Universal Quantum Graphs

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1. Introduction

A central topic in Quantum Chaos: The study of the distribution of spacings of eigenvalues of classically chaotic Hamiltonian systems.

BGS conjecture: Spectral fluctuation properties of (Hamiltonian) quantum systems that are chaotic (mixing) in classical limit, coincide with those of random-matrix ensemble in same symmetry class (unitary, orthogonal, symplectic). O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. 42 (1984) 1.

Plenty of numerical evidence. Moreover, in semiclassical approximation (periodic-orbit expansion) shown to hold for level-level correlator of chaotic systems with orthogonal and unitary symmetry. S. Heusler *et al.*, New J. Phys. 11 (2009) 103205.
Also for level-level correlator of closed quantum graphs. S. Gnutzmann and A. Altland, PRE 72 (2005) 056215.
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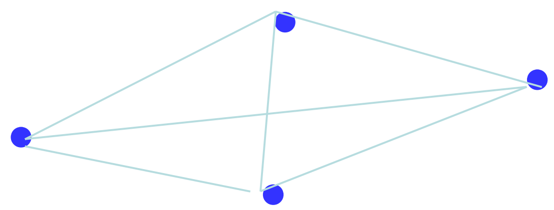
Today: Prove BGS conjecture for time-reversal invariant graphs in most general form. For closed graphs that are classically chaotic (mixing), all spectral correlation functions coincide with those of the Gaussian orthogonal ensemble of random matrices (GOE). Analogous identities for all S-matrix correlation functions for open graphs. As a by-product, find complete distribution function for S matrix in Ericson regime.

Z. Pluhar and H. Weidenmüller, PRL 110 (2013) 034101, PRE 88 (2013) 022902, PRL (submitted).

2. Quantum Graphs

T. Kottos and U. Smilansky, Ann. Phys. (N.Y.) 274 (1999) 76.

System of V vertices connected by B bonds. Completely connected, simple, closed graphs:



Every pair of vertices connected by single bond. Then 2 B = V (V – 1). Vertices labeled α = 1, … V. Bonds labeled b = 1, …, B. Bond length L_b. Bond lengths incommensurate: No set of integer coefficients exists such that $\sum_b a_b L_b = 0$. Consider limit B >> 1. In that limit, all bond lengths bounded from above. On each bond Schrödinger wave [a exp (i k x) + b exp (- i k x)] with same wave number k on all bonds. Self-adjoined boundary conditions on each vertex α connect vectors of incoming and outgoing amplitudes on bonds linked to α:

$$\mathcal{O}^\alpha = \sigma^\alpha \mathcal{I}^\alpha$$

Extension to open graphs: Attach extra bond to vertices 1,2, …, Λ with Λ finite, fixed. These bonds extend to infinity. Each such bond defines a channel. Matrices σ^α defined as before but only for non-channel amplitudes. Then these are unitary (subunitary) for closed (open) graphs, have dimension (V-1) and are symmetric (time-reversal invariance).

Quantum graphs are simpler systems than typical chaotic dynamical systems. For the latter, proof of the BGS conjecture requires knowledge of the totality of periodic orbits. Numerical simulations of spectral fluctuations suggest that quantum graphs with incommensurate bond lengths are chaotic.

Spectral determinant and scattering matrix:

Block-diagonal vertex scattering matrix Σ contains all matrices σ^{α} in diagonal blocks.
Σ is unitary (subunitary) for closed (open) graphs. Dimension of Σ is V (V-1) = 2 B.

This fact requires doubling of number of bonds. Introduce „directed bond representation“. Bonds labeled (b d). Yields $\Sigma^{(B)}$. Reverse bond directions by matrix σ_1^d. Bond propagator is $\exp\{ik\mathcal{L}\}$ where $\mathcal{L}$ stands for all L_b .

Average level density is $\langle d_{\mathrm{R}} \rangle = \frac{1}{\pi} \sum_b L_b$.

Spectral determinant (zeros determine bound states): $\xi(k) = \det(1 - \exp\{ik\mathcal{L}\}\sigma_1^d \Sigma^{(B)})$

Scattering matrix: $S_{\alpha\beta}(k) = \rho_\alpha \delta_{\alpha\beta} + (\mathcal{T}\mathcal{W}^{-1}\mathcal{T}^T)_{\alpha\beta}$ where $\mathcal{W} = \exp\{-ik\mathcal{L}\}\sigma_1^d - \Sigma^{(B)}$

Intuitive picture: Matrix $\mathcal{T}$ couples to channels. Then propagation via $\mathcal{W}^{-1}$:

$$\mathcal{W}^{-1} = \exp\{ik\mathcal{L}\}\sigma_1^d + \exp\{ik\mathcal{L}\}\sigma_1^d \Sigma^{(B)} \exp\{ik\mathcal{L}\}\sigma_1^d + \ldots$$

Correlation functions: Defined as averages over wave number k (angular brackets).

For levels (closed graphs): $\left\langle \prod_{p=1}^{P} \frac{\mathrm{d}}{\mathrm{d}k} \ln \xi(k^{+} + \kappa_p) \prod_{q=1}^{Q} \frac{\mathrm{d}}{\mathrm{d}k} \ln \xi(k^{-} - \tilde{\kappa}_q) \right\rangle$

For S-matrix elements (open graphs): $\left\langle \prod_{p=1}^{P} S_{\alpha_p \beta_p}(k + \kappa_p) \prod_{q=1}^{Q} S^{*}_{\gamma_q \delta_q}(k - \tilde{\kappa}_q) \right\rangle$

Here P,Q arbitrary positive integers. Knowledge of all correlation functions is tantamount to knowledge of complete probability distribution. Assume $\kappa_p \langle d_{\mathrm{R}} \rangle \ll B \; , \; \tilde{\kappa}_q \langle d_{\mathrm{R}} \rangle \ll B \, .$

Classical Chaos (Mixing):

F. Barra and P. Gaspard, Phys. Rev. E 63 (2001) 066215.
P. Pakonski, K. Zyczkovski, and M. Kus, J. Phys. A 34 (2001) 9303.
S. Gnutzmann and U. Smilansky, Adv. Phys. 55 (2006) 527.

Perron-Frobenius operator is

$$\mathcal{F}_{bd,b'd'} = |(\sigma_1^d \Sigma^{(B)})_{bd,b'd'}|^2 \quad .$$

For closed graph $\mathcal{F}$ has single eigenvalue +1. Discrete time evolution defined by map $r \to \mathcal{F} r$ of density r in directed bond space. Not strictly a Hamiltonian system!
Mixing: Assume existence of spectral gap. Then +1 is only eigenvalue on unit circle in complex plane, all other eigenvalues of PF operator lie within unit circle and a finite distance away from it. Then m-fold repeated map $r \to \mathcal{F}^m r$ approaches uniform distribution in directed bond space for m >> 1 exponentially fast. For open graphs, use same assumption for eigenvalues of PF operator that differ from leading one (gap).

Further treatment:

(i) Using supersymmetry, write arguments of correlation functions as suitable derivatives of generating function G.

(ii) Use ergodicity to calculate k-averages of G as averages over phases $\phi_b = kL_b$. Do so with help of color-flavor transformation. These steps are exact. M. Zirnbauer, J. Phys. A 29 (1996) 7113.

(iii) Use saddle-point approximation to simplify $\langle G \rangle$. Matrix Y has dimension 4 P × 4 Q.

(iv) Massive modes (i.e., degrees of freedom not within saddle-point manifold): Within Gaussian approximation and for B >> 1 these do not contribute to observables (i.e., derivatives of $\langle G \rangle$). That follows from spectral gap of PF operator. Only matrices Y , $\tilde{Y}$ left.

(v) Result:

$$\langle G \rangle = \int \mathrm{d}(Y, \tilde{Y}) \Big(\dots \Big) \exp\{ SB_G + CC_G \}$$

$$SB_G = i\pi \langle d_{\rm R} \rangle \Big\{ \sum_p \kappa_p \mathrm{STr} \Big(\frac{1}{1 - Y\tilde{Y}} \Big)_{pp} + \sum_q \tilde{\kappa}_q \mathrm{STr} \Big(\frac{1}{1 - \tilde{Y}Y} \Big)_{qq} \Big\}$$

$$CC_G = -\frac{1}{2} \sum_\alpha \mathrm{STr} \ln \Big(1 + T^{(\alpha)} \frac{Y\tilde{Y}}{1 - Y\tilde{Y}} \Big)$$

$$T^{(\alpha)} = 1 - |\langle S_{\alpha\alpha} \rangle|^2$$

3. Random-Matrix Approach

K. B. Efetov, Adv. Phys. 32 (1983) 53; J. J. M. Verbaarschot, H. A. Weidenmüller, M. R. Zirnbauer, Phys. Rep. 129 (1985) 367.

GOE matrix has dimension N >> 1 and Gaussian-distributed matrix elements with zero mean values and second moments

$$\langle H_{\mu\nu} H_{\mu'\nu'} \rangle = \frac{N}{\lambda^2} (\delta_{\mu\mu'} \delta_{\nu\nu'} + \delta_{\mu\nu'} \delta_{\mu'\nu}) \qquad d = \frac{\pi\lambda}{N}$$

Correlation functions are (angular brackets denote ensemble averages)

Levels (closed system): $\left\langle \prod_{p=1}^{P} \mathrm{Tr}(E^+ + \epsilon_p - H)^{-1} \prod_{q=1}^{Q} \mathrm{Tr}(E^- - \tilde{\epsilon}_q - H)^{-1} \right\rangle$

S-matrix elements (open system): $\left\langle \prod_{p=1}^{P} S_{a_p b_p}(E + \epsilon_p) \prod_{q=1}^{Q} S^*_{c_q d_q}(E - \tilde{\epsilon}_q) \right\rangle$

where

$$S_{ab}(E) = \delta_{ab} - 2i\pi [W (E - H + i\pi W^\dagger W)^{-1} W^\dagger]_{ab}$$

and where $W_{a\mu} = W_{\mu a} = W^*_{a\mu}$ couples levels μ and channels a, b, …. Assume $\epsilon_p, \tilde{\epsilon}_q \ll \lambda$.

Further treatment

(i) Using supersymmetry, write arguments of correlation functions as suitable derivatives of generating function Z.

(ii) Calculate ensemble average. Use Hubbard-Stratonovich transformation. These steps are exact.

(iii) Use saddle-point approximation to simplify $\langle Z \rangle$. Matrix t_{12} has dimension 4 P × 4 Q.

(iv) Massive modes (i.e., degrees of freedom not within the saddle-point manifold): Within Gaussian approximation and for N >> 1 these do not contribute to observables (i.e., derivatives of $\langle Z \rangle$). Only matrices t_{12} , t_{21} left.

(v) Result: $$\langle Z \rangle = \int \mathrm{d}\mu(t_{12}) \Big(\dots \Big) \exp\{SB_R + CC_R\}$$

$$SB_R = \frac{i\pi}{d} \Big\{ \sum_p \epsilon_p \mathrm{STr} \Big(t_{12} t_{21} \Big)_{pp} + \sum_q \tilde{\epsilon}_q \Big(t_{21} t_{12} \Big)_{qq} \Big\}$$

$$CC_R = -\frac{1}{2} \sum_c \mathrm{STr} \ln \Big(1 + T^{(c)} t_{12} t_{21} \Big)$$

$$T^{(c)} = 1 - |\langle S_{cc} \rangle|^2$$

4. Equivalence

To show: $\langle G \rangle = \langle Z \rangle$ including source terms (i.e., for all P, Q). Equate ϵ_p/d with $\kappa_p \langle d_{\rm R} \rangle$, $\tilde{\epsilon}_q/d$ with $\tilde{\kappa}_q \langle d_{\rm R} \rangle$, $T^{(c)}$ with $T^{(\gamma)}$ for $c, \gamma = 1, \ldots, \Lambda$, and define

$$\tau = -it_{12} \frac{1}{\sqrt{1 + t_{21} t_{12}}} \ , \ \tilde{\tau} = it_{21} \frac{1}{\sqrt{1 + t_{12} t_{21}}}$$

Upon identifying $\tau = Y \ , \ \tilde{\tau} = \tilde{Y}$ we find $SB_R = SB_G \ , \ CC_R = CC_G$. Also source terms are equal for both closed and open systems.

Integration measure $\mathrm{d}(\tau, \tilde{\tau})$ is flat Berezinian (just as for $\mathrm{d}(Y, \tilde{Y})$). Matrices τ and Y have same dimension, share same symmetry properties, have same compact parametrization of Fermion-Fermion blocks, and parametrize same extension of Efetov's coset space. Therefore, there exists one-to-one map of τ onto Y. Hence

$$\langle Z \rangle = \langle G \rangle$$

and all (P, Q) correlation functions are equal, both for levels and for S-matrix elements.

5. Discussion

(i) Equivalence shown for orthogonal symmetry. Unitary symmetry should be straightforward.

(ii) Incommensurate bond lengths required for ergodicity and color-flavor transformation.

(iii) Graphs completely connected. Removal of finite number of bonds probably irrelevant for $B \to \infty$. But otherwise qualitative changes expected (Anderson localization, f.i.).

(iv) Graphs classically mixing: Gap in spectrum of PF operator. In some cases weaker conditions seem sufficient for vanishing of corrections due to massive modes. Implications for classical chaos?

S. Gnutzmann and A. Altland, Phys. Rev. E 72 (2005) 056215.
S. Gnutzmann, J. P. Keating, and F. Piotet, Ann. Phys. (N.Y.) 325 (2010) 2595.

(v) Correlation functions not worked out explicitly. Proof uses one-to-one map.

(vi) Field-theoretical approach to quantum chaos built entirely upon PF operator not sufficient. Properties of quantum amplitudes (elements of vertex scattering matrix) also needed to show that massive modes irrelevant.

A. V. Andreev, B. D. Simons, O. Agam, and B. L. Altshuler, Nucl. Phys. B 482 (1996) 536.