Random Analytic Chaotic Eigenstates

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Abstract

The statistical properties of random analytic functions $\psi(z)$ are investigated as a phase-space model for eigenfunctions of fully chaotic systems. We generalize to the plane and to the hyperbolic plane a theorem concerning the equidistribution of the zeros of $\psi(z)$ previously demonstrated for a spherical phase space ($SU(2)$ polynomials)). For systems with time reversal symmetry, the number of real roots is computed for the three geometries. In the semiclassical regime, the local correlation functions are shown to be universal, independent of the system considered or the geometry of phase space. In particular, the autocorrelation function of ψ is given by a Gaussian function. The connections between this model and the Gaussian random function hypothesis as well as the random matrix theory are discussed.

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I. INTRODUCTION

There are two basic descriptions of the eigenfunctions of fully chaotic systems. On the one hand, there is the random matrix theory of Wigner and Dyson as a model of the quantum behavior of chaotic motion [1,2]. In this model the Hilbert space is isotropic, i.e. an arbitrary eigenstate φ of the random matrix can point out in any direction of Hilbert space with equal probability. The only constraint comes from the normalization. Denoting by $\vec{a} = (a_0, a_1, \cdots, a_N)$ the components of φ in a given (and arbitrary) basis, the joint probability distribution for the amplitudes is [3]

$$\mathcal{D}_{RMT}(\vec{a}) = \frac{1}{|S_{2(N+1)}|} \delta \left[\sum_{m=0}^{N} |a_m|^2 - 1 \right] \ , \tag{1}$$

where $|S_n| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of a $(n-1)$-dimensional sphere of unit radius. We assume for the moment that $\vec{a}$ is a complex vector, this being the usual situation when the physical system has no time-reversal symmetry. By construction this theory concentrates on statistical aspects of the eigenstates, and the "ergodicity" in Hilbert space expressed by Eq.(1) is a direct consequence of the invariance of the ensemble of matrices under unitary transformations.

On the other hand, a semiclassical theory has also been developed [4–7]. Unlike the random matrix theory, its aim is to provide a system-dependent description of the eigenstates based on the solutions of the classical dynamics. Adopting a phase-space representation of quantum mechanics, in the semiclassical limit $\hbar \to 0$ the Wigner transform $\mathcal{W}_\psi(\vec{x}, \vec{p}, E)$ of an eigenstate ψ of an ergodic system at energy E takes to leading order the simple form [7]

$$\mathcal{W}_\psi(\vec{x}, \vec{p}, E) \approx \mathcal{N} \delta \left[E - H(\vec{x}, \vec{p}) \right] \ , \tag{2}$$

where $\mathcal{N}$ is a normalization constant, $(\vec{x}, \vec{p})$ parametrizes the phase space and $H(\vec{x}, \vec{p})$ is the classical Hamiltonian. Because in the semiclassical limit the Wigner function presents wild oscillations, Eq.(2) has to be interpreted as a smoothed version over a phase space region of the order of one Planck volume.

Besides the energy-shell ergodicity, Eq.(2) reveals a strong localization effect in phase space of which there is no trace in Eq.(1). Because of energy conservation $\mathcal{W}_\psi$ is, in the semiclassical limit, exponentially concentrated on the energy shell. This means that when expanding an eigenstate of H in an arbitrary basis ϕ_m of Hilbert space it will have components different from zero only on those states whose Wigner transform intersects – or is included into – the energy shell $\delta\left[E - H(\vec{x}, \vec{p})\right]$, while the amplitudes will vanish exponentially fast otherwise. Therefore, in general for a given chaotic Hamiltonian system the expected distribution of the eigenstates's amplitudes is quite different from the isotropic rule (1), because random matrix theory ignores the phase-space restrictions imposed by energy conservation. Eq.(1) has therefore to be interpreted as a valid statement only for the amplitudes a_m whose associated vector ϕ_m intersects the energy shell.

Having this in mind, our purpose is to further develop the connections between the two above mentioned descriptions. In particular, assuming a two-dimensional phase space we study the ergodicity of the eigenstates as implied by Eq.(1), as well as some correlation functions computed inside the energy shell. For that purpose, we use the Bargmann (or coherent state) representation which associates to each quantum state an analytic function $\psi(z)$ in the complex plane. This complex plane is interpreted as a parametrization of the energy shell manifold, and the three simplest one-dimensional complex manifolds are considered: the sphere (curvature $+1$), the plane (zero-curvature) and the hyperbolic plane or pseudosphere (curvature -1). After introducing the model in section II, we show in section III that the associated Bargmann functions given by Eq.(3) below have, when the coefficients a_m are distributed according to Eq.(1), a uniform distribution of zeros over the sphere, the plane and the hyperbolic plane, respectively. The uniformity of the distribution was shown in [8,9] for the spherical geometry, and the results of this paper constitute a generalization to the plane and to the hyperbolic plane. Some related theorems extending some of the results to higher dimensional phase spaces were recently obtained independently in Ref. [10].

We also consider the case when the W_1 and $SU(1,1)$ series in Eq.(3) are truncated to a

finite number of terms (thus introducing W_1 and $SU(1,1)$ polynomials). It is shown that for those polynomials the distribution of roots is now uniform over a finite region (a disk) of the complex plane and the hyperbolic plane, respectively.

The equidistribution of roots holds only if the vector $\vec{a}$ is complex. When it is real (which corresponds to time-reversal symmetric systems), a concentration of roots occurs over the real axis in the complex plane. We show that for the three random functions considered the number of real roots located inside a disk centered at the origin containing N zeros is proportional to $\sqrt{N}$.

Section IV is devoted to the study of the correlation properties of the random analytic functions between different points of the energy shell. We first argue that in the semiclassical limit all the correlation functions are locally universal, independent of the geometry of phase space. We then compute as a particular example the autocorrelation of $\psi(z)$.

II. RANDOM ANALYTIC FUNCTIONS

In order to study the statistical properties of eigenstates of chaotic systems, we first define an energy-shell (phase space) associated to the quantum mechanical system. We will not consider a specific dynamical system but make general statements which are expected to be valid for any system having at the classical level a strongly chaotic behavior.

Concerning the energy shell, the minimum dimensionality required in order to have chaotic behavior at the classical level is two. This two-dimensional manifold $\mathcal{M}$ could be for example the Poincaré surface of section of the energy shell of a two-freedoms Hamiltonian system. But we can alternatively choose a two-dimensional manifold and directly define on it a fully chaotic area-preserving map, thereby avoiding the problem of the reduction to the Poincaré section. In both cases one can define a unitary operator which quantizes the classical flow. In the following we study properties of the eigenstates of this unitary operator. Because in principle it corresponds to a section of it, we will still call the manifold $\mathcal{M}$ the "energy shell".

Independently of the definition of the dynamics, the three simplest possibilities for $\mathcal{M}$ are, in two dimensions, the sphere, the plane and the hyperbolic plane or pseudosphere [11] (the cylinder and the torus being just periodicized versions of the plane), having curvature $+1$, 0 and -1, respectively. The choice of the manifold will depend on the dynamical symmetry group underlying the system Hamiltonian under study. The three spaces considered are associated with the $SU(2)$, W_1 and $SU(1,1)$ symmetry, respectively. For example, for the motion of a spin in a variable magnetic field the appropriate phase-space manifold is the sphere (see Ref. [12] for illustrations in different geometries). Area preserving chaotic maps have been studied in the plane and its periodicized versions (e.g. the cat, standard and kicked Harper maps) and the sphere (the kicked top) but not, to our knowledge, in the pseudosphere.

The next step is to define a representation of the quantum states over the manifold $\mathcal{M}$. This can be achieved for the three geometries under consideration by means of a phase-space coherent state (Bargmann) representation of quantum mechanics consisting of overcomplete and non-orthogonal systems of Hilbert space vectors. An arbitrary quantum state can be written in the coherent state basis as [13,12]

$$\psi(z) = \begin{cases} \sum\limits_{m=0}^{N} \sqrt{C_N^m}\, a_m\, z^m\ , & SU(2) \\ \sum\limits_{m=0}^{\infty} \frac{1}{\sqrt{m!\hbar^m}}\, a_m\, z^m\ , & W_1 \\ \sum\limits_{m=0}^{\infty} \sqrt{C_{L+m-1}^m}\, a_m\, z^m\ , & SU(1,1) \end{cases} \tag{3}$$

where $C_k^l = k!/l!(k-l)!$ are Newton's binomial coefficients. For $SU(2)$ we have $N = 2j$, where $j = 1/2, 1, 3/2, \cdots$ is the usual spin label of the $(2j+1)$-dimensional representation. For $SU(1,1)$, $L = 2k$, with $k = 1, 3/2, 2, \cdots$ is the usual label of the discrete representation of the group [12]. Unlike the case of W_1 and $SU(1,1)$, the representation of the quantum state for $SU(2)$ is finite dimensional, reflecting the fact that the group is compact.

Eq. (3) provides a realization of the Hilbert space in terms of analytic functions $\psi(z)$. The complex variable appearing in Eq.(3) spans in each case the associated manifold $\mathcal{M}$. It is defined as

$$z = \begin{cases} \cot(\theta/2) \exp(i\varphi) \,, & SU(2) \\ (x - ip)/\sqrt{2} \,, & W_1 \\ \tanh(\tau/2) \exp(i\varphi) \,, & SU(1,1) \,. \end{cases} \qquad (4)$$

Here (θ, φ), (x, p) and (τ, φ) are the usual variables labeling these manifolds (see Fig.1). The measure of these three manifolds projected onto the z-plane is

$$\mu(z, \bar{z}) = \begin{cases} \dfrac{1}{\pi} \dfrac{d^2 z}{(1 + z\bar{z})^2} & SU(2) \\ d^2 z/(\pi\hbar) & W_1 \\ \dfrac{1}{\pi} \dfrac{d^2 z}{(1 - z\bar{z})^2} & SU(1,1) \,. \end{cases} \qquad (5)$$

The notation $d^2\omega$ where ω is a complex variable stands for $d(\mathrm{Re}\omega)d(\mathrm{Im}\omega)$.

Now comes the statistical ingredient entering in the definition of the model. If ψ is an eigenstate of the quantization of the map acting on $\mathcal{M}$ then, for a fully chaotic dynamics, we assume that the amplitudes $\{a_m\}$ in Eq.(3) are distributed according to Eq.(1). Of course this is an hypothesis whose validity has to be compared with the behavior of real dynamical systems. Including this element, we obtain therefore a phase-space model for chaotic eigenstates in terms of *random analytic functions*. For $SU(2)$ polynomials this model was introduced in Ref. [8] and further considered in [14,15,9,16,17].

Being analytic, the average properties of the random functions $\psi(z)$ cannot be stationary over the complex plane (unless the function is trivially a constant). For example, for $SU(2)$ the average square modulus of $\psi(z)$ grows like

$$\langle \psi(z)\overline{\psi(z)} \rangle = \sum_{m,n=0}^{N} \sqrt{C_m^N C_n^N} \langle a_m \bar{a}_n \rangle z^m \bar{z}^n = (1 + z\bar{z})^N \,. \qquad (6)$$

The brackets indicate average over a uniform distribution for the coefficients, $\langle f(\vec{a}) \rangle = \int d^2 a_0 \ldots d^2 a_N \mathcal{P}(\vec{a}) f(\vec{a})$. For convenience we use for the computation of the averages a different normalization from that of Eq.(1)

$$\mathcal{P}(\vec{a}) = \frac{2(N+1)}{|S_{2(N+1)}|} \delta \left[\sum_{m=0}^{N} |a_m|^2 - (N+1) \right] \,. \qquad (7)$$

We therefore have

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$$\langle a_m \bar{a}_n \rangle = \delta_{mn} \ , \tag{8}$$

from which we obtained Eq.(6). This change of normalization has no incidence in the results reported in this paper since the quantities considered (cf Eqs.(12) and (20)) are independent of the global normalization of ψ.

An analogous computation can be done for the other two geometries, the plane and the pseudosphere. When computing the averages for these two geometries, we truncate the series in Eq.(3) to a finite value $m = N$ and then we let $N \to \infty$. We get

$$\langle \psi(z)\overline{\psi(z)} \rangle = \begin{cases} (1 + z\bar{z})^N & SU(2) \\ \exp\left(z\bar{z}/\hbar\right) & W_1 \\ (1 - z\bar{z})^{-L} & SU(1,1) \ . \end{cases} \tag{9}$$

As was pointed out in Ref. [18], the statistical properties of the zeros of $\psi(z)$ are unaffected if instead of the distribution (7) a Gaussian independent distribution for the coefficients is used. This is also true in the large N limit for any statistical property of $\psi(z)$, because the distribution (7) also implies in that limit a Gaussian law for the distribution of a given coefficient. For most purposes, Eq.(7) (or alternatively (1)) is equivalent to a Gaussian independent distribution.

The random analytic model then implies that at each point of phase space the distribution $P(\psi)$ of the function $\psi(z)$ is Gaussian, with a local variance given precisely by Eq.(9) (in agreement with the old conjecture of Berry [5])

$$P(\psi) = \frac{1}{\sqrt{2\pi\langle|\psi(z)|^2\rangle}} \exp\left(-|\psi(z)|^2 / 2\langle|\psi(z)|^2\rangle\right) \ . \tag{10}$$

The relationship between the statistical properties of ψ in different representations was recently considered in [19].

III. ERGODICITY

The simplest question concerning the statistical properties of the functions $\psi(z)$ is related to the ergodicity. We would like to establish more clearly the connection between the Hilbert

space ergodicity of Eq.(1) on the one hand and the phase-space ergodicity over the energy shell on the other. From the results of the previous section this connection is now easy to establish.

As we mention, the functions $\psi(z)$ are not stationary in the complex z-plane. However, the quantity which is physically associated to the presence of the particle in a neighborhood of a phase space cell (and therefore more closely related to the notion of ergodicity) is not $\psi(z)$ but rather the (quasi) probability density

$$\mathcal{H}_\psi(z, \bar{z}) = \frac{|\psi(z)|^2}{\langle |\psi(z)|^2 \rangle} \ , \tag{11}$$

usually called the Husimi function. It is simply obtained from the Wigner transform through a Gaussian convolution. Exactly as for the average Wigner function Eq.(2), Eq.(11) implies that the average of the Husimi function is constant in the complex z-plane. Because of this, the (quasi) probability $\mathcal{H}_\psi d\mu$ is going to be stationary on the manifold $\mathcal{M}$ (and not in the complex z-plane where the measure is not uniform (cf Eq.(5)).

In particular, the previous statement applies to the distribution of the zeros of $\mathcal{H}_\psi$, i.e. the density of zeros of $\mathcal{H}_\psi$ should be uniform over the manifold $\mathcal{M}$. This is going to be true for the zeros of $\psi(z)$ as well, because both functions have the same set of zeros [20]. In fact, the uniformity over $\mathcal{M}$ of the density of zeros $\rho(z) = \delta\left[\psi(z)\right] |d\psi(z)/dz|^2$ of $\psi(z)$ can be directly checked for the three geometries from the general formula [15]

$$\langle \rho(z) \rangle = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} [\ln \langle \psi(z) \overline{\psi(z)} \rangle] \ . \tag{12}$$

From (9) and (12) we find, for each geometry

$$\langle \rho(z) \rangle \, d^2z = \begin{cases} \dfrac{N}{\pi} \dfrac{d^2z}{(1 + z\bar{z})^2} = \dfrac{N}{4\pi} \sin\theta \, d\theta d\varphi & SU(2) \\[2ex] d^2z/(\pi\hbar) & W_1 \\[2ex] \dfrac{L}{\pi} \dfrac{d^2z}{(1 - z\bar{z})^2} = \dfrac{L}{4\pi} \sinh\tau \, d\tau d\varphi & SU(1,1) \ . \end{cases} \tag{13}$$

For the sphere and the pseudosphere, the density has been transformed to the usual coordinates spanning the corresponding manifold. Eq.(13) shows that the density of zeros

of the random-$\psi(z)$ model is indeed uniform over the associated manifold $\mathcal{M}$ (this being true for arbitrary values of the parameters N, L or $\hbar$).

The uniformity of the density of zeros was also demonstrated for random analytic functions defined on the two-dimensional torus [21]. There exist also some related results obtained without using the random hypothesis. For example it was proved [21] that the Schnirelman property for the Husimi density implies a (weak) convergence of the density of zeros towards the uniform measure. The uniformity of the distribution of zeros was checked numerically for certain maps on the sphere [8,9] and also for two-dimensional chaotic systems with W_1 symmetry [22].

In spite of its aesthetical beauty, Eq.(13) has an unpleasant feature for the W_1 and the $SU(1,1)$ groups, since a uniform density of zeros for $\psi(z)$ over the whole (non-compact) manifold $\mathcal{M}$ is not realistic from a dynamical point of view. For chaotic Hamiltonian systems, non-compact energy shells like the plane or the pseudosphere are somewhat artificial because energy conservation implies in general a compact manifold (or Poincaré surface of section). A more appropriate model for the eigenfunctions of a system having a W_1 or $SU(1,1)$ symmetry would then be a truncated version of Eq.(3) to a finite degree N. For the plane for example, we have the W_1 polynomial

$$\psi(z) = \sum_{m=0}^{N} \frac{a_m}{\sqrt{m!\hbar^m}}\, z^m \; . \tag{14}$$

In practice the integer N at which the truncation is made fixes the approximate number of zeros lying in the classically allowed region of phase space at a fixed energy (outside which the associated Husimi function has a rapid decay). It is given by the semiclassical rule $N \approx$ (area of classically allowed region)$/2\pi\hbar$. The average density of roots for this polynomial can be computed from (12), with the result

$$\langle \rho(z) \rangle = \frac{1}{\pi\hbar} \left\{ 1 - g\left(\frac{z\bar{z}}{\hbar}\right) \left[1 + N - \frac{z\bar{z}}{\hbar} + \frac{z\bar{z}}{\hbar} g\left(\frac{z\bar{z}}{\hbar}\right) \right] \right\} \; . \tag{15}$$

where $g(x) = (x^N/N!)/ \sum_{m=0}^{N} (x^m/m!)$. The function (15) has a step-like shape with a constant density of zeros $1/(\pi\hbar)$ inside a circle of radius $\sim \sqrt{N\hbar}$ and a vanishing density outside (see

Fig.2). It tends, in the large N-limit, to the unit step function $\Theta(1-r)$ where $r = |z|/\sqrt{N\hbar}$. The convergence is however slow since for large values of r we find a power law dependence $\langle \rho(r) \rangle \approx 1/(\pi\hbar Nr^4) + \mathcal{O}(1/r^6)$.

We therefore recover for the W_1 random polynomials the appealing feature that the Bargmann transform of a chaotic eigenstate has a uniform density of zeros in a finite region of the phase space plane (over which the associated Husimi function is stationary). The decay of the density outside the classically allowed region is also of interest, because it describes the evanescence of the eigenfunction ψ in the classically forbidden region [23]. But we don't expect the decay predicted by Eq.(15) to be very accurate due to the abrupt truncation of ψ at $m = N$.

Analogous results are obtained for the $SU(1,1)$ polynomials defined, like the W_1 polynomial, by the truncation of the series in Eq.(3) to a finite value of m

$$\psi(z) = \sum_{m=0}^{N} \sqrt{C_{L+m-1}^m} \, a_m \, z^m \ . \tag{16}$$

In the limit $N \gg L$ we find a constant density of zeros in the hyperbolic plane inside a region of size $\cosh \tau \approx N/L$.

Time reversal symmetry

The previous results are valid when the coefficients of the random functions are complex. This generally corresponds to systems without time reversal symmetry. For systems having that symmetry, the coefficients $\{a_m\}$ can be chosen to be real, and $\psi(z)$ satisfies now the functional equation

$$\psi(z) = \overline{\psi(\bar{z})} \ .$$

As discussed in detail in [9], the existence of a functional equation generically produces a concentration of a certain number N_R of roots over the symmetry line (here the real axis in the complex z-plane, labeled by x). The distribution and correlations of roots of real polynomials was studied in [24]. For $SU(2)$ polynomials, explicit expressions for the correlations existing between real roots where obtained in [17].

The number $N_R = \int \langle \rho(x) \rangle \, dx$ of real roots of a polynomial whose coefficients are real random Gaussian independently distributed with variance σ_m^2 may be computed from a general formula obtained by Kac. The average density of real roots is given by [18]

$$\langle \rho(x) \rangle = \frac{1}{\pi} \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)} \ , \qquad (17)$$

with

$$
\begin{aligned}
A(x) &= \sum_{m=0}^{N} \sigma_m^2 x^{2m} \\
B(x) &= A'(x)/2 \\
C(x) &= [A''(x) + A'(x)/x]/4 \ ,
\end{aligned}
\qquad (18)
$$

where the prime indicates derivative with respect to x.

For $SU(2)$ polynomials we have $\sigma_m^2 = C_N^m$, $A(x) = (1 + x^2)^N$ and from (17) we obtain $\langle \rho(x) \rangle \, dx = \frac{N}{\pi} \frac{dx}{1+x^2}$. By a stereographic projection $x = \cot(\theta/2)$ the density simplifies to $\langle \rho(\theta) \rangle \, d\theta = \frac{\sqrt{N}}{2\pi} d\theta$. After integration we obtain $N_R^{(SU(2))} = \sqrt{N}$, a result which was obtained in [14].

For real W_1 random functions, $\sigma_m^2 = 1/(k!\hbar^k)$, $A(x) = \exp(x^2/\hbar)$ and $\langle \rho(x) \rangle \, dx = dx/(\pi\sqrt{2\hbar})$ (remember that the measure is $dx/\sqrt{2}$). Since for the complex W_1 function we have, according to Eq.(13), N roots in a disk of radius $r = \sqrt{2\hbar N}$, we find that the number of real roots when the coefficients are real is $N_R^{(W_1)} = \frac{2}{\pi}\sqrt{N}$.

Finally, for $SU(1,1)$ we have $\sigma_m^2 = C_{L+m-1}^m$, $A(x) = 1/(1-x^2)^N$ and $\langle \rho(x) \rangle \, dx = \frac{N}{\pi} \frac{dx}{1-x^2}$. Going to the hyperbolic plane via $x = \tanh(\tau/2)$, the density is simply written $\langle \rho(\tau) \rangle \, d\tau = \frac{\sqrt{N}}{2\pi} d\tau$. Since for the complex $SU(1,1)$ function we have, according to Eq.(13), a number L of roots in a region of size $\tau = \text{arccosh}(3)$, we find a number of real roots given by $N_R^{(SU(1,1))} = \frac{\text{arccosh}(3)}{\pi}\sqrt{N}$.

IV. PHASE-SPACE CORRELATIONS – UNIVERSALITY

Besides the average density of roots, other relevant statistical properties are correlation functions of $\psi(z)$ or of its zeros. One might think that this quantities are geometry-

dependent. However, in the large-N limit, the three different geometries treated here tend to the same flat-space result. Indeed, for fixed m we have

$$\lim_{X \to \infty} C_X^m = \lim_{X \to \infty} C_{X+m-1}^m = \frac{X^m}{m!} \ , \tag{19}$$

where $X = N$ for the $SU(2)$ group and $X = L$ for the $SU(1,1)$. This implies that, with the scaling $z \to \sqrt{X}z$, the $SU(2)$ and truncated $SU(1,1)$ polynomials tend to the truncated W_1 polynomial in a region around the origin of the z-plane (the argument can be repeated for an arbitrary point lying inside the constant-density region). This is easy to understand since in the large-N limit the local density of zeros of the rescaled polynomials tend to diverge (the mean spacing between zeros in rescaled units goes like $1/\sqrt{N}$). On a scale of distances of a few mean spacings between zeros the space will look, locally, flat. As a consequence the *local* fluctuation properties of all these functions are, asymptotically, the same [25]. This can be rephrased as a general statement concerning the local correlation properties of eigenstates of fully chaotic systems (for points lying inside the energy shell): they are *universal, independent of the system under consideration or the geometry of phase space.*

The n-point correlation functions among zeros have been computed analytically by J. Hannay for random $SU(2)$ polynomials [15]. In particular, it was found that there is a cubic repulsion between zeros. The previous considerations imply that in the large-N limit these correlation functions should be valid locally for any geometry. The two-point correlation function as well as a computation of the nearest neighbor spacing distribution for the random analytic model have been compared with numerical results obtained for several chaotic maps in different phase space geometries in [16]. Very good agreement has been found, thus supporting the universality of the correlations. Further support comes from a computation of the two-point correlation function and the nearest neighbor spacing distribution for the zeros of a two-dimensional chaotic system having a W_1 symmetry [22].

Concerning $\psi(z)$ itself (and not its zeros), the simplest quantity of interest is the auto-correlation function

$$C(\xi, z) = \frac{\langle \psi(z - \xi/2) \overline{\psi(z + \xi/2)} \rangle}{\sqrt{\langle |\psi(z - \xi/2)|^2 \rangle \langle |\psi(z + \xi/2)|^2 \rangle}} \ . \tag{20}$$

As before, this quantity will be asymptotically independent of the geometry. For simplicity, we restrict the computation to the W_1 case. Using Eqs.(3), (8) and (9) one finds

$$C(\xi, z) = \exp\left[-|\xi|^2/2 + i\,\mathrm{Im}(z\bar{\xi}) \right] \ . \tag{21}$$

Unlike the phase, the modulus of the autocorrelation is independent of z and is isotropic. This result should be compared with an analogous computation in configuration space [5], where it was found that C is given by a Bessel function (we find that Eq.(21) holds also for systems with time reversal symmetry; analogous quantities for systems with and without time-reversal symmetry have been considered more recently in [26]).

V. CONCLUSIONS

The distribution of the zeros of W_1 random polynomials of degree N is strongly reminiscent to that of the eigenvalues of a Gaussian ensemble of $N \times N$ complex matrices discussed by Ginibre (see chapter 15 in [1]). Indeed, for large values of N both distributions have a uniform density in the complex plane inside a disk of radius $|z| \approx \sqrt{N}$, and rapidly decaying to zero outside of it (for simplicity, we set $\hbar = 1$ in Eq.(16)). The distributions are however different. The density has a Gaussian tail for $|z| \gg \sqrt{N}$ in the case of the eigenvalues of random complex matrices, while it has a power law decay for W_1 polynomials. Moreover, in the case of Ginibre the distribution may be interpreted as a two-dimensional Coulomb gas, while the two-dimensional gas associated to zeros of random polynomials includes, apart from the Coulomb interaction, N-body terms [16]. A detailed comparison between these two interacting systems was recently made in [27].

Several extensions of the results reported here are conceivable, in particular to higher dimensional complex manifolds. The random matrix distribution (1) is in fact valid independently of the number of degrees of freedom of the underlying physical system. It is therefore

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natural to expect that results analogous to those obtained here for the distribution of the zeros in two dimensional phase spaces also hold in higher dimensions (like for example for polynomials with several complex variables). And indeed a generalization to $SU(m + 1)$ polynomials with $m > 1$ was recently done in Ref. [10].

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Captions to Figures

Figure 1: (a) The sphere and, (b) the pseudosphere (upper sheet of the hyperboloid $-x_0^2 + x_1^2 + x_2^2 = -1$) having curvature $+1$ and -1, respectively. For $SU(2)$, the definition of z corresponds to a stereographic projection which maps the unit sphere labelled by (θ, φ) onto the equatorial complex plane labelled by z, the north pole of the sphere being the center of projection. In the case of $SU(1, 1)$, a projection of the pseudosphere onto the unit disk in the complex z-plane located at $x_0 = 0$ is done through the point $(x_1, x_2, x_0) = (0, 0, -1)$.

Figure 2: The normalized density of zeros Eq.(15) for the W_1 random polynomials as a function of $r = |z|/\sqrt{N\hbar}$ for $N = 100$.

a)

(θ, φ)

z

b)

$\langle \rho \rangle * \pi$

1.0

0.5

0.0

x_0

(τ, φ)

z

x_2

x_1

0.0   0.5   1.0   1.5

r