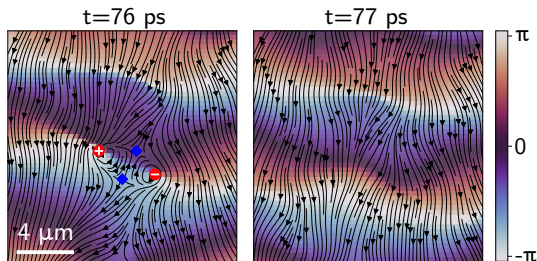
$$\sigma = -1 \quad \text{and} \quad \mu \in \mathbb{R}$$

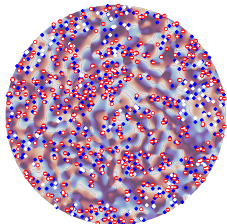


Node collision
(fold-Hopf with $\sigma = 1$)

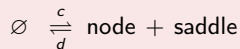
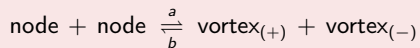


Bristol mechanism
(fold-Hopf with $\sigma = -1$)

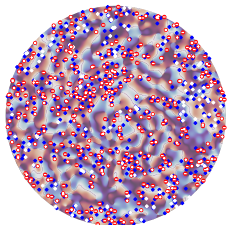




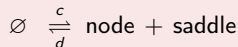
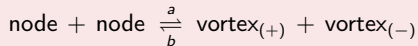
simplest possible processes



The (positive) quantities a , b , c , and d are the reaction rates.



simplest possible processes



The (positive) quantities a , b , c , and d are the reaction rates.

Rate equations:

$$\frac{dV_{\pm}}{dt} = aN^2 - bV_{+}V_{-},$$

$$\frac{dS}{dt} = c - dNS,$$

$$\frac{dN}{dt} = c - 2aN^2 + 2bV_{+}V_{-} - dNS,$$

where $N(t)$ denotes the number of nodes, $S(t)$ the number of saddles, and $V_{+}(t)$ [$V_{-}(t)$] the number of vortices with positive [negative] vorticity.

In the following $b = 0$

$$(b = 0) \quad \frac{dV_{\pm}}{dt} = aN^2, \quad \frac{dS}{dt} = c - dNS, \quad \frac{dN}{dt} = c - 2aN^2 - dNS,$$

No imparted angular momentum:

$$V_+(t) - V_-(t) = C^{st} \ll \text{typical} \left(V_+(t) + V_-(t) \right) \rightsquigarrow V_+(t) = V_-(t) \equiv \frac{V(t)}{2}$$

rescaled quantities

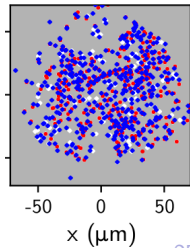
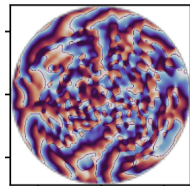
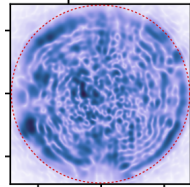
$$\tau = t/t_0, \quad n = N/N_0, \quad v = V/N_0, \quad s = S/N_0$$

$$\text{with } t_0 = 1/\sqrt{2ac} \text{ and } N_0 = \sqrt{c/2a}$$

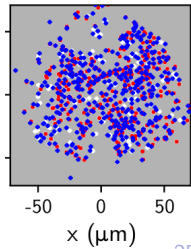
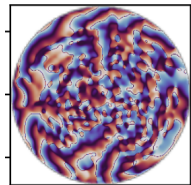
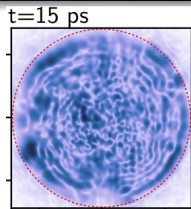
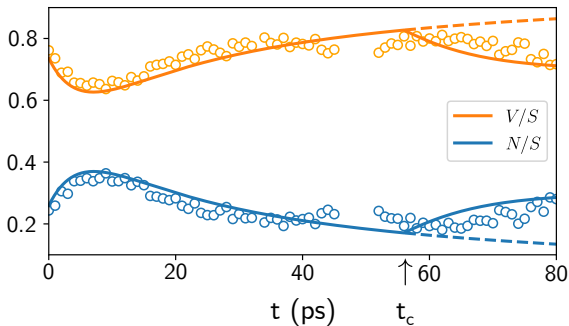
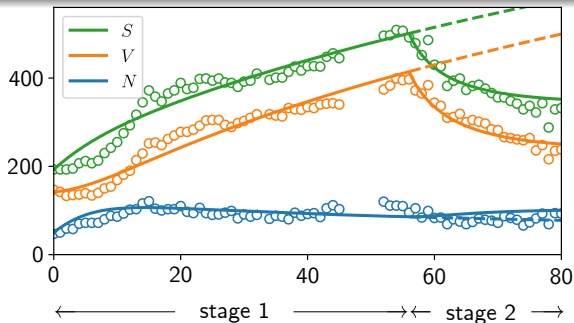
$$\boxed{\frac{dv}{d\tau} = n^2, \quad \frac{ds}{d\tau} = 1 - \gamma ns, \quad \frac{dn}{d\tau} = 1 - n^2 - \gamma ns,}$$

where the single parameter $\gamma \equiv d/2a$ governs the qualitative features of the dynamical system.

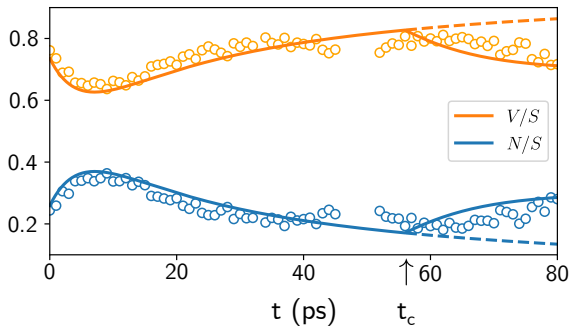
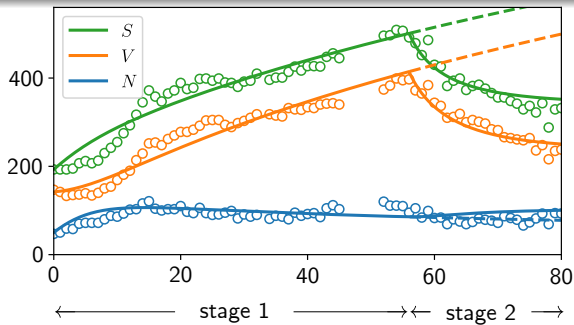
$t=15$ ps



experimental results



experimental results



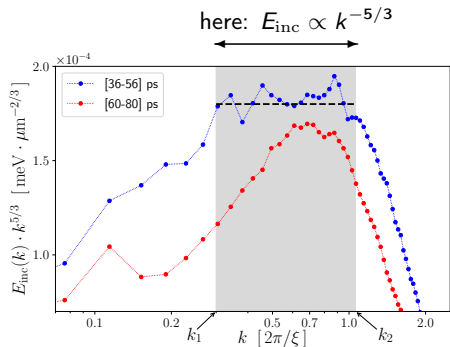
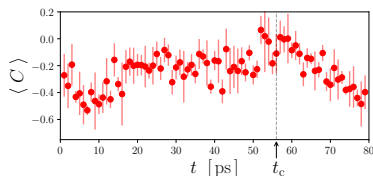
$$I_P = N(t) + V(t) - S(t) \\ = C^{st} \simeq 0$$

$t = t_c$: onset of turbulence decay

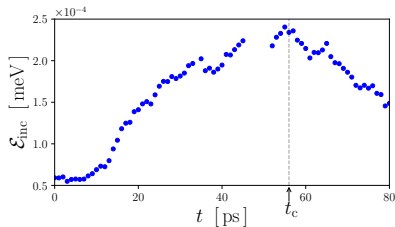
Quantitative evaluation of clustering:

$$C = \frac{1}{V} \sum_{i=1}^V c_i \quad (c_i = \pm 1)$$

White, Barenghi, Proukakis, PRA (2012)



$$\mathcal{E}_{\text{inc}}(t) = \int_{k_1}^{k_2} E_{\text{inc}}(k, t) dk$$

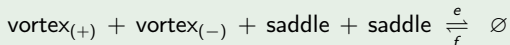


End of clustering

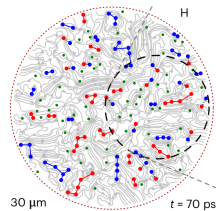
The vortices are no longer grouped in packs of the same vorticity.

Nodes don't seem to play a significant role.

→ Bristol mechanism:



for simplicity, take $f = 0$.

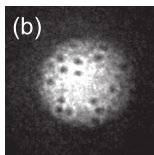
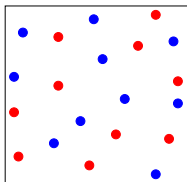
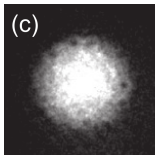
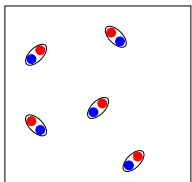
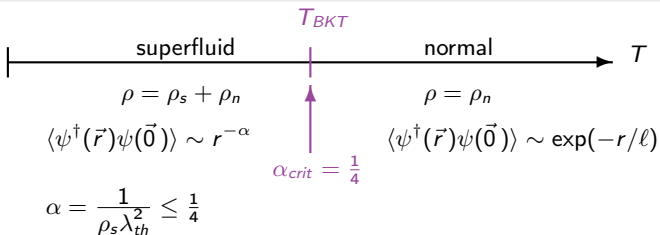


$$\frac{dv}{d\tau} = n^2 - \varepsilon v^2 s^2, \quad \frac{ds}{d\tau} = 1 - \gamma ns - \varepsilon v^2 s^2, \quad \frac{dn}{d\tau} = 1 - n^2 - \gamma ns,$$

where $\varepsilon = \frac{1}{2} e N_0^3 t_0 = ec / (8a^2)$ is the rescaled rate of annihilation of saddles and vortices through the Bristol mechanism.

$S \approx V \rightsquigarrow$ Bristol is effectively a 4 vortices mechanism (also 3 and 2 since $V \approx S + N$ with $N \approx \text{cst}$).

Berezinskii-Kosterlitz-Thouless transition



Shin group (Seoul)
PRL 2013

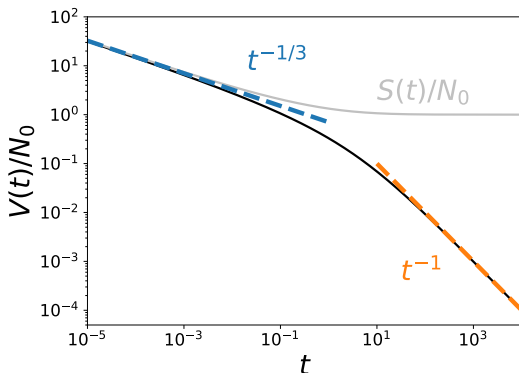
Bristol mechanism: vortex₍₊₎ + vortex₍₋₎ + saddle + saddle $\xrightarrow{e}$ $\emptyset$

$V_{(+)} = V_{(-)} = \frac{1}{2}V$, also $S = V + N$ and $N = N_0 \ll V(t=0)$

$$\frac{dV}{dt} = -\frac{e}{2}V^2S^2 = -\frac{e}{2}V^2(N_0 + V)^2$$

Initially, $V \gg N_0$: annihilation ≈ 4 vortices process $\sim V \propto t^{-1/3}$

At late time, $V \ll N_0$: annihilation ≈ 2 vortices process $\sim V \propto t^{-1}$



2D quantum vortices

carry 2 topological indices

Two mechanisms of vortex formation/annihilation:

- fulfill **topological** and **quantum** constraints
- Experimentally relevant

Involve critical points $\neq$ vortices:

candidate observables for studying the transition to turbulence and its decay

2D quantum vortices

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👍 concepts of singular optics relevant for quantum fluids

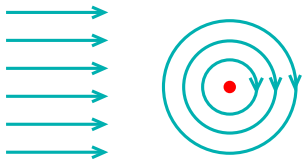
➡ effective kinetic description $\approx$ Landau mean field

➡ Topological constraints $\rightarrow$ non trivial spatial correlations between (different types of) critical points?

➡ Thermodynamic counterpart in BKT transition ?
Relevant for quench dynamics !

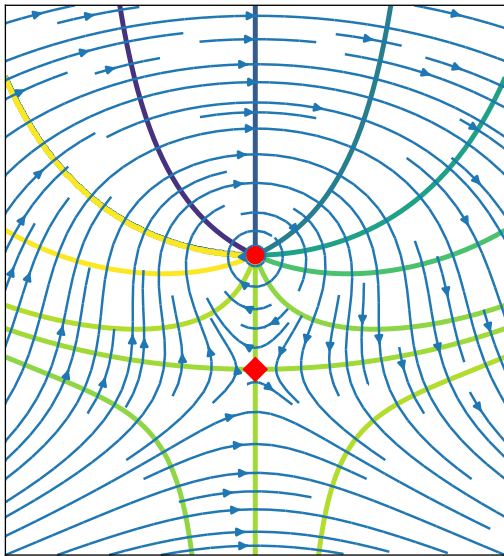
Thank you for your attention

Saddles: model case

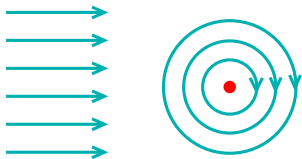


flow + vortex

$$\psi = e^{ikx} (x - iy)$$

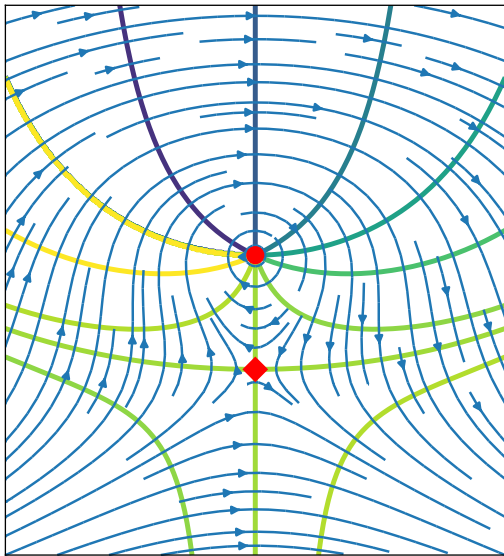
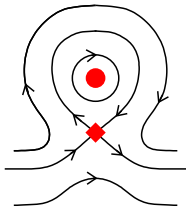


Saddles: model case

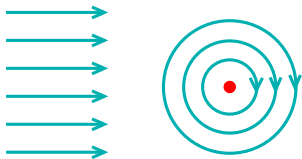


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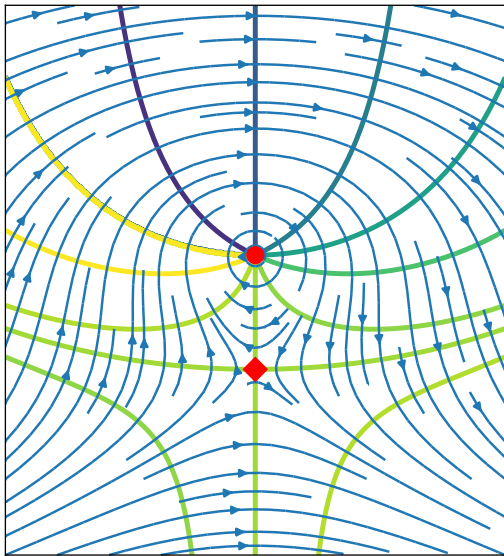
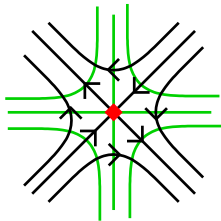


Saddles: model case



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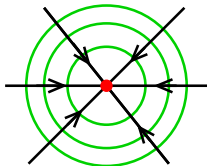
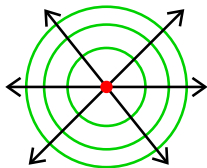
In a quantum fluid

$$\Psi(\vec{r}) = A(\vec{r}) \exp\{i\chi(\vec{r})\}$$

$$\vec{v} = \vec{\nabla}\chi$$

phase χ : velocity potential

min or max of χ :



Other points with zero velocity: (elusive) nodes

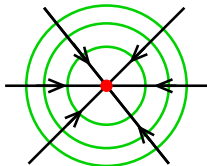
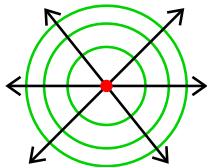
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$$\vec{v} = \vec{\nabla}\chi$$

phase χ : velocity potential

min or max of χ :



Nodes are forbidden:

- in an incompressible fluid:

$$\vec{\nabla} \cdot \vec{v} = 0 \implies \vec{\nabla}^2 \chi = 0$$

Hence not seen, e.g., in a Coulomb gas model

- in a stationary configuration:

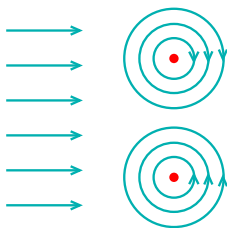
No sink nor source !

$$\vec{\nabla}^2 \psi + [f(A) + U(\vec{r}) - \mu]\psi = 0 \quad (1)$$

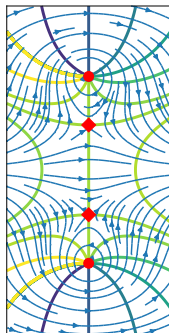
where $\psi = A \exp(i\chi)$:

$$\vec{\nabla}^2 \psi = \left\{ \vec{\nabla}^2 A - A|\vec{\nabla}\chi|^2 + i(A\vec{\nabla}^2 \chi + 2\vec{\nabla}A \cdot \vec{\nabla}\chi) \right\} e^{i\chi}$$

at a stagnation point where $\vec{\nabla}\chi = 0$,
the **imaginary part** of (1) yields $\vec{\nabla}^2 \chi = 0$



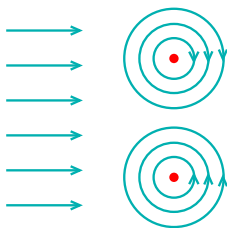
flow + 2 vortices



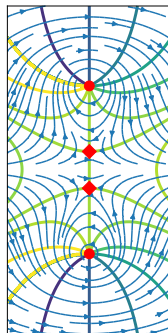
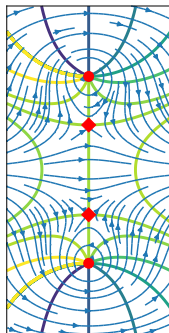
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$$k^2 b = \left| 1.3 \right|$$



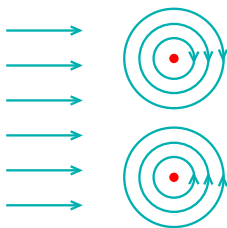
flow + 2 vortices



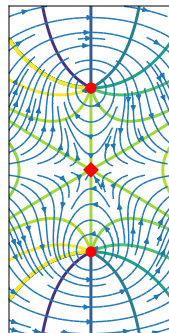
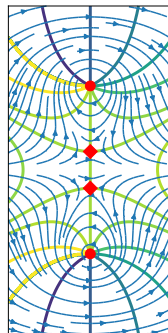
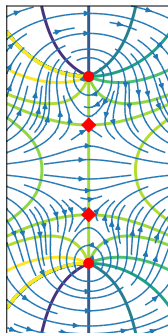
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$$k^2 b = \left| \begin{array}{c|c|c} 1.3 & 1.05 & \end{array} \right|$$



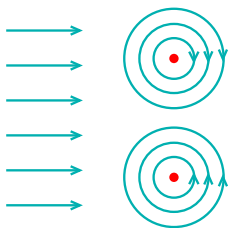
flow + 2 vortices



solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$$k^2 b = \begin{array}{|c|c|c|} \hline 1.3 & 1.05 & 1 \\ \hline \end{array}$$

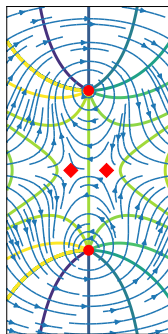
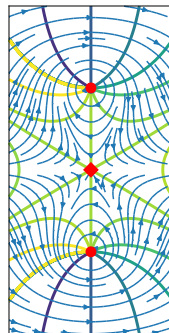
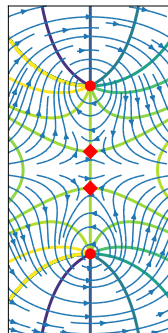
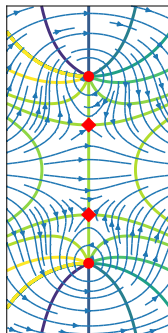


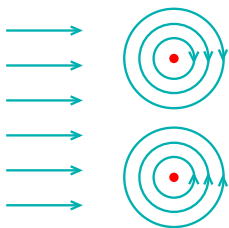
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$$k^2 b = \begin{vmatrix} 1.3 & 1.05 & 1 \\ 0.95 & & \end{vmatrix}$$



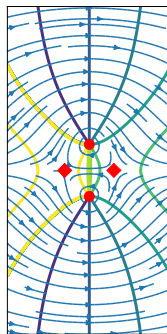
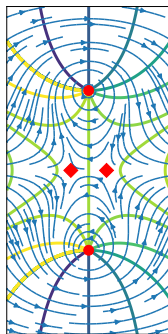
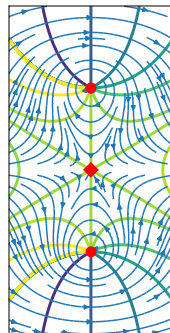
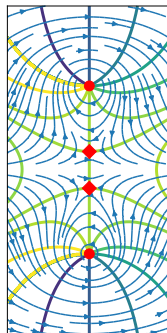
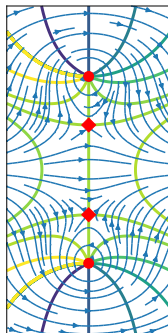


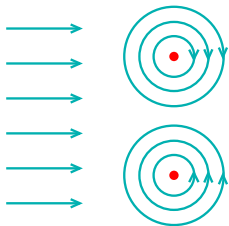
flow + 2 vortices

solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$$k^2 b = \begin{array}{|c|c|c|} \hline 1.3 & 1.05 & 1 \\ \hline 0.95 & 0.1 & \\ \hline \end{array}$$





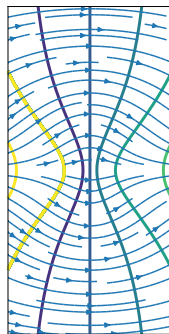
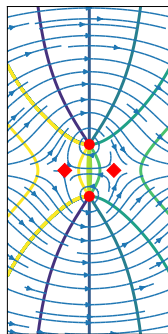
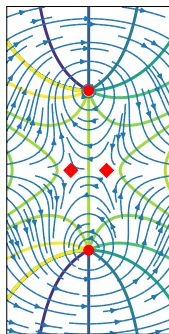
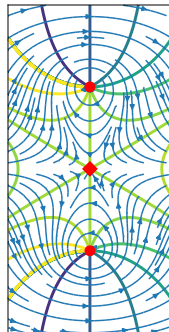
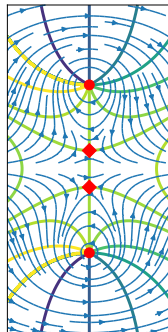
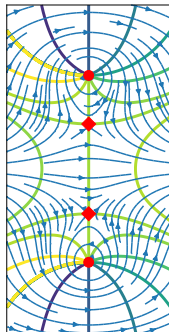
flow + 2 vortices

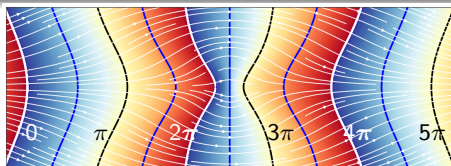
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx} [x - ik(y^2 - b)]$$

$k^2 b =$

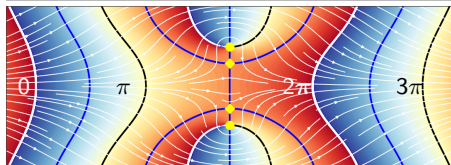
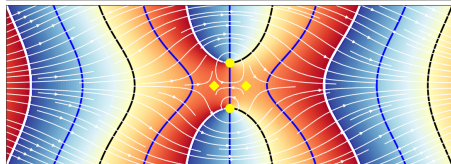
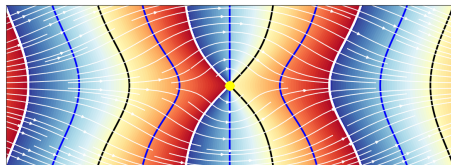
1.3	1.05	1
0.95	0.1	-0.1





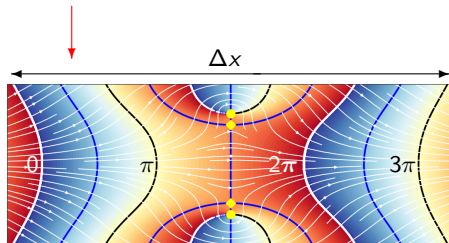
Initially:

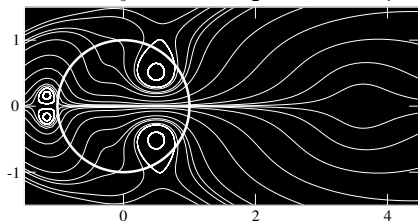
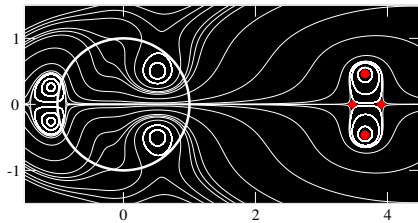
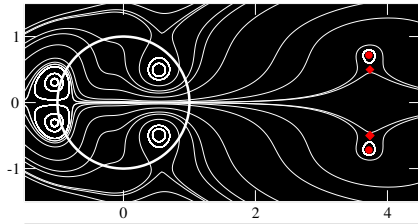
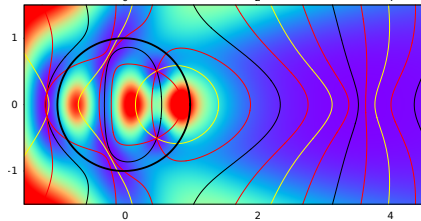
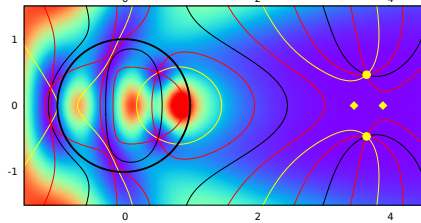
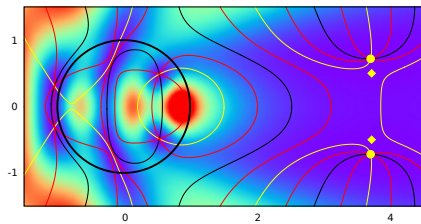
$\Delta\chi = 5\pi$ along $y = 0$ and fixed Δx



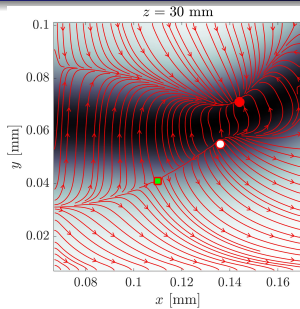
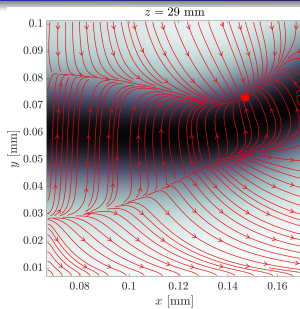
Eventually:

$\Delta\chi = 3\pi$ along $y = 0$, same Δx



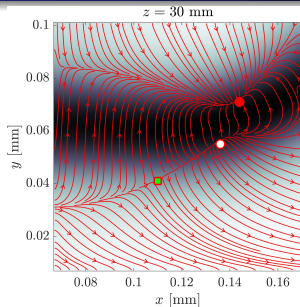
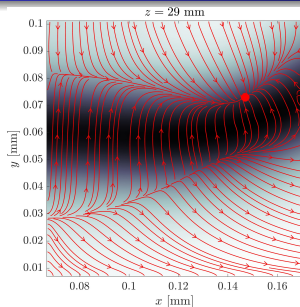


Formation of saddles and nodes: saddle-node bifurcation



$$\Phi_2 = 0.96 \pi$$

Formation of saddles and nodes: saddle-node bifurcation

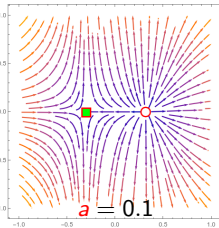
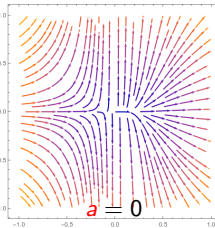
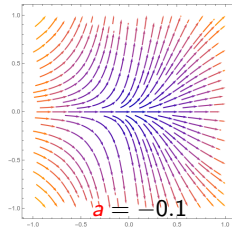


$$\Phi_2 = 0.96 \pi$$

Saddle-node:

$$\begin{cases} v_x = x^2 - a \\ v_y = y \end{cases}$$

$$a \in \mathbb{R}$$



$$\vec{v} = \vec{\nabla}(\frac{1}{3}x^3 - ax + \frac{1}{2}y^2)$$

orbitally equivalently: $\vec{v} = \vec{\nabla}\chi$ where

$$\chi(\vec{r}) = (\text{almost}) \text{any fct of } Z = \frac{1}{3}x^3 - ax + \frac{1}{2}y^2$$