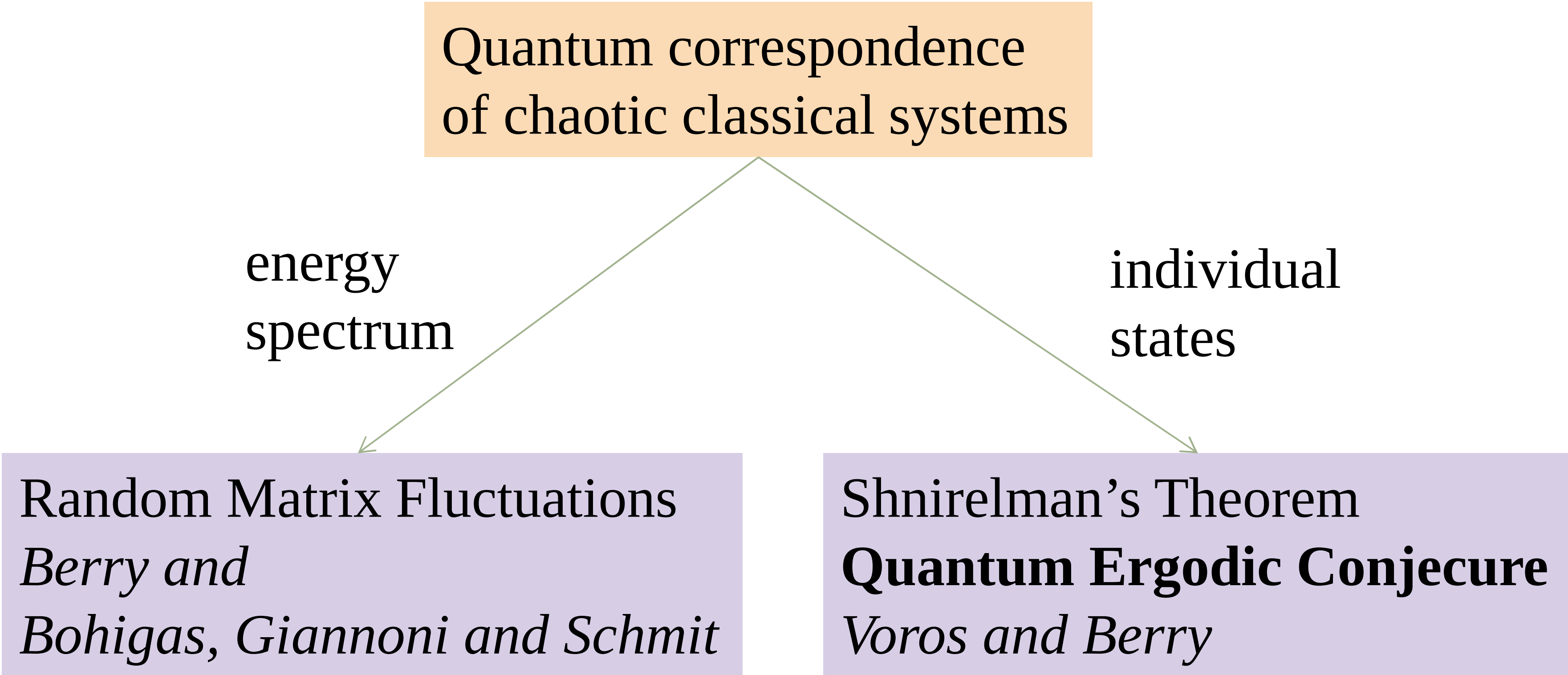
The Quantum Ergodic Conjecture Revisited

It would have been nice to discuss this with Oriol



Reviewed in Bohigas, Tomsovic and Ullmo (1993)

Shnirelman's Theorem (Colin de Verdière, Zelditch...)

a pedestrian version

The expectation of smooth functions of position, $F(\hat{q})$, for a quantum billiard, which is classically ergodic, is

$$\langle n | F(\hat{q}) | n \rangle = \frac{1}{A} \int_{bil} dq \, F(\hat{q})$$

for almost all eigenstates, $|n\rangle$.

no phase space, $\mathbf{R}^{2N} : x = (p, q)$, *no Wigner function.*

The Quantum Ergodic Conjecture:

The Wigner function for a typical eigenstate, $|n\rangle$, coincides approximately with the corresponding ergodic probability density on the energy shell of energy E_n :

$$W_\pi(\mathbf{x}) = \frac{\delta(E - H(\mathbf{x}))}{\int d\mathbf{x}\ \delta(E - H(\mathbf{x}))}$$

Expectation of observables:

$$\langle n|F(\hat{p},\hat{q})|n\rangle = \int dpdq\, F(p,q)\, W_\pi(p,q)$$

Wigner functions need adaptations for billiards

More delicate consequences of QEC:

Universal structure of local correlations for wave functions:

$$C_w(\xi_q; q) = \int dq \frac{\exp[-(q-\xi_q)^2 / 2\Delta^2]}{2\pi\Delta} \left\langle q + \frac{\xi_q}{2} \middle| n \right\rangle \left\langle n \middle| q - \frac{\xi_q}{2} \right\rangle$$

$$\propto J_0\left(\frac{\sqrt{2m(E - V(q))}}{\hbar} \xi_q \right)$$

Coincidence with the correlations for the hypothesis of Random Gaussian Wave Functions!

Considerable computational verification (mostly billiards) and interpretation of experiments in quantum dots.

Problems with a full acceptance of QEC:

i. **Hudson's theorem:** The only pure state with a non-negative Wigner function is that of a coherent state,

$$W_\eta(\mathbf{x}) = \frac{1}{\pi\hbar} e^{-(\mathbf{x}-\eta)^2/\hbar},$$

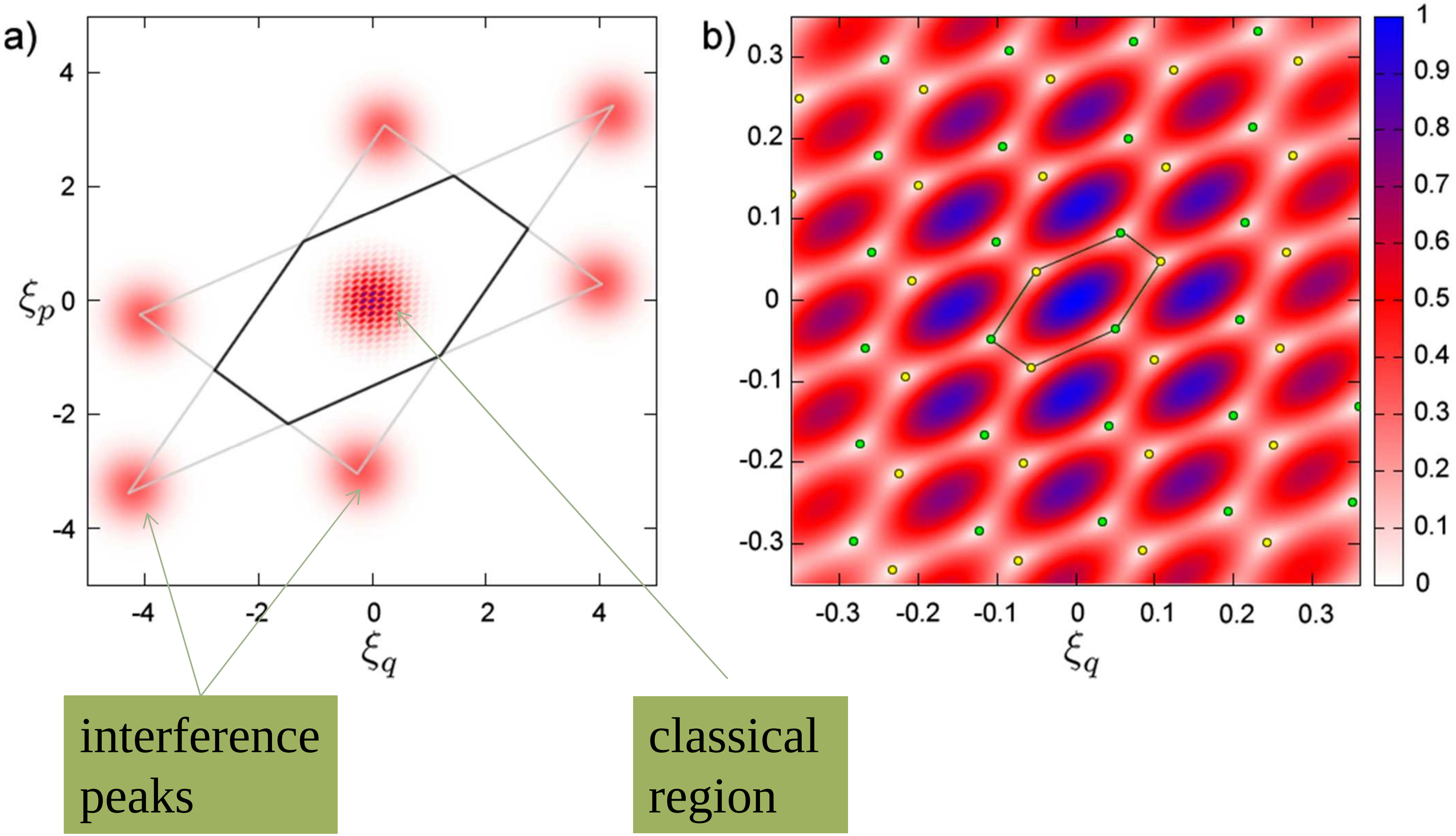
whereas $W_\pi(x)$ results from a sequence of non-negative functions:
The proposed density operator cannot be a pure state.

ii. Exact Fourier invariance of **phase space correlations**,

$$C(\xi) = (2\pi\hbar)^L \int \mathrm{d}\mathbf{x}\, W(\mathbf{x})\, W(\mathbf{x}-\xi) = |\langle n|\hat{T}_\xi|n\rangle|^2$$

for pure states: *Large structures (eg. the energy shell) need be accompanied by fine fluctuations, absent in QEC.*

Phase space correlations for a triplet of coherent states:
Fourier invariance is reflected in the duality between large and small scales.



iii. Positivity of the density operator: If an operator $\hat{A}$ has no negative eigenvalues (eg. the density operator), then

$$\langle n|\hat{A}|n\rangle = \int dx\, A(x) W_n(x) > 0.$$

Negativity witness for the QEC:

Construct a density operator with its Wigner function, $W_w(x)$, such that

$$\int dx\, W_\pi(x) W_w(x) < 0.$$

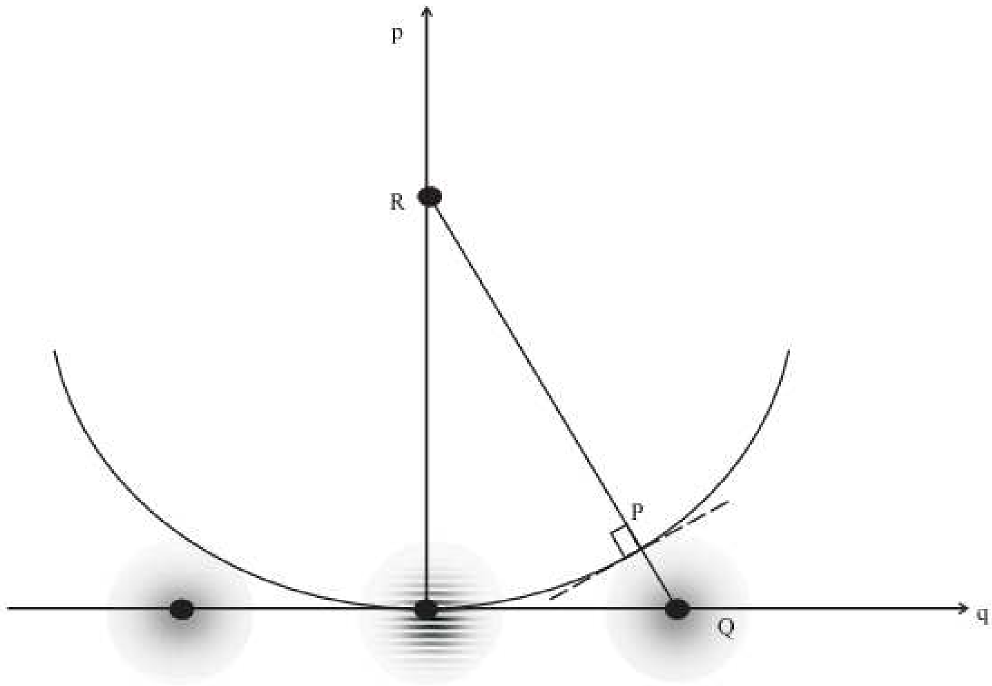
Then the operator, $\hat{\rho}_\pi$, represented by $W_\pi(x)$, is not positive!

Nandor Balazs (1980), but only for a single degree of freedom: The system is both ergodic and integrable.

New negativity witness: A Schrödinger cat state

$$W_{\pm}(\mathbf{x}) = \frac{1}{2\pi\hbar\,(1 \pm e^{-\eta^2/\hbar})}\left[e^{-(\mathbf{x}-\eta)^2/\hbar} + e^{-(\mathbf{x}+\eta)^2/\hbar} \pm 2e^{-\mathbf{x}^2/\hbar}\cos\left(\frac{2}{\hbar}\mathbf{x}\wedge\eta\right)\right]$$

The QEC operator
is not positive for any curved
energy shell, for any number
of degrees of freedom.



Choosing the minus sign and placing the central negative trough on the tangent to the energy shell, its curvature avoids the positive Gaussians at the coherent states.

What about a change of representation?
Can one avoid these pitfalls?

The chord function,

$$\chi(\xi) = \frac{1}{(2\pi\hbar)^N} \int \mathrm{d}\mathbf{x} \, \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathbf{x} \wedge \boldsymbol{\xi} \right\} W(x),$$

is an alternative complete representation of the density operator. Its Taylor series is

$$\chi(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k}{(i\hbar)^n} \binom{n}{n-k} \left\langle \mathcal{M}\left(\hat{q}^{n-k}\hat{p}^k\right)\right\rangle \xi_q^k \xi_p^{n-k},$$

where,

$$\mathcal{M}\left(\hat{q}^n\hat{p}^k\right) = \frac{1}{n+k} \sum_{P_{nk}} \hat{q}^n\hat{p}^k$$

and P_{nk} all possible permutations of products of q^n and p^k.
Hence, the expectation of any smooth observable, $F(\hat{p},\hat{q})$, depends only on the neighbourhood of the origin of $\chi(\xi)$.

A *soft* Quantum Ergodic Conjecture:

Near the origin, the chord function is approximately:

$$\chi_\pi(\xi) = \frac{1}{(2\pi\hbar)^N} \int \mathrm{d}\mathbf{x}\ \exp\left\{\frac{\mathrm{i}}{\hbar}\mathbf{x} \wedge \boldsymbol{\xi}\right\} \frac{\delta(E - H(\mathbf{x}))}{\int d\mathbf{x}\ \delta(E - H(\mathbf{x}))}$$

Thus, one obtains classical ergodic expectations for observables, without assuming full knowledge of the chord function. No restrictions are violated.

Can one also obtain the universal local correlations from the soft QEC?

$$C_w(\xi_q;q) = \int dq \frac{\exp[-(q-\xi_q)^2 / 2\Delta^2]}{2\pi\Delta} \left\langle q+\frac{\xi_q}{2} \middle| n \right\rangle \left\langle n \middle| q-\frac{\xi_q}{2} \right\rangle$$

$$= \int d\xi_p \; \exp[-\Delta^2 \xi_p^2] \, \chi(\xi_p, \xi_q) \; \exp[i\xi_p \cdot q / \hbar]$$

Thus, again, one needs only knowledge of the chord function in a minimum uncertainty region around the origin.

Is there any new prediction within the soft QEC?

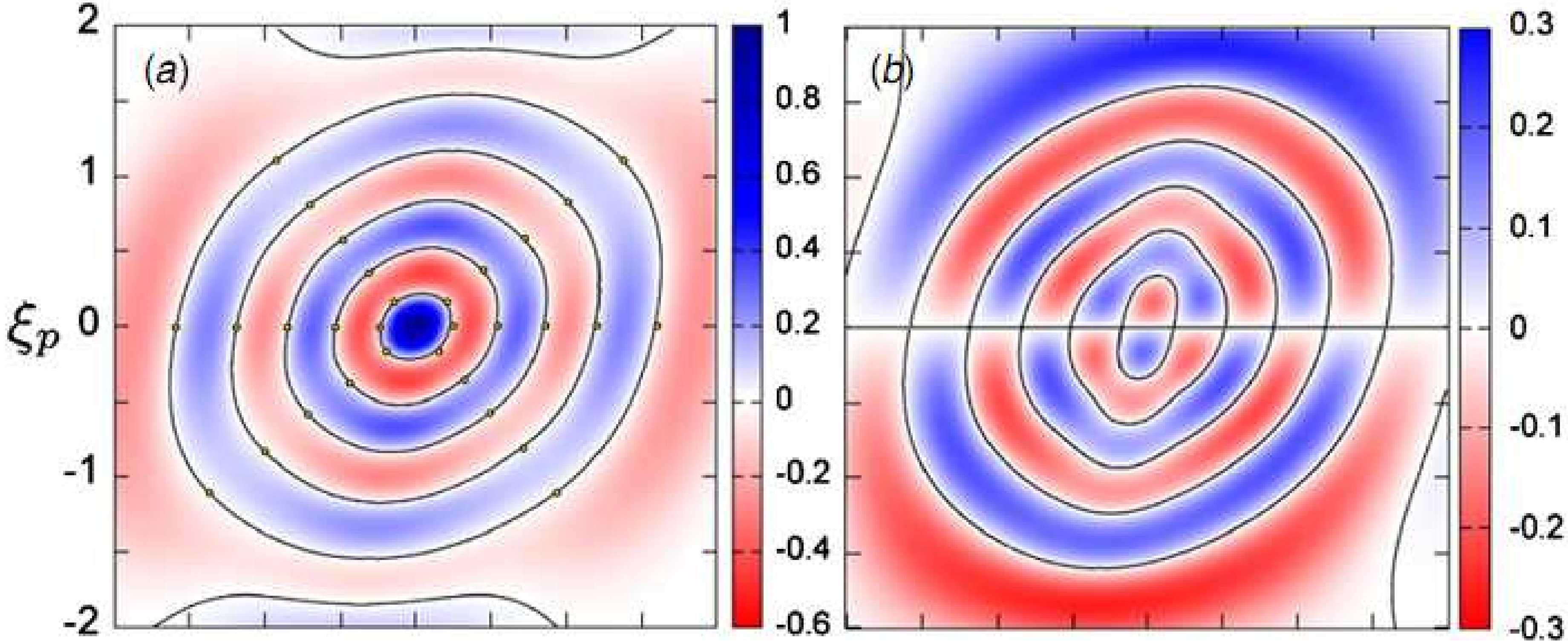
Blind Spots: Zeroes of the phase space correlations

$$C(\xi) = |\langle n|\hat{T}_\xi|n\rangle|^2 = |\chi(\xi)|^2$$
$$= \int \frac{\mathrm{d}\eta}{(2\pi\hbar)^L} e^{i\eta\wedge\xi/\hbar} |\chi(\eta)|^2$$

The Fourier integral over $|\chi(\eta)|^2$ *is just the correlation of the Wigner function (no zeroes close to the origin for QEC), but we cannot perform it, assuming soft QEC.*

Zeroes of the chord function for QEC are approximate blind spots for small chords.

Nodal lines for the real and imaginary parts of the chord function for the eigenstate of a deformed harmonic oscilator.
Blind spots are their intersection



For one degree of freedom, they are well approximated by soft QEC, near the origin.

Conclusion:

Soft QEC supplies the same classical ergodic expectations and the same wave function correlations as hard QEC, without breaking any rules.

Blind spots for small displacements,
also zeroes of the FT of the Husimi function,
are an entirely new prediction.

Blind spot graphics result from collaboration with Eduardo Zambrano