RANDOM-MATRIX THEORY AND WAVE TRANSPORT IN DISORDERED WAVEGUIDES WITH FINITE-SIZE SCATTERERS [a]

Pier A. Mello

Instituto de Física, UNAM-México

Collaborators:

M. Díaz, Instituto de Física, UNAM

S. Tomsovic, Univ. of Washington, Seattle

M. Yépez, Instituto de Física, UNAM

[a] Orsay, March 13, 2014, *Wandering from Nuclei to Chaos*, "In Memoriam" of Oriol Bohigas

Introduction

Wave transport in disordered systems with uncorrelated disorder in the scattering potential has been extensively studied

Common feature of the problems investigated by our group: size of the individual scatterers: the smallest one in the problem: much smaller than the wavelength and is of no physical relevance.

Individual potentials are statistically independent and modeled by delta functions
Distance between successive scatterers is then taken very small
This allows considering the dense weak-scattering limit (DWSL)

We studied the conductance, its fluctuations, and the individual transmission coefficients

We found insensitivity of the results to details of the individual-scatterer statistical distribution, expressed in the form of a central-limit theorem

However, statistical correlations in the disordered potential, and finite size of individual scatterers and their separation, are known to have important effects in the transport properties.

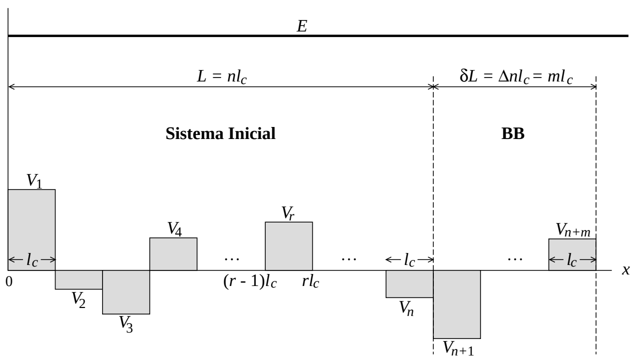
- E.g., in the so called "random-dimer model", fully transparent (delocalized) states have been discovered (Dunlap, Wu, Phillips, Hilke, et al., (1990-93))
- In the presence of long-range correlated disorder in 1D systems, Izrailev et al. (1999) found a mobility edge.
- De Moura et al. (1998): Anderson-like metal-insulator transition
- Schomerus and Titov (2005) report violations of single-parameter scaling due to short-range correlations.
- Izrailev et al. (1995) have discussed delocalization in the continuous random-dimer model
- and Lifshitz et al (1997) in continuous disordered systems consisting of δ-potentials and barrier-well sequences

Here we build on previous work [a]

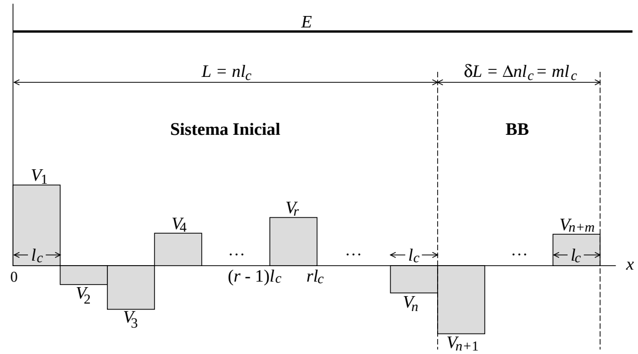
to study the simplest extension of the problems we studied earlier:

wave transport in 1D disordered systems, in which the various scatterers have a finite size:

succession of n barriers and wells, to be called steps, with a finite width and weak compared with energy E



[a] M. Díaz, P. A. Mello, M. Yépez and S. Tomsovic, *Europh. Lett.* **97**, 54002 (2012)



- *Fixed step width* l_c fits an arbitrary number of wavelengths $\delta/2\pi$; k= wave number; $\delta = kl_c \equiv$ *phase parameter*

- *Random heights* V_r $(r = 1, \cdots n)$, statist. indep. of one another; n distributions are uniform, with zero average, and all identical

This potential has a very simple type of correlation: it is perfectly correlated for points inside each step of length l_c (l_{corr} the correlation length), but uncorrelated for points lying at a distance larger then l_c.

Our interest in the model:

- It allows investigating the effect of a finite size of the scatterers. The model could be interpreted as simulating a potential described by a random process with a correlation length l_c.

- It exhibits peculiar transport properties which, to the best of our knowledge, have not been discussed in the literature.

- We wish to encourage experimental realization of the system in the laboratory

Summary

We study, for the model we just described:

• the Landauer resistance of the chain and

• its transmission coefficient,

averaged over an ensemble of realizations,

as functions of the number of scatterers n and phase parameter δ.

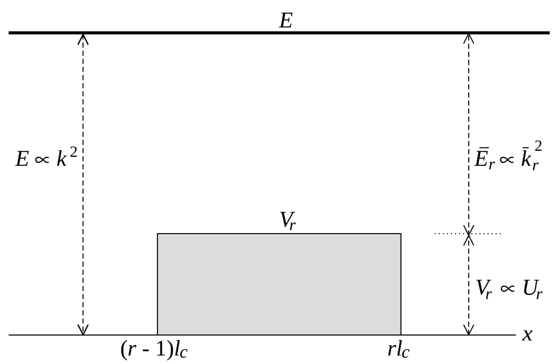
• We use a *random transfer-matrix* method

to perform the theoretical analysis of the problem

• We verify the results by means of computer simulations.

The Theoretical Model

The r-th scatterer of the chain: $E > V_r$



$$U_r = \frac{2mV_r}{\hbar^2}\ , \qquad y_r = U_r l_c^2 = \frac{U_r}{k^2}\,(kl_c)^2 \equiv \frac{U_r}{k^2}\,\delta^2\ , \qquad \delta = kl_c$$

Transfer matrix for the r-th scatterer:

$$\boldsymbol{M}_r = \begin{bmatrix} \alpha_r & \beta_r \\ \beta_r^* & \alpha_r^* \end{bmatrix}, \qquad |\alpha_r|^2 - |\beta_r|^2 = 1$$

$\boldsymbol{M}_r$ depends on three parameters: E, U_r, l_c: these occur in the dimensionless combinations δ and y_r.

The chain of steps

Transfer matrix for a chain of n (non-overlapping) steps:

$$\boldsymbol{M}^{(n)} = \boldsymbol{M}_n \cdots \boldsymbol{M}_r \cdots \boldsymbol{M}_2 \boldsymbol{M}_1 = \begin{bmatrix} \alpha^{(n)} & \beta^{(n)} \\ \left(\beta^{(n)}\right)^* & \left(\alpha^{(n)}\right)^* \end{bmatrix}$$

i) Landauer resistance of the chain

$$R^{(n)}/T^{(n)} = |\beta^{(n)}|^2 \equiv \lambda^{(n)}$$

ii) Transmission coefficient of the chain

$$T^{(n)} = \frac{1}{1+\lambda^{(n)}} \; .$$

The ensemble of chains

Choose the $y_r = U_r l_c^2$'s $(r = 1, \cdots, n)$ statistically independent and uniformly distributed in $(-y_0, y_0)$.

In our description by means of transfer matrices, we end up with an *ensemble of transfer matrices*

The recursion relation

The original system of n scatterers is extended by one scatterer

$$\boldsymbol{M}^{(n+1)} = \boldsymbol{M}_{n+1}\boldsymbol{M}^{(n)}.$$

Recursion relation for ⟨Landauer's resistance of the chain⟩

$$\begin{cases} \left[1+2\langle|\beta^{(n+1)}|^2\rangle\right] - \left[1+2\langle|\beta^{(n)}|^2\rangle\right] \\ \qquad = 2\langle|\beta_{n+1}|^2\rangle\left[1+2\langle|\beta^{(n)}|^2\rangle\right] + 2\left[\langle\alpha_{n+1}\beta^*_{n+1}\rangle\langle\alpha^{(n)}\beta^{(n)}\rangle + \text{c.c.}\right], \\ \\ \langle\alpha^{(n+1)}\beta^{(n+1)}\rangle - \langle\alpha^{(n)}\beta^{(n)}\rangle \\ \qquad = \langle\alpha_{n+1}\beta_{n+1}\rangle\left[1+2\langle|\beta^{(n)}|^2\rangle\right] + \left(\langle\alpha^2_{n+1}\rangle - 1\right)\langle\alpha^{(n)}\beta^{(n)}\rangle \\ \qquad\qquad + \langle\beta^2_{n+1}\rangle\langle\alpha^{(n)}\beta^{(n)}\rangle^* \end{cases}$$

Notice: $\langle|\beta^{(n)}|^2\rangle$ coupled to $\langle\alpha^{(n)}\beta^{(n)}\rangle$.

Recursion relation has the structure:

$$\boxed{z(n+1) = \Omega\ z(n)}$$

$$z(n) = \left[A(n)/2,\ ib(n)/\sqrt{2},\ -ib^*(n)/\sqrt{2}\right]^T \quad ; \qquad z(0) = [1/2, 0, 0]^T$$

We used the definitions

$$A(n) = 1 + 2\langle|\beta^{(n)}|^2\rangle, \qquad b(n) = e^{2in\delta}\langle\alpha^{(n)}\beta^{(n)}\rangle.$$

Matrix Ω is *complex symmetric* and *independent of* n.

$$\boxed{z(n) = \Omega^n z(0)}$$

If Ω has *no double characteristic values*, it can be diagonalized by a *complex orthogonal transformation*

$$\Omega = \Omega_0 + \Delta\Omega(y_0, \delta);\ \Omega_0 = \begin{bmatrix} \mu_1^{(0)} = 1 & 0 & 0 \\ 0 & \mu_2^{(0)} = e^{2i\delta} & 0 \\ 0 & 0 & \mu_3^{(0)} = e^{-2i\delta} \end{bmatrix}$$

Unperturbed matrix $\Omega_0 \equiv$ limiting value for $V \equiv 0$

Average Landauer resistance vs n for δ in regime I

Let δ be far from π. E.g.: $\delta = \pi/2$,

the three unperturbed eigenvalues: $\{\mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}\} = \{1, -1, -1\}$.

Regime I $\equiv \{\mu_2^{(0)}, \mu_3^{(0)}\}$ far away from $\mu_1^{(0)}$:

they may be considered *effectively decoupled* when we turn on a weak interaction, $y_0^2 \ll 1$.

We then *restrict to the* 1×1 *block of* Ω:

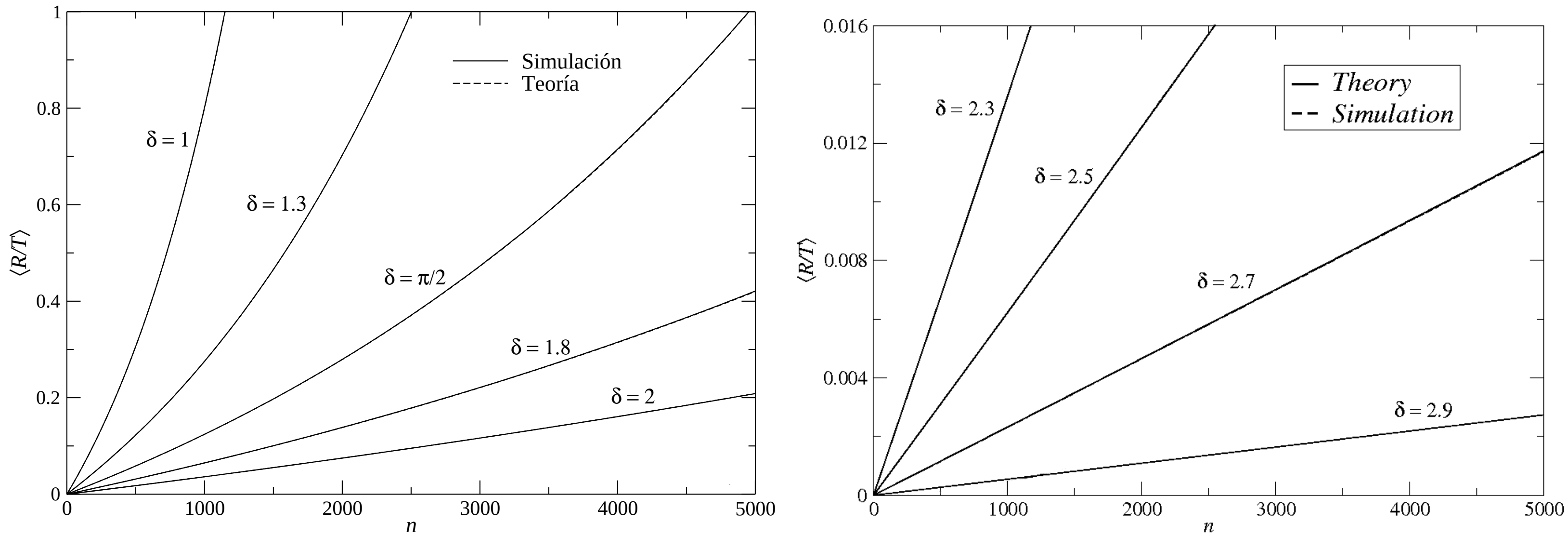
$$A(n) \approx (\Omega_{11})^n A(0) = \left(1 + 2\langle|\mathring{\beta}_1|^2\rangle\right)^n = e^{2n\frac{1}{2}\ln\left(1+2\langle|\mathring{\beta}_1|^2\rangle\right)} \equiv e^{2nl_c/\ell},$$

the well known *Landauer's exponential behavior*, where, now:

$$\frac{l_c}{\ell} = \frac{1}{2}\ln\left(1 + 2\langle|\mathring{\beta}_1|^2\rangle\right).$$

The new feature is the dependence of the mfp on δ. In the WSL:

$$\langle|\mathring{\beta}_1|^2\rangle = \frac{l_c}{\tilde{\ell}} + O\left(\frac{y_0}{\delta^2}\right)^4, \qquad \frac{l_c}{\tilde{\ell}} = \frac{y_0^2}{12\delta^4}\sin^2\delta.$$



Average Landauer resistance $\langle (R/T)^{(n)} \rangle$ vs. n, for δ in regime I; $y_0 = 0.09$.

a) $1 < \delta < 2$ b) $2.3 < \delta < 2.9$.

Theory :
$$\langle |\beta^{(n)}|^2 \rangle = \frac{1}{2}\left(e^{2nl_c/\ell} - 1\right).$$

$$\langle |\beta_1|^2 \rangle = l_c/\tilde{\ell} + O\left(y_0/\delta^2\right)^4, \qquad l_c/\tilde{\ell} = (y_0^2/12\delta^4)\sin^2\delta.$$

Numerical simulation: ensemble of 10^6 realizations.

As δ increases towards π, the average resistance decreases and the mfp increases; i.e., the system tends to delocalize.

Excellent agreement

Average Landauer resistance vs n for δ in regime II

For $2.9 \leq \delta \leq 3.4$,

$\{\mu_2^{(0)} = e^{2i\delta},\ \ \mu_3^{(0)} = e^{-2i\delta}\}$ are not far enough away from $\mu_1^{(0)} = 1$

to be effectively decoupled.

E.g., for $\delta = \pi$, $\quad \mu_1^{(0)} = \mu_1^{(0)} = \mu_1^{(0)} = 1$

i.e., the three unperturbed eigenvalues are degenerate

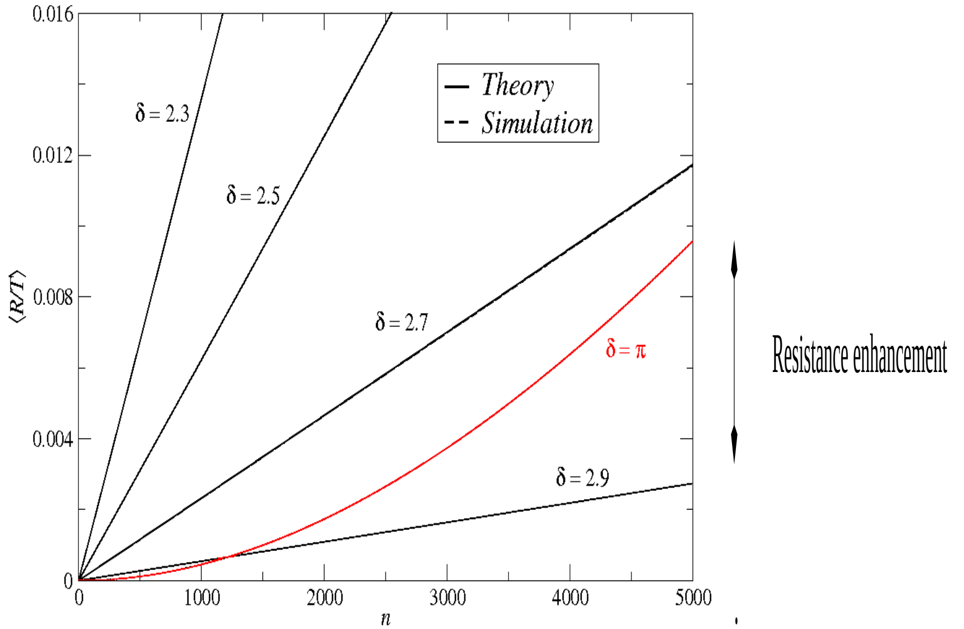
If δ is very close to π, Ω has to be diagonalized exactly:

novel behavior shows up as a consequence of the coupling

Exact solution has the structure:

$$A(n) = \sum_{a=1}^{3} (O_{1a})^2 \ \mu_a^n$$

μ_a = a-th eigenvalue of Ω



Average Landauer resistance $\langle (R/T)^{(n)} \rangle$ vs. n, for $\delta = \pi$, in red; $y_0 = 0.09$.

Theory: above Ω diagonalized exactly
Numerical simulation: ensemble

Excellent agreement

Results for $2.3 < \delta < 2.9$ (regime I) shown in black *for comparison*

Notice resistance enhancement as a result of the coupling:
system is *more localized for* $\delta = \pi$ *than for neighboring values of* δ
i.e., *the tendency to delocalize as we move towards* $\delta = \pi$

is reversed in the vicinity of $\delta = \pi$.

Write $\delta = \pi - \epsilon$

For ϵ not too small, so that the unperturbed eigenvalues do not become degenerate,

we may use *perturbation theory (PT) in the parameter y_0*:

$$A(n) \approx A_1 e^{n \ln(1+\Delta M_{11})} + \left[A_2 e^{n \ln(e^{2i\delta}+\Delta M_{22})} + cc \right]$$

$e^{2i\delta}$ dominates over ΔM_{22}: we estimate

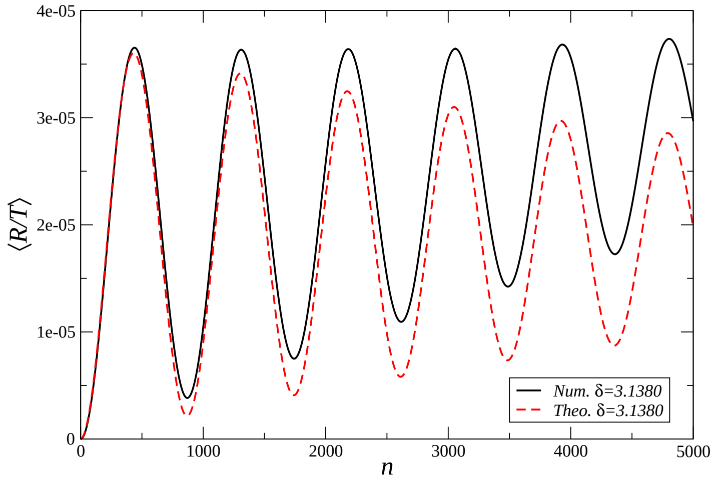
$$\text{2nd term} \approx e^{n \ln e^{2i\delta}} = e^{n 2i\delta} = e^{2in(\pi+\epsilon)} = e^{2in\epsilon}$$

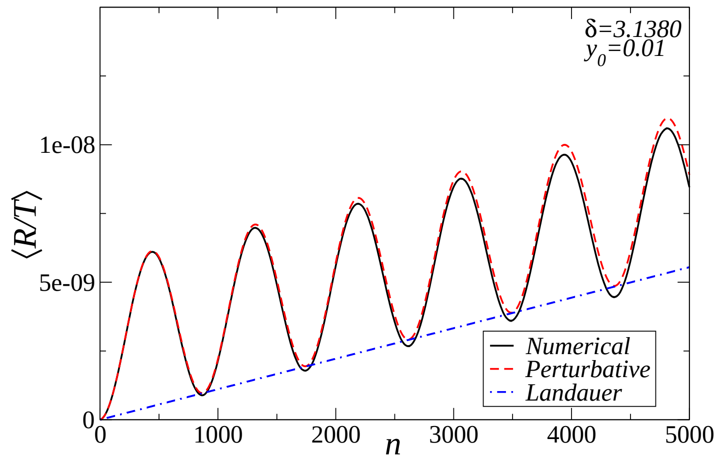
This result *oscillates with n*, with a period Δn that satisfies $2\ \Delta n\ \epsilon = 2\pi$, so that

$$\Delta n \sim \frac{\pi}{\epsilon}$$

i) Period Δn independent of y_0

ii) Period decreases as we go away from $\delta = \pi$



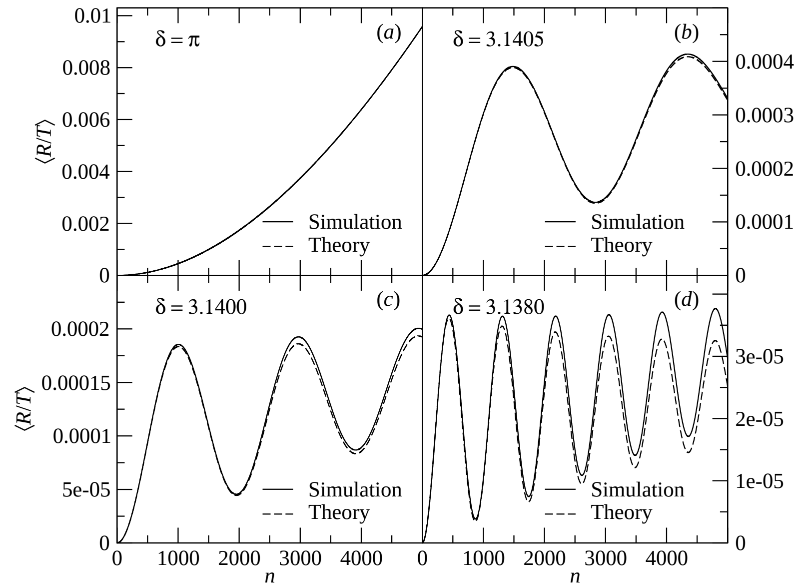


Average Landauer resistance vs n average Landauer resistance vs n, for $\delta = 3.1380$

Theory: Perturbation theory in y_0
Numerical simulations: ensemble

a) $y_0 = 0.09$, b) $y_0 = 0.01$.

Period Δn is independent of y_0, as predicted by theory



Average Landauer resistance $\langle (R/T)^{(n)} \rangle$ vs. n, for $\delta \approx \pi$; $y_0 = 0.09$.

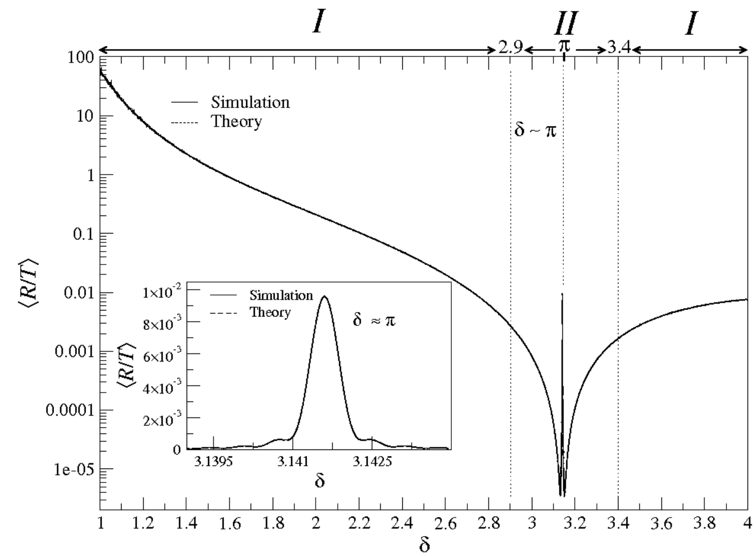
Theory: Continuous approximation from earlier theory [a]

Computer simulations: ensemble of 10^6 realizations

Period Δn becomes smaller as we go away from $\delta = \pi$, as predicted by theory

[a] M. Díaz, P. A. Mello, M. Yépez and S. Tomsovic, *Europh. Lett.* **97**, 54002 (2012)

Average Landauer resistance for fixed n vs δ in regimes I and II



Average Landauer resistance $\langle R/T \rangle^{(n)}$ vs δ, for $n = 5000$ scatterers; $y_0 = 0.09$.

Theory: *Regime I*: present theory

Regime II: Continuous approximation from earlier theory.

Computer simulations: ensemble of 10^5 realizations

Main figure: i) *gross-structure* behavior in a semilog scale: regimes I and II, ii) Well inside regime II: *enhancement of nearly three orders of magnitude*: *incipient "forbidden region"*

Inset: region $\delta \approx \pi$ in a linear scale.

Description

Regime I: average resistance decreases as $\delta \Rightarrow \pi$:

recall earlier finding that the system becomes more delocalized

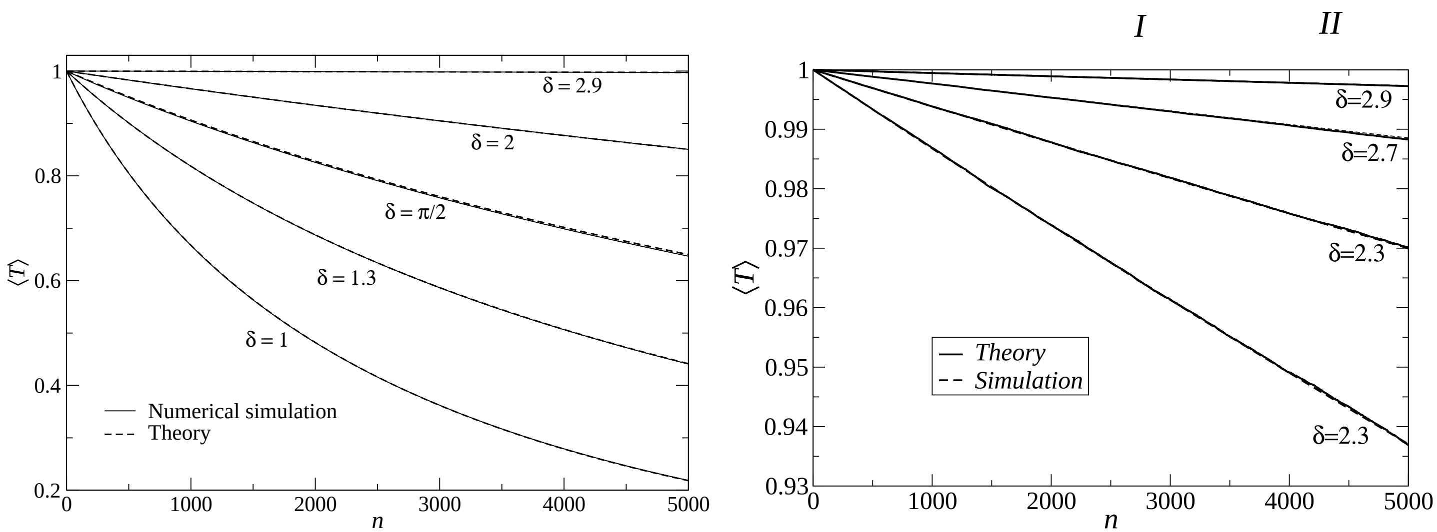
Well inside regime II: this tendency is reversed

We show analytically and verify numerically, the existence of an *"incipient forbidden region"* I.e., in a *scattering experiment*:

- for $\delta \approx \pi$, resistance presented by the system has a *dramatic enhancement of nearly three orders of magnitude*
- incipient: peak-to-valley enhancement in the resistance *keeps becoming greater for larger n's*
- It results from coherent contribution of all the barriers and wells with same width l_c [a].

[a] Gastón García C., *private communication*, and Phys. Rev. B**56**, 4845 (1997), Phys. Rev. A**79**, 052103 (2009): S-matrix pole structure: a possible explanation?

Average Transmission Coefficient vs n for δ in regime I



Average transmission coefficient $\langle T^{(n)} \rangle$ vs. n, for δ in regime I; $y_0 = 0.09$.

a) $1 < \delta < 2$ b) $2.3 < \delta < 2.9$.

Theory: $\langle T \rangle = 2\mathrm{e}^{-\tilde{s}/4} \int_0^\infty \mathrm{e}^{-\tilde{s}t^2} \pi t [\tanh(\pi t)/\cosh(\pi t)] \mathrm{d}t, \qquad \tilde{s} = L/\tilde{l}$

Numerical simulations: ensemble of 10^6 realizations

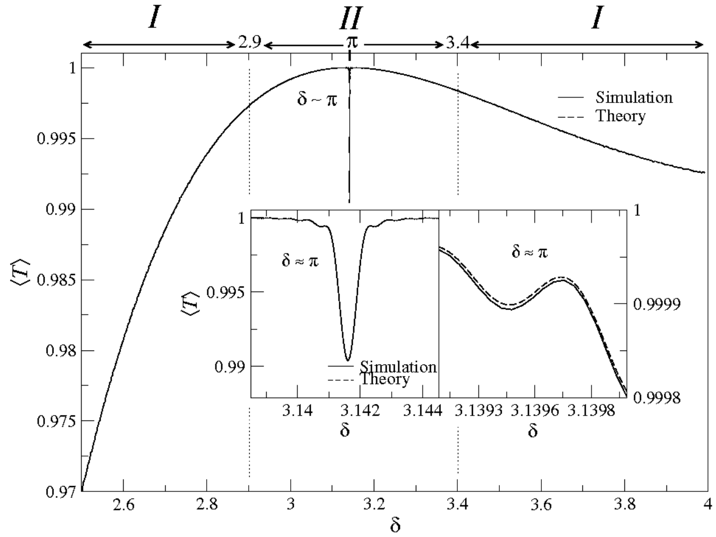
Error bar $\sim 10^{-3}$ for $\delta = \pi/2$ and $n = 5000$

As δ increases towards π, $\langle T \rangle$ increases and mfp increases;

i.e., the system tends to delocalize.

Excellent agreement

Average Transmission for fixed n vs δ in regimes I and II



Average transmission $\langle T \rangle$ vs δ, for $n = 5000$ scatterers and $y_0 = 0.09$

Theory: *Regime I*: present theory

Regime II: continuous approximation from earlier theory.

Computer simulations: ensemble of 10^5 realizations

Main figure: 1) gross-structure behavior in regimes I and II

ii) Well inside regime II: significant drop: "incipient forbiden" region

Insets: zoom of the region $\delta \approx \pi$

- $\langle T \rangle$ exhibits a gross-structure behavior in the form of a "bump". For weak scatterers, system is almost transparent in regime II, and regime I is more localized.

- Trend to delocalize as $\delta = \pi$ is approached reverses in an extremely narrow window around $\delta = \pi$

- For $\delta \approx \pi$, we see analytically and numerically an "incipient" *forbidden region* which becomes ever deeper as n increases
- In a scattering experiment, $\langle T \rangle$ suffers a *dramatic reduction*, with a peak to valley ratio that increases with n. Consistent with the behavior of $\langle R/T \rangle$ already noticed

- Incipient forbidden region has features in common with a finite stretch of a periodic Kronig Penney potential:

i) it becomes deeper as n increases
ii) it becomes wider as the strength of the potential increases (y_0),
iii) it shows interference fringes at the edges

CONCLUSIONS

- We discussed wave transport in 1D disordered chains of barriers and wells with constant width l_c and random strength

- Weak scatterers: i) system almost transparent for $\delta = kl_c \approx \pi$
  ii) less delocalized farther away

- iii) for $\delta \approx \pi$: incipient "forbidden region" where
  a) $\langle T \rangle$ suffers a dramatic reduction
  b) Landauer resistance increases by various orders of magnitude

- These phenomena are described very well by the theoretical analysis and are verified by computer simulations

- Results suggest interest in experimental realization of the system

- Same model has been studied recently by Izrailev et al. [a] using a mapping to a "classical phase space" and iterating there.

[a] I. F. Herrera-Gonzalez, F.M. Izrailev and N.M. Makarov, Phys. Rev. E 88, 052108 (2013)