MAGNETIC MOMENT and PERTURBATION THEORY with SINGULAR MAGNETIC FIELDS

Alain Comtet(+)1,
Stefan Mashkevich(++) and **Stéphane Ouvry** (+)1

(+)Division de Physique Théorique [2], IPN, Orsay Fr-91406
(++) Institute for Theoretical Physics, Kiev 252143, Ukraine

Abstract : The spectrum of a charged particle coupled to Aharonov-Bohm/anyon gauge fields displays a nonanalytic behavior in the coupling constant. Within perturbation theory, this gives rise to certain singularities which can be handled by adding a repulsive contact term to the Hamiltonian. We discuss the case of smeared flux tubes with an arbitrary profile and show that the contact term can be interpreted as the coupling of a magnetic moment spinlike degree of freedom to the magnetic field inside the flux tube. We also clarify the ansatz for the redefinition of the wave function.

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e-mail : COMTET@IPNCLS.IN2P3.FR, MASH@PHYS.UNIT.NO, OUVRY@FRCPN11

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[1] *and LPTPE, Tour 12, Université Paris 6*
[2] *Unité de Recherche des Universités Paris 11 et Paris 6 associée au CNRS*

The fact that the problem of N noninteracting anyons [1], for $N > 2$, is exactly solvable only in the two limit cases of bosons and fermions, gives rise to the idea of applying perturbation theory in order to get at least some information in the vicinity of these two limit cases. However, perturbation theory meets certain difficulties near Bose statistics, as originally noticed in [2]. In order to overcome these difficulties, it was pointed out in [3] [4], that certain modifications of the singular N-anyon Hamiltonian are required.

In the regular gauge, anyons may be viewed [5] as charged particles with attached singular Aharonov-Bohm [6] flux tubes. In this letter, we discuss a generalization of the perturbative algorithm discussed in [3] [4] to the case of smeared flux tubes with any profile. This will bring some light on the singular case itself. In particular, the contact repulsive interaction $\delta^2(\vec{r})$ added to the singular Hamiltonian will be reinterpreted as a magnetic moment coupling of the particle to the magnetic field inside the flux tube. Characteristic features of the singular perturbative algorithm, as for example cancellation of singular 2-body as well as regular 3-body interactions in the transformed Hamiltonian, will be shown to be easily generalized provided that such magnetic moment couplings are properly taken into account.

Let us first remind to the reader what happens in the paradigm Aharonov-Bohm (A-B) problem, or equivalently, in the relative 2-anyon problem [3] [4]. This is convenient since a complete checking of the perturbative results at all stages is possible for this problem by comparison against the exact ones. We work in the regular gauge, in which the wave functions are single-valued and the A-B statistical parameter α explicitly appears in the Hamiltonian. We consider a particle of charge e and mass m moving in a plane and coupled to the gauge potential of a singular flux tube ϕ located at the origin

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 \tag{1}$$

where $\vec{A}(\vec{r}) = \frac{\alpha}{e}\frac{\vec{k}\times\vec{r}}{r^2}$ and $\vec{k}$ is the unit vector perpendicular to the plane. The A-B statistical parameter is $\alpha = \frac{e\phi}{2\pi}$. The Hamiltonian (1), in polar coordinates, is

$$H = \frac{1}{2m}(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{2i\alpha}{r^2}\frac{\partial}{\partial \phi} + \frac{\alpha^2}{r^2}) + \frac{1}{2}m\omega^2 r^2 \tag{2}$$

where one has added a harmonic attraction in order to discretize the spectrum. The complete set of exact eigenstates for this Hamiltonian, up to a normalization, is given by

$$E_{\ell n} = (2n + |\ell - \alpha| + 1)\omega, \tag{3}$$

$$\psi_{\ell n}(r, \phi) = r^{|\ell-\alpha|}\,{}_1F_1\left(-n, |\ell - \alpha| + 1, -m\omega r^2\right)\exp\left(-\frac{m\omega}{2}r^2\right)\exp\left(i\ell\phi\right) \tag{4}$$

The ground state wave function is obtained by setting $\ell = 0$, $n = 0$

$$\psi_{00}(r, \phi) = r^{|\alpha|}\exp\left(-\frac{m\omega}{2}r^2\right) \tag{5}$$

Its energy is

$$E = (|\alpha| + 1)\omega \tag{6}$$

It is, however, impossible to get (6) in perturbation theory near Bose statistics, treating the α dependent terms in (2) as perturbations. Indeed, in the s-wave sector, non-zero peturbative corrections turn out to be logarithmically divergent. For example, the unperturbed ground state wave function $\psi_{00}^{(0)} = \sqrt{\frac{m\omega}{\pi}}\exp\left(-\frac{m\omega}{2}r^2\right)$ gives

$$\left\langle \psi_{00}^{(0)} \right| \frac{\alpha^2}{2mr^2} \left| \psi_{00}^{(0)} \right\rangle = \int_0^\infty \frac{\omega\alpha^2}{r} e^{-m\omega r^2} dr \tag{7}$$

The reason of this divergence may be traced back to the fact that the unperturbed $\ell = 0$ wave function does not vanish at the origin while the perturbed one does and therefore one cannot get the latter as a perturbative series starting from the former.

In order to get a meaningful perturbation expansion, a modification of the Hamiltonian is required. Adding to H a short range repulsive interaction [4], one defines

$$H' = H + \frac{\pi|\alpha|}{m}\delta^2(\vec{r}) \tag{8}$$

The contact term clearly does not affect the exact wave functions, since they vanish at the origin, except in the Bose case $\alpha = 0$, but then there is no contact interaction. Still, this new Hamiltonian makes it possible to use perturbation theory, with the parameter $\frac{\pi|\alpha|}{m}$ in (8) precisely chosen for this aim. Indeed, the first order correction to the ground state energy from the contact term is just

$$\left\langle \psi_{00}^{(0)} \right| \frac{\pi|\alpha|}{m} \delta^2(\vec{r}) \left| \psi_{00}^{(0)} \right\rangle = \frac{\pi|\alpha|}{m} \left| \psi_{00}^{(0)}(0) \right|^2 = |\alpha|\omega, \tag{9}$$

and it turns out that, while the higher order corrections due to this term are divergent, they nevertheless exactly cancel the divergent corrections coming from the $\frac{\alpha^2}{r^2}$ term. More precisely, the singular perturbative problem is solved in the sense that, if a short range regulator is introduced to give an unambiguous meaning to the perturbative divergences, they do cancel in the limit where the regulator vanishes [7].

It would however be more satisfactory to have a perturbative algorithm where perturbative divergences do not exist from the very beginning [3]. If, willing to take into account the small r behavior of the ground state wave function (5), one redefines [3]

$$\psi(r, \phi) = r^{|\alpha|} \tilde{\psi}(r, \phi) \tag{10}$$

then the Hamiltonian $\tilde{H}$ acting on $\tilde{\psi}$ no longer contains the dangerous $\frac{\alpha^2}{r^2}$ term

$$\tilde{H} = \frac{1}{2m}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{2i\alpha}{r^2}\frac{\partial}{\partial \phi} - \frac{2|\alpha|}{r}\frac{\partial}{\partial r}\right) + \frac{1}{2} m\omega^2 r^2 \tag{11}$$

The last term in the brackets in (11), which appears in place of the singular one, does not lead to any perturbative singularities. The first-order correction to the ground state energy

$$\left\langle \psi_{00}^{(0)} \right| - \frac{|\alpha|}{mr}\frac{\partial}{\partial r} \left| \psi_{00}^{(0)} \right\rangle = |\alpha|\omega, \tag{12}$$

does coincide with the exact answer, while the higher order perturbative corrections are finite and cancel. The fact that first order perturbation theory gives here the exact answer is of course due to the fact that one has “guessed” the correct ansatz (10) by looking at the exact solution (5).

In perturbation theory for the N-anyon problem [3], the ansatz analogous to (10),

$$\psi = \prod_{j<k} r_{jk}^{|\alpha|}\tilde{\psi}, \tag{13}$$

eliminates in $\tilde{H}$ not only the singular 2-body terms, but also the 3-body terms, thus considerably simplifying the treatment. This complete cancellation can be understood if one remarks that the prefactor $\prod_{j<k} r_{jk}^{|\alpha|}$ is nothing but a pseudo gauge transformation factor, whose parameter is the real part of the analytic function $|\alpha|\sum_{j<k}\ln z_{jk}$. The imaginary part of the same analytic function is precisely the singular gauge transformation parameter which defines the anyonic N-body vector potential $\vec{A}(\vec{r}_i) = \frac{\alpha}{e}\vec{\partial}_i\sum_{j<k}\phi_{jk}$. It is not difficult to realize that, due to the Cauchy-Riemann relations, $\sum_i \vec{A}^2(\vec{r}_i)$ indeed disappears in $\tilde{H}$ [3] [8].

To gain a more complete understanding of the singular perturbative algorithm, let us now try to see how it applies in a regular case. We consider first a smeared flux tube version of the singular Aharonov-Bohm problem -possible generalizations involve flux tubes of finite size- and we concentrate on a flux smeared over a certain region of space, with a given profile. The effective change of statistics of the particles then depends on the distance between them [9]. Thus consider the vector potential

$$\vec{A}(\vec{r}) = \frac{\alpha}{e}\frac{\vec{k}\times\vec{r}}{r^2}\varepsilon(r) \tag{14}$$

where $\varepsilon(r)$ satisfies the boundary conditions $\varepsilon(\infty) = 1$ (hence at large distances one has effectively anyons with statistics α) and $\varepsilon(0) = 0$, in order to avoid singularities at the origin. The physical meaning of $\varepsilon(r)$ is rather obvious : $\Phi(r) = 2\pi\frac{\alpha}{e}\varepsilon(r)$ is the flux through a circle of radius r, and

$$B(r) = \frac{\alpha}{er}\frac{d\varepsilon(r)}{dr} \tag{15}$$

is the magnetic field profile of the smeared flux tube.

The Hamiltonian now reads

$$\mathcal{H} = \frac{1}{2m}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{2i\alpha\varepsilon(r)}{r^2}\frac{\partial}{\partial \phi} + \frac{\alpha^2\varepsilon^2(r)}{r^2}\right) + \frac{1}{2}m\omega^2 r^2 \tag{16}$$

In the problem at hand, there is always a characteristic parameter R, which is essentially the size of the flux tube, such that $\varepsilon(r) \sim 1$ for $r \gg R$. All the results of the ideal anyon model should be recovered in the limit $R \to 0$. Since the Hamiltonian $\mathcal{H}$ tends to H in this limit, the problem of perturbation theory does manifest itself for $\mathcal{H}$. Indeed, the first order correction to the ground state energy

$$\left\langle \psi_{00}^{(0)} \right| \frac{\alpha^2\varepsilon^2(r)}{2mr^2} \left| \psi_{00}^{(0)} \right\rangle = \int_0^\infty \frac{\omega\alpha^2\varepsilon^2(r)}{r} e^{-m\omega r^2} dr \tag{17}$$

is finite, but diverges as $R \to 0$, whereas it should tend to $|\alpha|\omega$.

What stands, in this smeared case, in place of the singular A-B perturbative algorithm? Let us recall that for ideal anyons, the magnetic field inside the singular flux tube is $B(r) = 2\pi\frac{\alpha}{e}\delta^2(\vec{r})$. The $\frac{\pi|\alpha|}{m}\delta^2(\vec{r})$ contact term added to H may be interpreted as the coupling to the singular magnetic field of a magnetic moment μ associated to the particle[3]

$$\mu = -\frac{e}{2m}\frac{\alpha}{|\alpha|} \tag{18}$$

Coming back to the smeared flux case, this suggests to introduce

$$\mathcal{H}' = \mathcal{H} - \mu B(r) \tag{19}$$

corresponding to the magnetic moment coupling

$$-\frac{e}{2m}g\frac{\sigma_3}{2}B(r) \tag{20}$$

[3] Such magnetic moment couplings have already been introduced in the anyon model [10], as relics of a relativistic formulation, but were shown to be associated to attractive δ^2 interactions.

with the gyromagnetic factor $g = 2$. What is now the appropriate generalization of the ansatz (10) for the wave function? In the singular case, the idea was to extract the short distance ground state behavior. It happens that the ground state wave functions for a 2-dimensional particle with the gyromagnetic factor $g = 2$ in a magnetic field B are (up to a holomorphic function) [11]

$$\psi_{00} = e^{-2m\mu a(r)} \tag{21}$$

where $a(r)$ is such that

$$\Delta a(r) = B(r) \tag{22}$$

In [11], spin-$\frac{1}{2}$ particles have been considered, altogether with a Pauli Hamiltonian viewed as the nonrelativistic limit of the relativistic Dirac Hamiltonian. In the present context, however, spin is an additional degree of freedom simply introduced by hand. Taking into account (15), one has

$$a(r) = \frac{\alpha}{e} \int_0^r \frac{\varepsilon(r')}{r'}\, dr' \tag{23}$$

and the generalized ansatz is

$$\psi(r,\phi) = \exp\left[\int_0^r |\alpha| \frac{\varepsilon(r')}{r'}\, dr'\right] \tilde{\psi}(r,\phi). \tag{24}$$

Transforming $\mathcal{H}'$, one obtains

$$\tilde{\mathcal{H}} = \frac{1}{2m}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{2i\alpha\varepsilon(r)}{r^2}\frac{\partial}{\partial \phi} - \frac{2|\alpha|\varepsilon(r)}{r}\frac{\partial}{\partial r}\right) + \frac{1}{2}m\omega^2 r^2 \tag{25}$$

where the $\vec{A}^2$ have again disappeared.

In a sense, coming back to the singular case, one has now at hand a clearer point of view on certain subtleties associated with the contact term, and also a more precise understanding of the ansatz for the redefinition of the wave function. A "naive" $R \to 0$ limit $\varepsilon(r) \equiv 1$ would imply that *both* $\mathcal{H}$ and $\mathcal{H}'$ would coincide with H. However, if one insists on the non-singular boundary condition $\varepsilon(0) = 0$, then in the limit $R \to 0$ one should rather take $\varepsilon(r) = \eta(r)$, where $\eta(r)$ is the step function. Then $\frac{d\varepsilon(r)}{dr} = \delta(r)$, and $\mathcal{H}'$ coincides with H', not H. Here, to ignore the difference between unity and the step function would be the same as, say, to consider that $\Delta \ln r = 0$, rather than $\Delta \ln r = 2\pi\delta^2(\vec{r})$, thus "losing" the $\delta^2(\vec{r})$ contact term [4]. Note also that once the correct ansatz is made, i.e. once one works with $\tilde{\mathcal{H}}$, this subtlety does not anymore play any role : in the limit $R \to 0$, it does not matter whether to put $\varepsilon = 1$ or $\varepsilon = \eta(r)$ to get (11), since the correct short-distance behavior has already been properly taken into account.

Generalizing further, consider now the Hamiltonian

$$2m\mathcal{H}'_{\pm} = (\vec{p} - e\vec{A})^2 \mp e(\partial_1 A_2 - \partial_2 A_1) \tag{26}$$

and go to the Coulomb gauge $A_1 = -\partial_2 a, \quad A_2 = \partial_1 a$. In 2 dimensions, this is a general choice of gauge. One has

$$2m\mathcal{H}'_{\pm} = -\Delta - 2ei\epsilon_{ij}\partial_j a\partial_i + e^2(\vec{\partial}a)^2 \mp e\Delta a \tag{27}$$

Redefine[4] $\psi = e^{\mp ea}\tilde{\psi}$. If $\tilde{\psi} = \tilde{\psi}(z)$ (respectively $\tilde{\psi} = \tilde{\psi}(\bar{z})$), then ψ is the zero energy ground state wave function of H_+ (respectively H_-). Otherwise, one gets, acting on $\tilde{\psi}$,

$$2m\tilde{\mathcal{H}}_{\pm} = -\Delta - 2ei\epsilon_{ij}\partial_j a\partial_i \pm 2e\partial_i a\partial_i \tag{28}$$

The connection with what has been said above is transparent if one specializes to the rotationally invariant case $\epsilon_{ij}\partial_j a\partial_i = -\frac{da(r)}{dr}\partial_\phi$. Focusing on the s-wave sector, only the

[4]Note that the inverse transformation $\psi = e^{\pm ea}\chi$ leads to the Fokker-Planck equation associated to $\mathcal{H}'_{\pm}$

$$-\Delta\chi + \partial_i(\chi K_i) = E\chi$$

$$K_i = \pm 2e(\partial_i a + i\epsilon_{ij}\partial_j a)$$

term $\pm 2e\partial_i a\partial_i$ contributes to the energy shift

$$E - E_0 = \pm\frac{e}{m}\int \psi_{00}^{(0)}\partial_i a\partial_i\psi_{00}^{(0)} d^2\vec{r} = \pm\frac{e}{2m}\int \partial_i|\psi_{00}^{(0)}|^2\partial_i a d^2\vec{r} = \pm\frac{e}{2m}\int |\psi_{00}^{(0)}|^2\Delta a d^2\vec{r} \quad (29)$$

If one wishes to generalize to the N-body case, one starts from

$$2m\mathcal{H}'_{\pm} = \sum_{i=1}^{N}(\vec{p}_i - e\vec{A}_i)^2 \mp eB(\vec{r}_i) \quad (30)$$

with

$$\vec{A}(\vec{r}_i) = -\vec{\partial}_i \times \sum_{j<k} a(r_{jk}), \quad B(\vec{r}_i) = \Delta_i \sum_{j<k} a(r_{jk}) \quad (31)$$

and redefines

$$\psi = \prod_{j<k} e^{\pm ea(r_{jk})}\tilde{\psi} \quad (32)$$

to get a Hamiltonian without 3-body interactions, exactly as in the N-anyon case.

To conclude, and as an explicit illustration, let us carry out the calculation in the simple case where the magnetic field is uniform within a circle of radius R. One has

$$\varepsilon(r) = \begin{cases} \frac{r^2}{R^2}\,, & r \leq R, \\ \\ 1\,, & r \geq R. \end{cases} \quad (33)$$

The first-order correction from the last term of $\mathcal{H}'$ in (19) is

$$\left\langle \psi_{00}^{(0)} \right| \frac{|\alpha|}{2mr}\frac{d\varepsilon(r)}{dr} \left| \psi_{00}^{(0)} \right\rangle = \frac{1 - \exp(-q)}{q}|\alpha|\omega, \quad (34)$$

where

$$q = m\omega R^2 \quad (35)$$

is the squared ratio of the flux tube radius to the length scale of the harmonic potential: The particle is well outside the flux tube if $q \ll 1$. In the limit $q \to 0$, the exact result (6) is recovered. Alternatively, one may proceed with the Hamiltonian (25) to get

$$\left\langle \psi_{00}^{(0)} \right| -\frac{|\alpha|\varepsilon(r)}{mr}\frac{\partial}{\partial r}\left|\psi_{00}^{(0)}\right\rangle = \frac{1-\exp(-q)}{q}|\alpha|\omega, \tag{36}$$

the same answer as above.

In conclusion, hard core boundary prescriptions in the singular A-B/anyon cases can be naturally understood in the context of Aharonov-Casher Hamiltonians for spin 1/2 particles coupled to 2-d magnetic field, with the gyromagnetic factor $g = 2$.

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