Spectral and entanglement statistics in quantum spin chains

JP Keating (Bristol)

N Linden (Bristol)

HJ Wells (Bristol)

In memory of Oriol Bohigas

Motivation

- Statistical properties of many-body quantum systems
- RMT for many-body quantum systems with finite interactions – c.f. the embedded ensembles (French & Wong, Bohigas & Flores)
- New questions from quantum information theory and quantum statistical mechanics/thermodynamics

References

1. Random Matrices and Quantum Spin Chains, J.P. Keating, N. Linden and H.J. Wells, Markov Processes and Related Fields, in press. (**arXiv: 1403.1114 [math-ph]**)
2. Spectra and eigenstates of spin chain Hamiltonians, J.P. Keating, N. Linden and H.J. Wells (**arXiv:1403.1121 [math-ph]**)

Quantum Spin Chains

- *n* quantum spins
- Consider initially a 1-D chain with periodic boundary conditions and nearest-neighbour interactions
- General Hamiltonian

$$H_n = \sum_{j=1}^{n} \left(\sum_{a,b=1}^{3} \alpha_{a,b}^{(j)} \sigma_j^{(a)} \sigma_{j+1}^{(b)} + \sum_{a=1}^{3} \alpha_{a,0}^{(j)} \sigma_j^{(a)} \right)$$

Quantum Spin Chains (cont)

- We will sometimes also consider translationally invariant Hamiltonians

$$H_n^{\text{inv}} = \sum_{j=1}^{n} \left(\sum_{a,b=1}^{3} \alpha_{a,b} \sigma_j^{(a)} \sigma_{j+1}^{(b)} + \sum_{a=1}^{3} \alpha_{a,0} \sigma_j^{(a)} \right)$$

- Notation: $\sigma_j^{(a)} = I_2^{\otimes(j-1)} \otimes \sigma^{(a)} \otimes I_2^{\otimes(n-j)}$ where $\sigma^{(a)}$

is the *a*th Pauli spin matrix.

Random Matrices

- Take the coefficients $\alpha_{a,b}^{(j)}$ to be random variables.
- This gives 9*n* (no local term) or 12*n* (with local term) random variables. (9/12 for H^{inv})
- E.g. the $\alpha_{a,b}^{(j)}$ could be drawn from a normal distribution with mean 0 and variance $\propto 1/n$
- Ensembles are not invariant and not Wigner
- NB the size of the matrices is 2^nx2^n

Density of States

Density: $\mathrm{d}\mu_n(\lambda) = \left\langle \frac{1}{2^n} \sum_{k=1}^{2^n} \delta(\lambda - \lambda_k) \right\rangle \mathrm{d}\lambda$

Fourier transform: $\psi_n(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} \mathrm{d}\mu_n(\lambda) = \left\langle \frac{1}{2^n} \mathrm{Tr}\, \mathrm{e}^{\mathrm{i}tH_n} \right\rangle$

Theorem 1: for $\alpha_{a,0}^{(j)} = 0,\ \alpha_{a,b}^{(j)} \in N(0, 1/9n)$

$$\left|\psi_n(t) - \exp\left(-t^2/2\right)\right| \leq t^2 \sqrt{n} \frac{1}{9n}\left(36\sqrt{2} + 81\right)$$

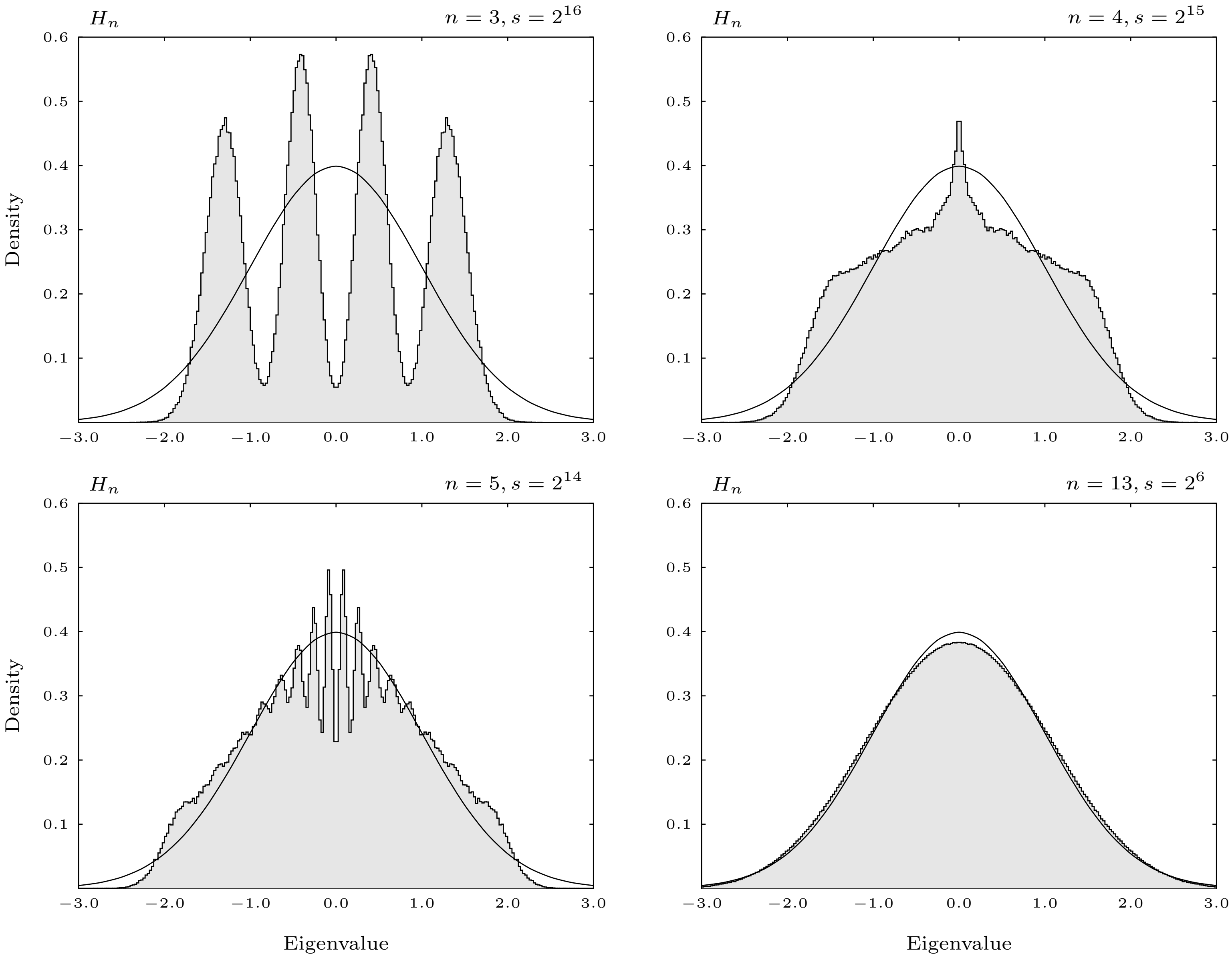
Density of States (cont)

Proof:

- Separation into commuting subsets $\sum_{j-\text{odd}} + \sum_{j-\text{even}}$
- Use the identity

$$e^{it(X+Y)} - e^{itX}e^{itY} = \int_0^1 \int_0^1 t^2 e^{i(1-s)t(X+Y)} e^{i(1-r)stX} [X,Y] e^{irstX} e^{istY} \, dr ds$$

- plus Cauchy-Schwartz to bound commutators



H_n
$n = 3, s = 2^{16}$
H_n
$n = 4, s = 2^{15}$
H_n
$n = 5, s = 2^{14}$
H_n
$n = 13, s = 2^{6}$
Density
Density
0.6
0.5
0.4
0.3
0.2
0.1
−3.0
−2.0
−1.0
0.0
1.0
2.0
3.0
Eigenvalue
Eigenvalue

Density of States (cont)

Theorem 2: the same CLT for the density of states holds for fixed (i.e. non-random) couplings when *n* tends to infinity with the normalization $\mathrm{Tr} H_n^2 = 2^n$ and provided $\left|\alpha_{a,b}^{(j)}\right| < C$

(Proof uses a more efficient lattice splitting due to Hartmann, Mahler & Hess)

Density of States (cont)

e.g., as conjectured by Atas & Bogomolny, if

$$H_n = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{1+\gamma^2+\delta^2}} \left(\sum_{j=1}^{n} \sigma_j^{(1)} \sigma_{j+1}^{(1)} + \gamma \sigma_j^{(1)} + \delta \sigma_j^{(3)} \right)$$

then

$$\lim_{n\to\infty} \mathrm{Tr}\left(H_n\right)^k = \begin{cases} 0 & k \text{ odd} \\ \left(2k-1\right)!! & k \text{ even} \end{cases}$$

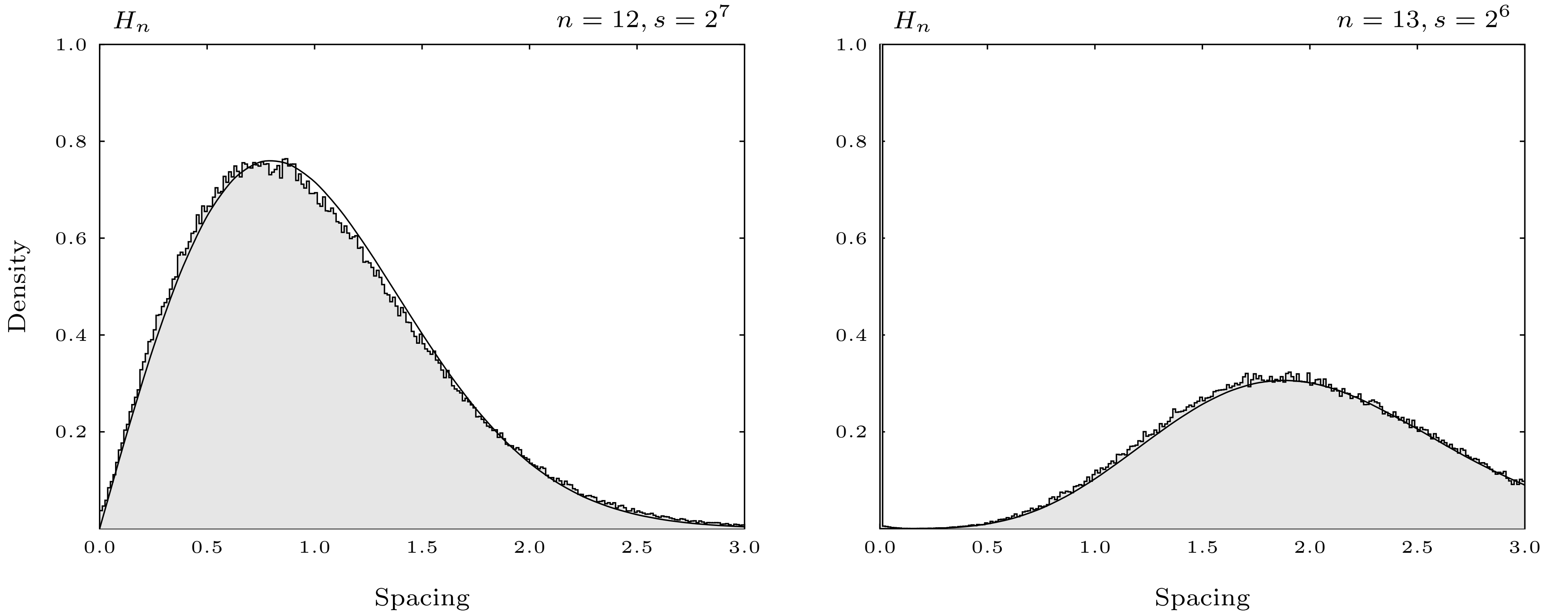
but the theorem also holds for Hamiltonians that are non-translationally invariant.

Density of States (cont)

Further generalizations: the CLT

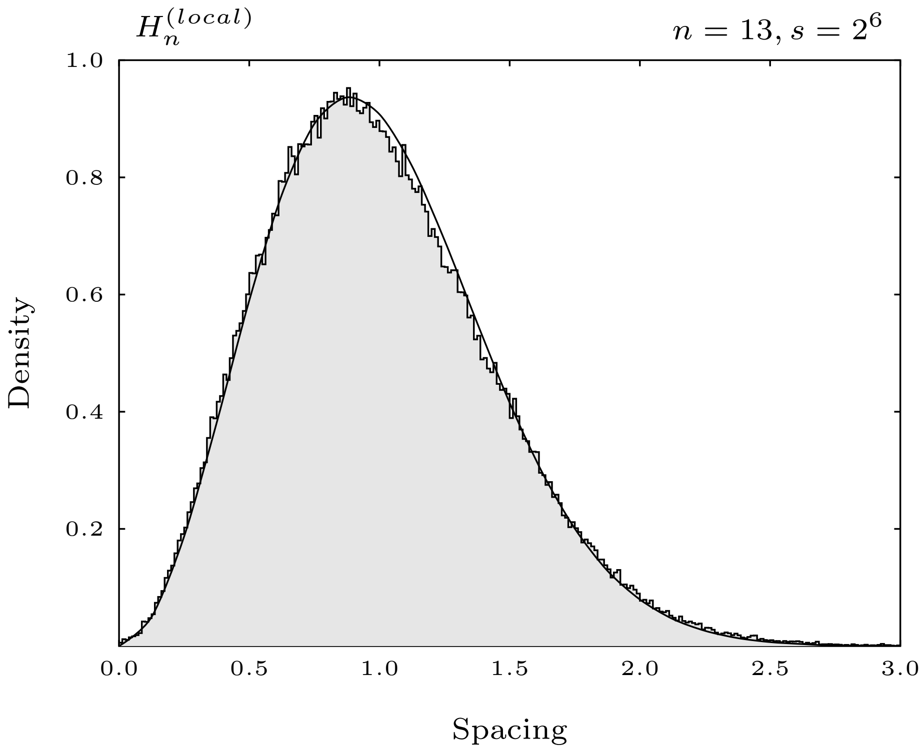
- is universal for a wide class of probability distributions;
- holds for all finite range interactions;
- extends to other geometries, e.g. all *c*-colourable graphs

Spectral Statistics

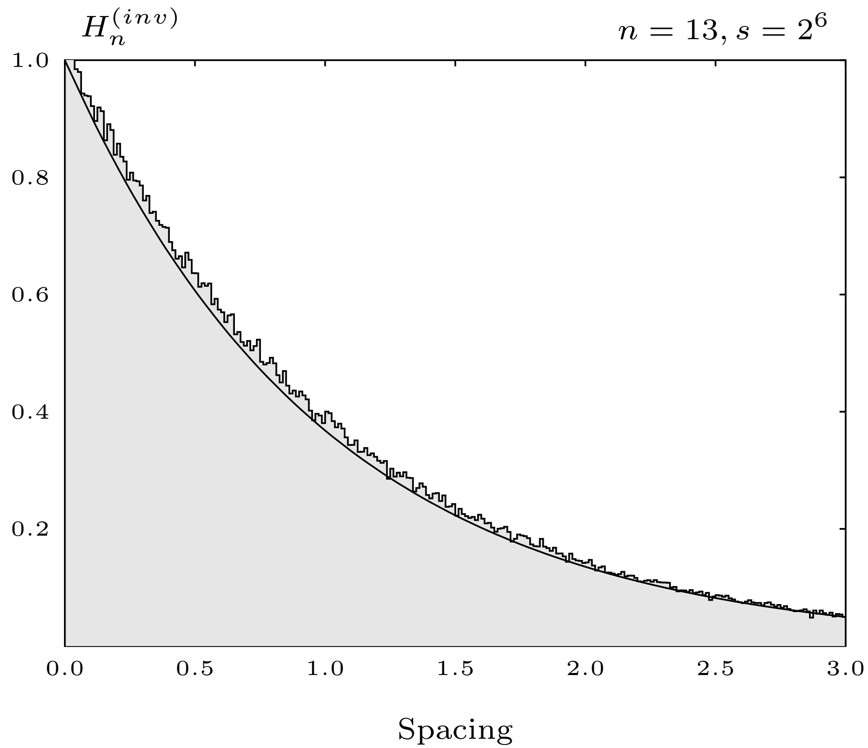


No local terms

Spectral Statistics (cont)



with local terms



translation-invariant

Spectral Statistics (cont)

In the absence of local terms, H_n commutes with the antiunitary operator

$$B = K\left(\sigma^{(2)}\right)^{\otimes n}$$

where K denotes complex conjugation. NB

$$B^2 = (-1)^n$$

So when n is odd we have a Kramers degeneracy

Spectral Statistics (cont)

Theorem 3: for odd prime values of *n*

$$H_n^{\text{inv}} = \sum_{j=1}^{n} \left(\sum_{a,b=1}^{3} \alpha_{a,b} \sigma_j^{(a)} \sigma_{j+1}^{(b)} + \sum_{a=1}^{3} \alpha_{a,0} \sigma_j^{(a)} \right)$$

generically has a non-degenerate spectrum

Proof uses two ingredients:

Spectral Statistics (cont)

Lemma: for odd prime values of *n*, there exists some ε such that

$$H_n^{\text{inv}} = \sum_{j=1}^{n} \left(\varepsilon \sigma_j^{(1)} \sigma_{j+1}^{(2)} + \sigma_j^{(3)} \right)$$

has a non-degenerate spectrum

Proof – perturbation theory

Spectral Statistics (cont)

Then (a la Pastur): the Vandermonde matrix V formed from the energy levels of H^{inv}

- satisfies $\det VV^{T}=0$ if any of the eigenvalues is degenerate
- is a polynomial in the parameters α

The lemma implies that the Lebesgue measure of the zeros of this polynomial is zero

Entanglement

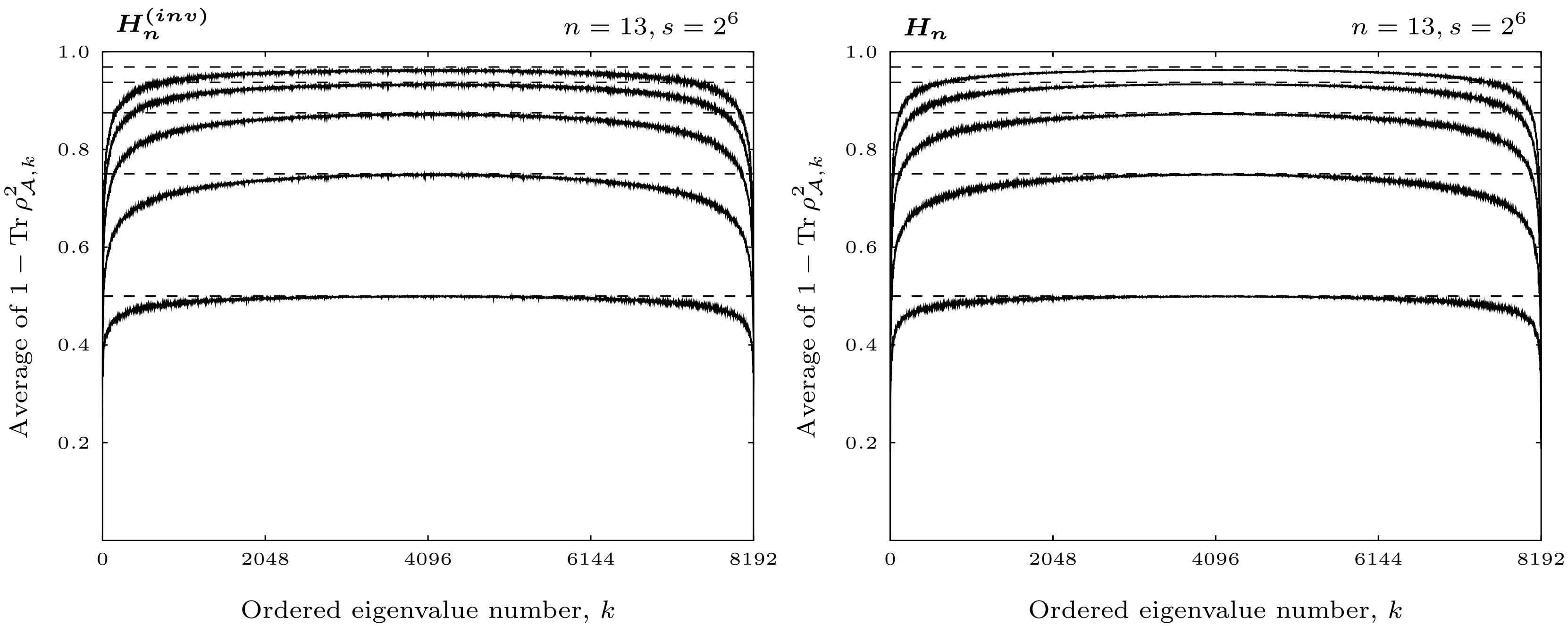
Split the chain into subsystems *A* (*m* spins) and *B* (*n*-*m* spins), and compute how these are entangled via the eigenfunctions in the limit when $n \to \infty$.

Entanglement

Eigenstate purity: if $\rho_k = |\psi_k\rangle\langle\psi_k|$ and $\rho_k^{(A)} = \mathrm{Tr}_B \rho_k$ then the purity is $\mathrm{Tr}_A\left(\rho_k^{(A)}\right)^2$. This satisfies

$$\frac{1}{2^m} \le \mathrm{Tr}_A\left(\rho_k^{(A)}\right)^2 \le 1$$

lower bound – maximally entangled state across *A* and *B*; upper bound – product state across *A* and *B*. So (1 – purity) measures entanglement.



(a) Translationally invariant

(b) Generic

Entanglement (cont)

Theorem: for translation-invariant spin chains

$$\frac{1}{2^m} \leq \frac{1}{2^n} \sum_{k=1}^{2^n} \mathrm{Tr}_A \left(\rho_k^{(A)} \right)^2 \leq \min\left(\frac{1}{2^m} + \frac{2^m}{n},\ 1 \right)$$

Proof – expand the density matrix in the basis of spin matrices and use translation invariance to estimate the expansion coefficients

Entanglement (cont)

Corollary: for any fixed $\varepsilon > 0$, the proportion of eigenstates for which

$$\mathrm{Tr}_A\left(\rho_k^{(A)}\right)^2 \geq \frac{1}{2^m} + \varepsilon$$

tends to zero as $n \to \infty$

i.e. almost all states are maximally mixed

Conclusions

- Gaussian density of states (as if components are weakly interacting)
- Maximally entangled eigenstates (as if components are strongly interacting)
- Random matrix model exhibits standard (universal) RMT statistics, despite having exponentially fewer free parameters.