Stellar parametrization for chaotic eigenfunctions

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(S.N.– A.V., J. Stat. Phys. **92** (1998) 431–518)

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Initial problem

A largely unresolved question in quantum mechanics:

Semiclassical ($\hbar \to 0$) *form* of *quantum bound states*
when the corresponding classical dynamics is far from integrable ("chaotic"):

the "chaotic eigenfunction" problem.

In mathematical terms:

$$\hat{H}\Psi = E\Psi$$

when the classical dynamics at energy E is *strongly coupled*
(ergodic, mixing, hyperbolic, etc.)

- for Schrödinger operators $\hat{H} = -\hbar^2 \Delta_q + V(q) \qquad (q \in \mathbb{R}^n)$
- for Laplacians $\hat{H} = -\hbar^2 \Delta_q$ in bounded domains [billiards] or
  on compact manifolds [geodesic motion].

Reduced setting

Models exist on 2D, *compact*, phase spaces:

$$
\begin{array}{rcl}
\text{continuous time} & \longrightarrow & \text{discrete time} \\
\text{Hamiltonian flow} & \longrightarrow & \text{area-preserving map} \\
\text{quantum Hamiltonian } \hat{H} & \longrightarrow & \textit{quantum map } \hat{U} \text{ (unitary)} \\
\hat{H}\Psi = E\Psi \quad \text{on } L^2(\mathbb{R}^2) & \longrightarrow & \hat{U}\psi = \mathrm{e}^{\mathrm{i}\theta}\,\psi \quad \text{on } \mathcal{H} = \mathbb{C}^N
\end{array}
$$

$$
\boxed{\text{Semiclassical regime} \iff N \equiv \text{Area}/h \to +\infty.}
$$

Simplest 2D compact phase spaces:

- S^2 (→ quantum spin)
- $\mathbb{T}^2$ (→ quantum torus), e.g. (Weyl):

$$\psi(q+1) = \psi(q) \quad \text{and} \quad \tilde{\psi}(p+1) = \tilde{\psi}(p),$$

$$(\tilde{\psi}(p) \equiv (2\pi\hbar)^{-1} \int \psi(q)\, \mathrm{e}^{-\mathrm{i}pq/\hbar}\, \mathrm{d}q, \qquad h = 2\pi\hbar \equiv 1/N),$$

the simplest case carrying proven chaotic classical maps,

moreover quantizable.

Phase-space representations on quantum torus

$$\mathbb{T}^2 \approx [0,1]^2 \qquad \mapsto \qquad \{\psi(q+1)=\psi(q),\, \tilde{\psi}(p+1)=\tilde{\psi}(p)\}.$$

• **Bargmann** ($z \equiv \frac{q-\mathrm{i}p}{\sqrt{2}}$) : $\quad \psi(z) = \langle z \mid \psi \rangle \quad$ *entire*, quasi-periodic,

$$\psi(z+\tfrac{1}{\sqrt{2}}) = \mathrm{e}^{\pi N(\frac{1}{2}+\sqrt{2}z)}\,\psi(z) \quad \text{and} \quad \psi(z+\tfrac{\mathrm{i}}{\sqrt{2}}) = \mathrm{e}^{\pi N(\frac{1}{2}-\mathrm{i}\sqrt{2}z)}\,\psi(z)$$

$\Rightarrow \quad \psi(z)$ has *exactly* N *zeros* in any unit (q,p) square.

• **Husimi** ($\vec{x} \equiv (q,p)$) : $\quad H_\psi(\vec{x}) = \dfrac{|\langle z \mid \psi \rangle|^2}{|\langle z \mid z \rangle|} \equiv \mathrm{e}^{-2\pi N|z|^2}\,|\psi(z)|^2$

a *probability density* on $\mathbb{T}^2$, with exactly N zeros $\vec{x}_j$.

• **Stellar**: the torus density $\quad \rho_\psi(\vec{x}) = N^{-1} \sum_{j=1}^{N} \delta^2(\vec{x}-\vec{x}_j)$;

a genuine *parametrization of pure states*, thanks to Hadamard product:

$$\psi(z) = Z_\psi \text{ “}\prod_k (z-z_k)\text{”}.$$

Stellar parametrization ρ_ψ

	PROS	CONS
$\psi \mapsto \rho$	true parametrization	indirect
$\rho \mapsto \psi$	explicit formula (Hadamard product)	linear features obscured (Σ, $\perp$, dynamics)
Semiclassical limit	phase-space analysis	dual to zeros
Generality	Kähler phase spaces	partial; abstruse in higher D
For eigenfunctions	new, unorthodox picture	zeros are elusive, not directly computable
As nodal sets	purest	residual dependence on complex structure
	NONLINEAR	**NONLINEAR**

Eigenfunctions: integrable case

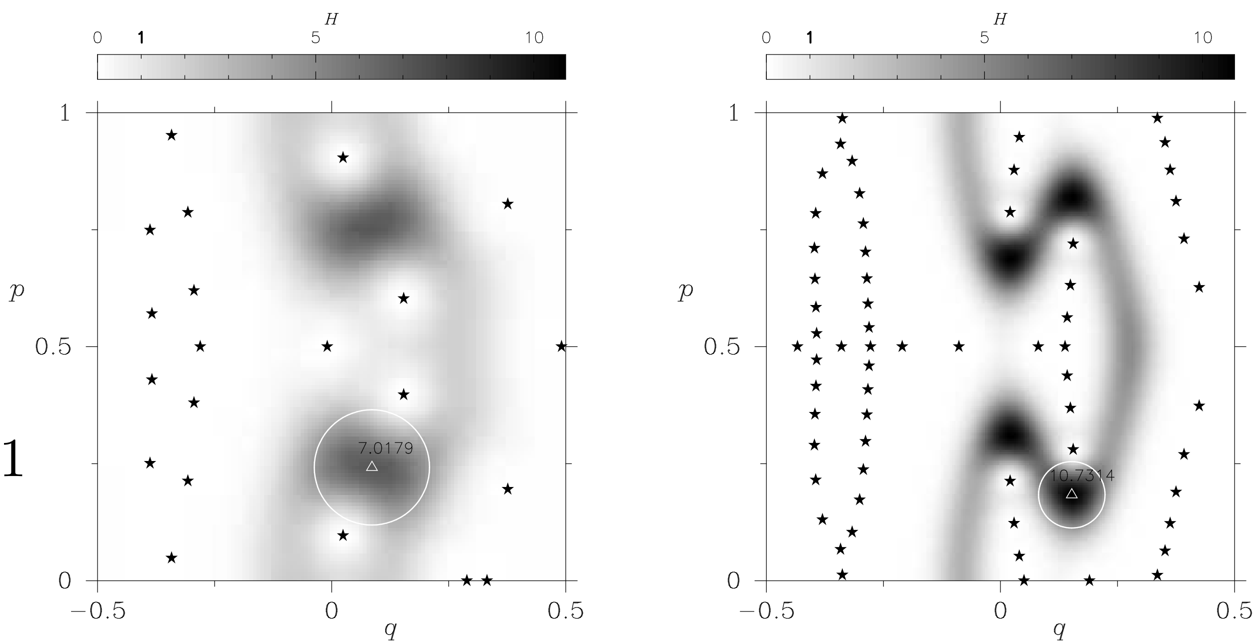
E.g., $\hat{P}\psi(q) = \lambda\psi(q)$ (*ordinary* differential equation).

Algebraically, Bargmann/$(z, \bar{z}) \approx$ Schrödinger/(q, p) $\Longrightarrow$

$\psi(z) \sim \sum \mathrm{e}^{\frac{1}{\hbar}\int_{\{P=\lambda\}}^{z} \bar{z}\, \mathrm{d}z}$ (WKB form) for $\hbar \to 0$ $\Longrightarrow$

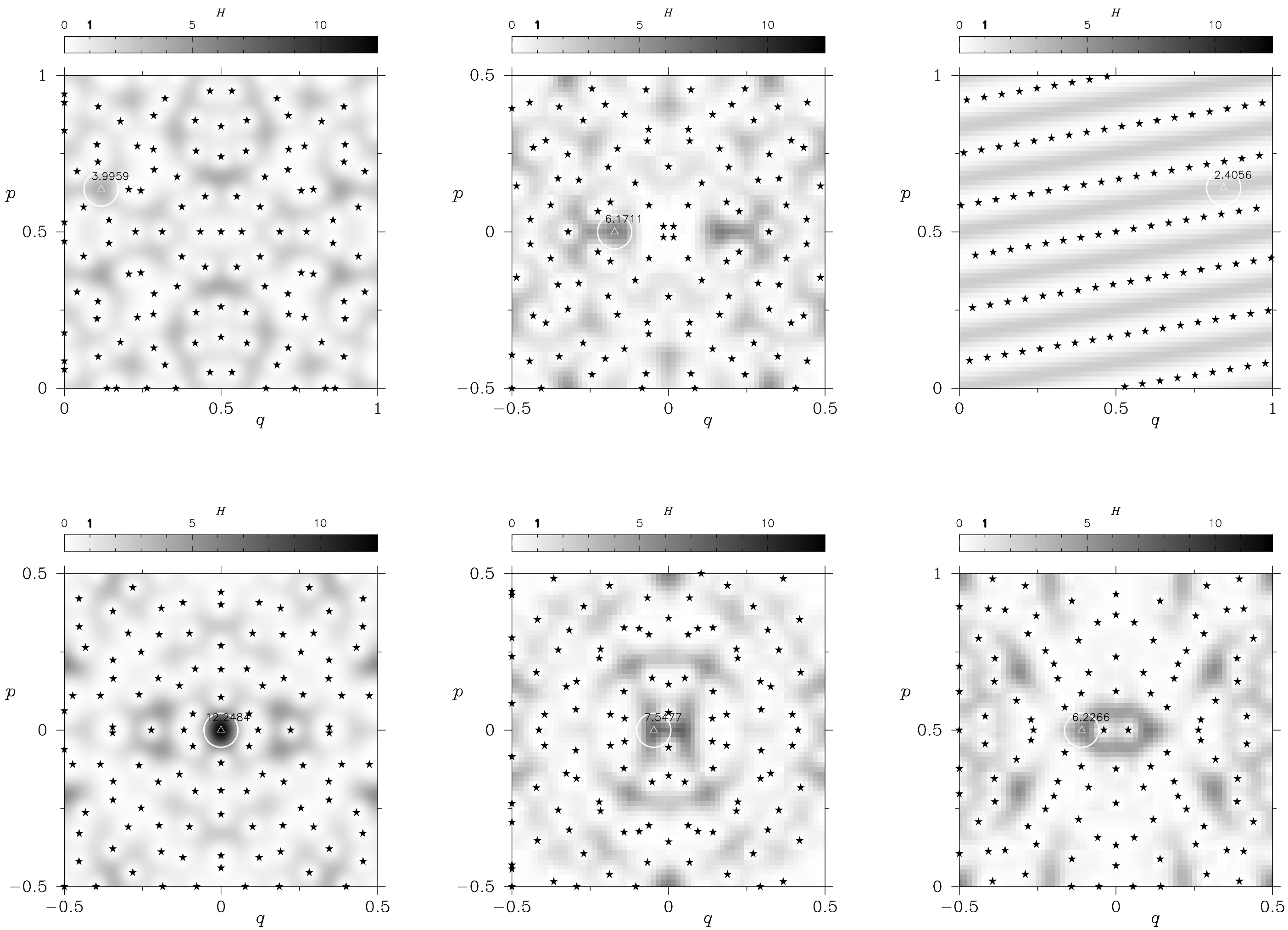
in the $N \to \infty$ limit, the N zeros concentrate on **curves** (anti-Stokes lines)

$N = 21$



$N = 63$

Eigenfunctions: a chaotic case (cat map, $N = 107$)



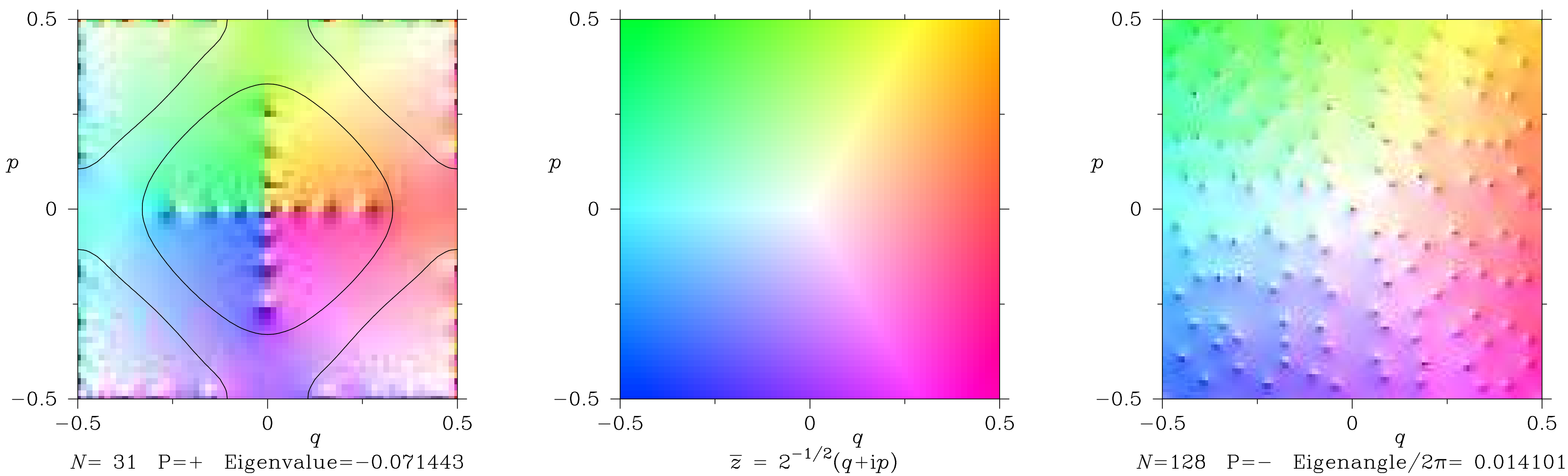
Eigenfunctions: chaotic case

- **Theorem** (conjectured by Lebœuf–A.V., proved by Nonnenmacher):

as $N \to \infty$, if the Husimi density $H_\psi \xrightarrow{\text{w-}*} 1$ (quantum ergodicity),
then the **stellar** density $\rho_\psi \xrightarrow{\text{w-}*} 1$ (zeros *equidistribute* in phase space).

- **Many open issues:**

- bulk-singular limit (e.g., $\hbar\,[\psi'/\psi](z) \xrightarrow{L^s} \bar{z}$ for $1 \le s < 2$)



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then the **stellar** density $\rho_\psi \xrightarrow{\text{w-}*} 1$ (zeros *equidistribute* in phase space).

• **Open issues (continued):**

- converse results? Speed of convergence?
- analysis of finite-N fluctuations - under $\int H_\psi(\vec{x})\,\mathrm{d}^2x = 1$, what to use?

$$\begin{aligned} ||H_\psi||_2 &= \left(\textstyle\int H_\psi(\vec{x})^2\,\mathrm{d}^2x\right)^{1/2} && \text{(cf. inverse participation ratio)} \\ \text{other } L^r\text{-norms } ||H_\psi||_r &= \left(\textstyle\int H_\psi(\vec{x})^r\,\mathrm{d}^2x\right)^{1/r} && (r = \infty \text{ to measure scars?}) \\ \text{H-function}[H_\psi] &= \textstyle\int H_\psi(\vec{x}) \log H_\psi(\vec{x})\,\mathrm{d}^2x && (= -[\text{Wehrl's entropy}]) \\ \text{GM}[H_\psi] &= \exp \textstyle\int \log H_\psi(\vec{x})\,\mathrm{d}^2x && \text{(geometric mean)} \end{aligned}$$

or

Fourier coefficients $\rho_{\vec{k}} = N^{-1} \sum_{j=1}^{N} \mathrm{e}^{-2\pi\mathrm{i}\vec{k}\cdot\vec{x}_j}$ for $\vec{k} \in \mathbb{Z}^2$ (save $\rho_{\vec{0}} \equiv 1$)

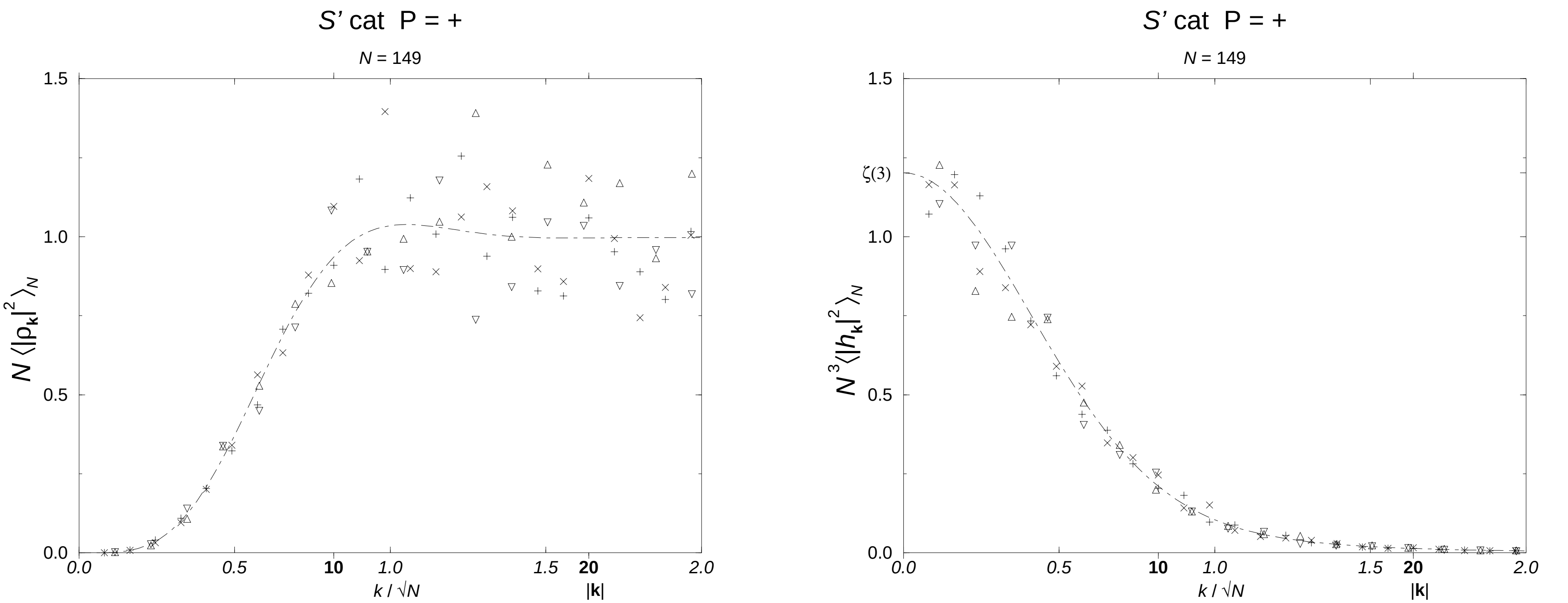
Equidistribution of zeros $\iff$ $\rho_{\vec{k}} \to 0$ for $N \to \infty$ at fixed $\vec{k}$

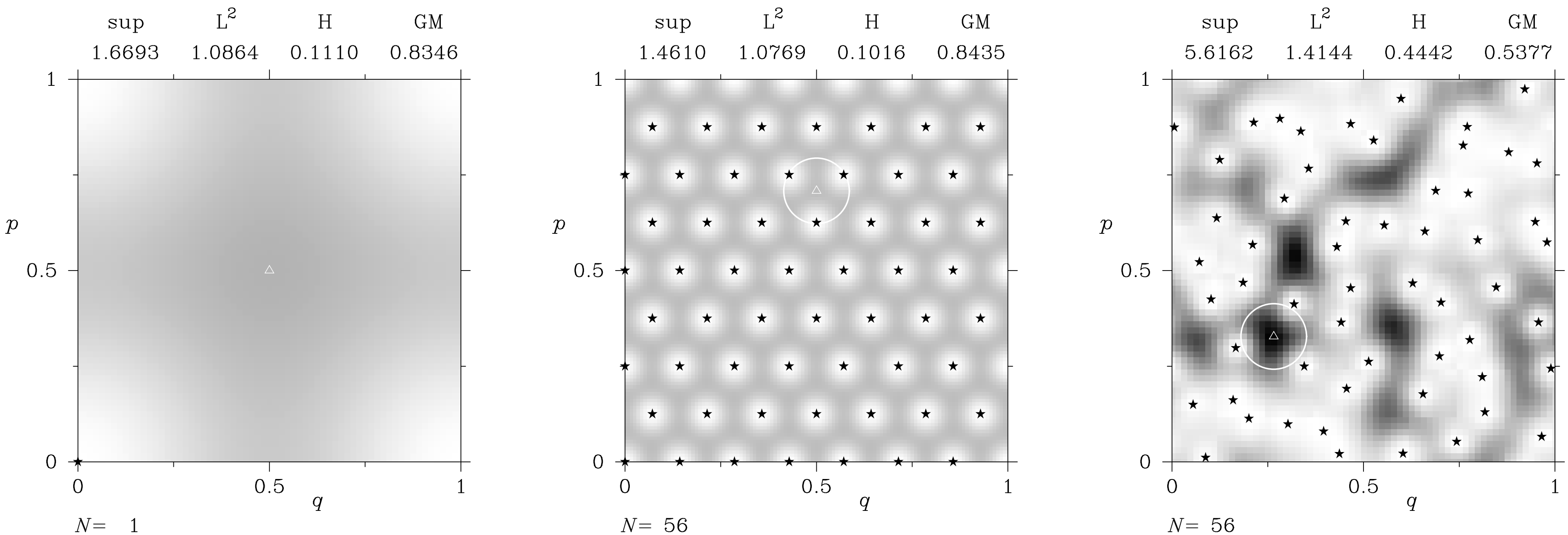
Conjecture: *quantum ergodicity* $\iff$ $\rho_{\vec{k}} = o(1/N)$

Statistical model: random polynomials

For chaotic eigenfunctions in the $N \to \infty$ limit, the zeros very accurately follow the statistics of SU(2) random polynomials (a **conjecture**).
Ex.: form factor $\langle |\rho_{\vec{k}}|^2 \rangle_N \overset{N\to\infty}{\sim} N^{-1} F_2(|\vec{k}|/\sqrt{N}) \propto N^{-3} \quad (\vec{k} \neq \vec{0} \text{ fixed})$

$$F_2(\kappa) = -2\pi\kappa^2 \int_0^\infty y \; J_2(\kappa\sqrt{8\pi y})\,(1 - \coth y)\,dy.$$





sup L2 H GM
1.6693 1.0864 0.1110 0.8346
p
q
N= 1
sup L2 H GM
1.4610 1.0769 0.1016 0.8435
p
q
N= 56
sup L2 H GM
5.6162 1.4144 0.4442 0.5377
p
q
N= 56

Abrikosov constants for vortex lattices

(superconductors / rotating Bose–Einstein condensates)

For a 2D superconductor in a uniform transverse magnetic field (or for a rotating Bose–Einstein condensate), the lowest Landau level is described by a Bargmann space for a quantum torus.

In Ginzburg–Landau picture, the field vortex distribution ψ minimizes

$$\beta\{\psi\} = \frac{\int H_\psi(\vec{x})^2 \, \mathrm{d}^2 x}{(\int H_\psi(\vec{x}) \, \mathrm{d}^2 x)^2}.$$

Numerically, the square and equilateral-triangle lattices are critical points of the functional β, giving the values (*Abrikosov constants*)

$$\beta_\square \approx 1.1803, \qquad \beta_\triangle \approx 1.1596 \,.$$

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$$\beta_\square \approx 1.1803, \qquad \beta_\triangle \approx 1.1596 \; .$$

Analytically (Nonnenmacher–A.V.):

$$\beta_\square = \frac{\Gamma(1/4)^2}{2\,\pi^{3/2}}, \qquad \beta_\triangle = \frac{3\,\Gamma(1/3)^3}{2^{7/3}\pi^2}.$$

The Fourier coefficients $\rho_{\vec{k}}$ $(\vec{k} \in \mathbb{Z}^2)$

$$\rho_\psi(\vec{x}) = \frac{1}{N}\sum_{j=1}^{N} \delta^2(\vec{x}-\vec{x}_j) \;=\; \frac{1}{(2\pi)^2}\sum_{\vec{k}\in\mathbb{Z}^2} \rho_{\vec{k}}\, \mathrm{e}^{2\pi \mathrm{i}\vec{k}.\vec{x}}$$

$$\frac{1}{N}\log\frac{H_\psi(\vec{x})}{\mathrm{GM}[H_\psi]} \;=\; \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{0\}} -\frac{\rho_{\vec{k}}}{\pi|\vec{k}|^2}\, \mathrm{e}^{2\pi \mathrm{i}\vec{k}.\vec{x}}$$

Semiclassical features should mainly imprint *low Fourier coefficients.*

Mathematical difficulty:

$\dim\{\vec{x}_j\}_{j=1,\dots,N} = 2N$ — *zero* redundancy (good)

$\dim\{\rho_{\vec{k}}\}_{\vec{k}\in\mathbb{Z}^2} = \infty^2$: *infinite* redundancy (bad)

How to overcome the redundancy? (cf. crystallography).

Answer known in 1D ($\mathbb{T}^1$): the Newton relations, but we are on $\mathbb{T}^2$.

Analogous problem: remove the redundancy from the *Wigner* (instead of Husimi) representation.

Dynamical questions about the zeros

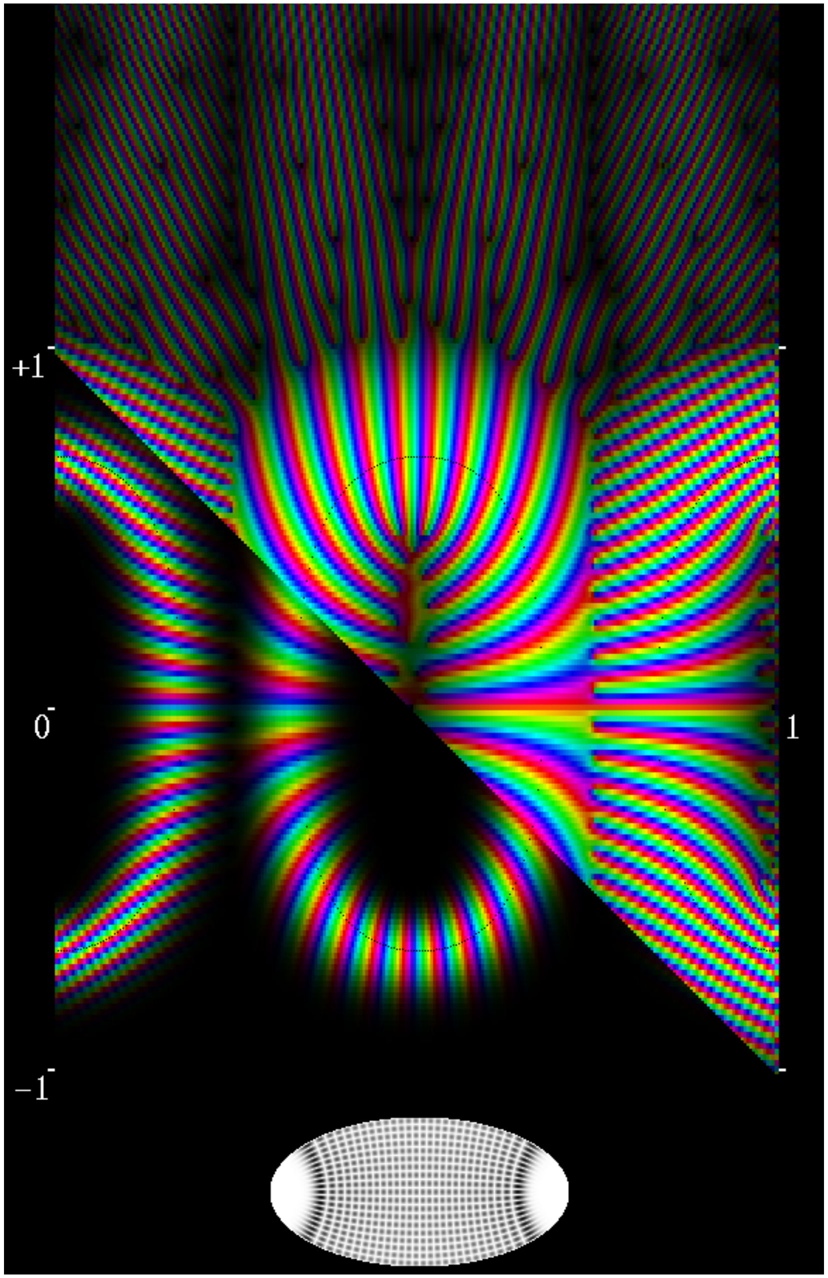
A time-dependent equation

$$i\dot{\Psi} = \hat{\mathcal{H}}\Psi(t)$$

yields an N-body symplectic system for the corresponding motion of the Husimi zeros of $\Psi(t)$.

For *quadratic* $\hat{\mathcal{H}}$ (quantized *linear* canonical transformations), it is a system of vortex-like evolution equations under *2-body* forces.

Its direct dynamical study could contribute to the understanding of:
- the effects of a change of complex structure (as interpolated by a continuous trajectory on the modular domain);
- the fine structure of the eigenconstellations of quantized cat maps.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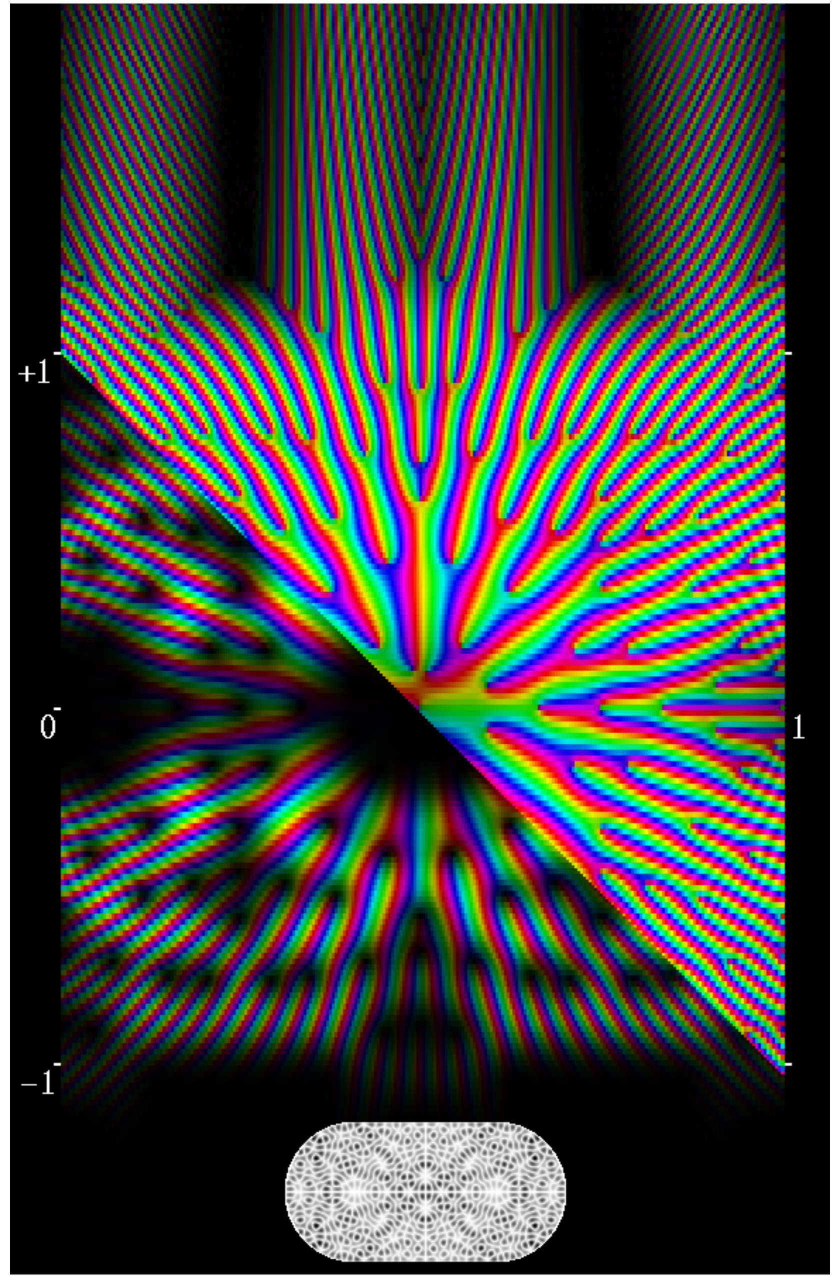
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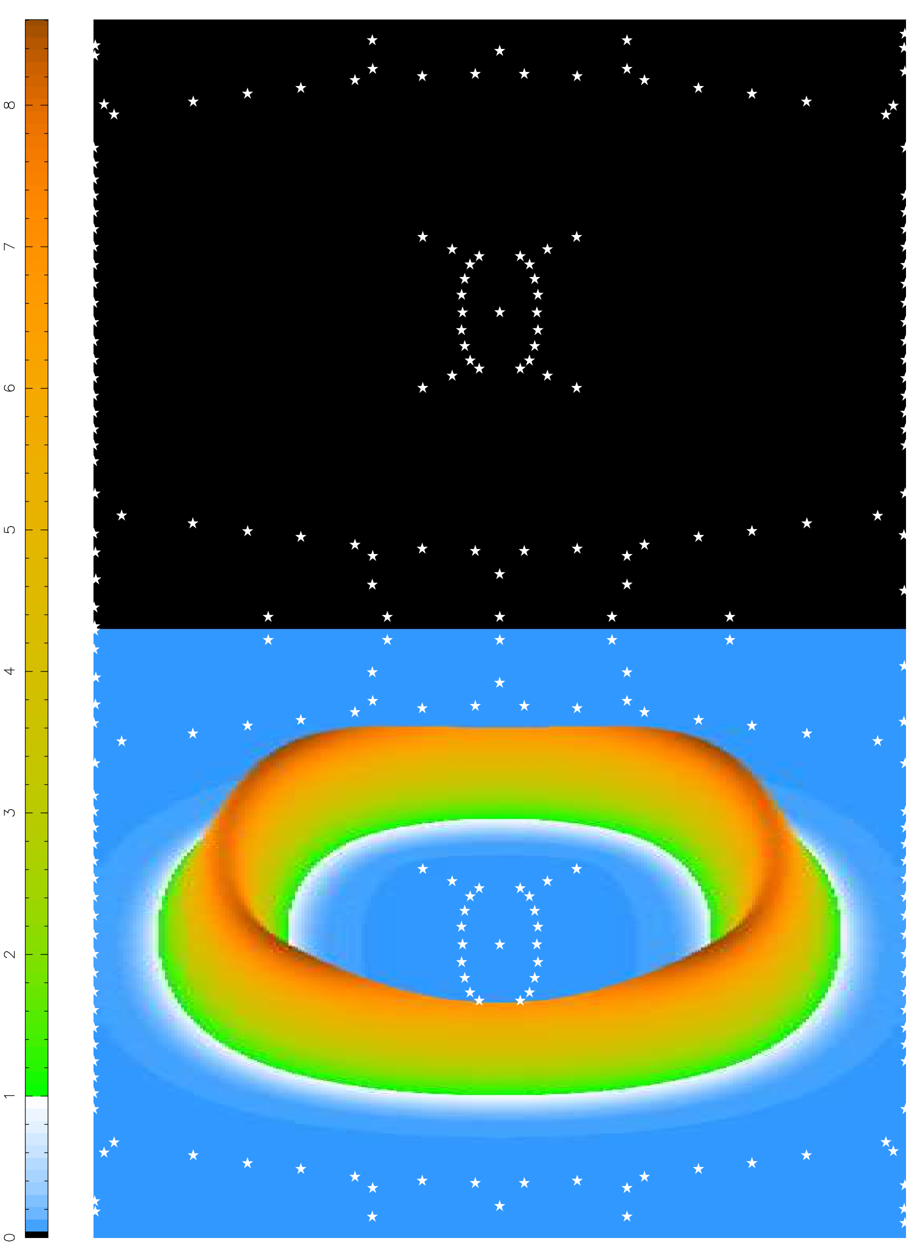

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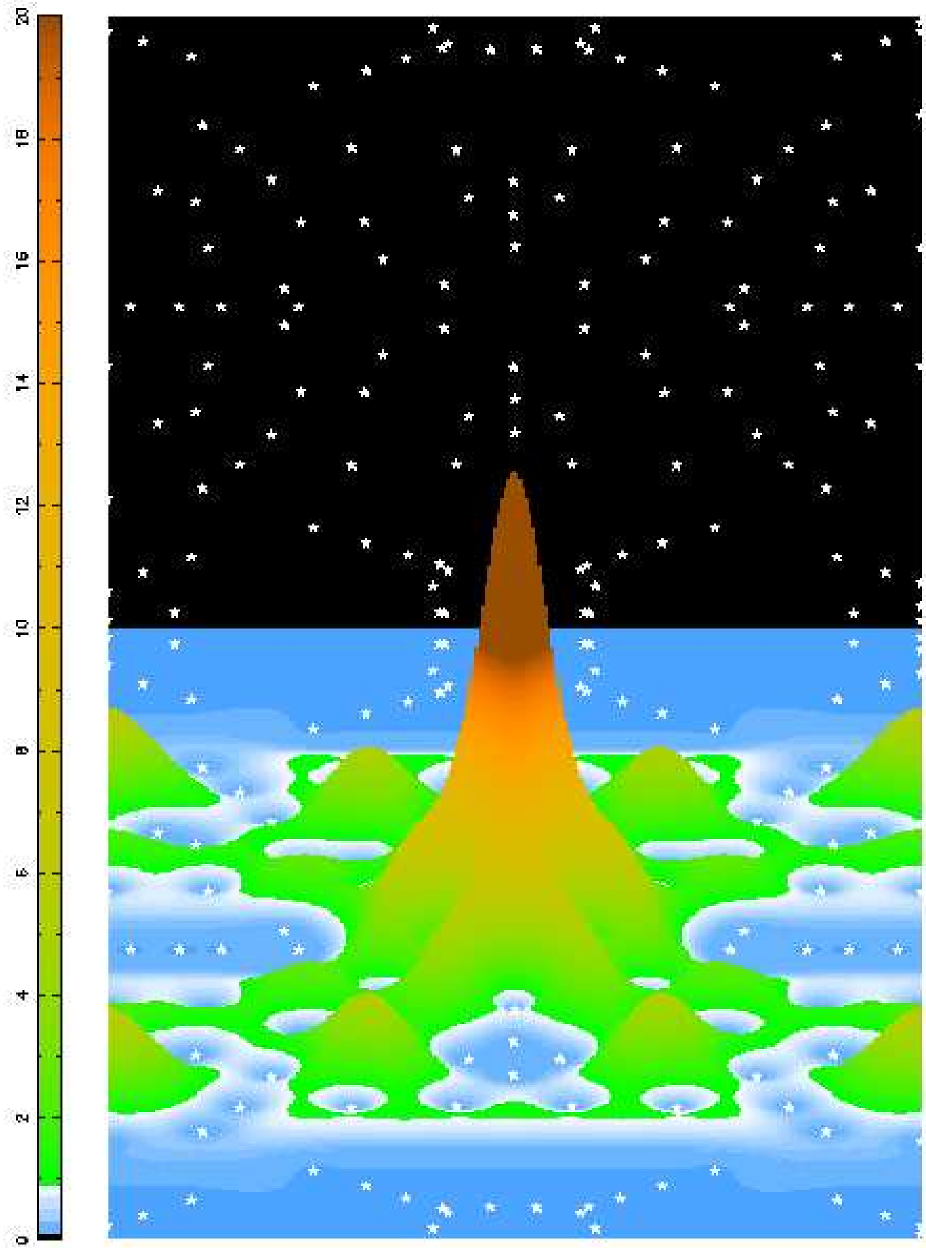

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