Random Walks on the Braid Group B_3 and Magnetic Translations in Hyperbolic Geometry

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June 8, 2007

Abstract

We study random walks on the three-strand braid group B_3, and in particular compute the drift, or average topological complexity of a random braid, as well as the probability of trivial entanglement. These results involve the study of magnetic random walks on hyperbolic graphs (hyperbolic Harper-Hofstadter problem), what enables to build a faithful representation of B_3 as generalized magnetic translation operators for the problem of a quantum particle on the hyperbolic plane.

1 Introduction and basic definitions

This paper presents different results concerning random walks on the three–strand braid group B_3. Besides the relevance of this work at a purely algebraic level (see [8] for a review of knot theory), its usefulness in the understanding of topological problems in statistical physics is also undeniable. We will keep in mind the picture of entanglements of chain–like objects, like polymers [6] or vortex lines in super conductors [3]. The tools involved in this work unexpectedly lead to the study of the quantum mechanics of a charged particle moving on the hyperbolic plane in a constant magnetic field, and

allow a discrete interpretation of an exact result [9] giving the probability distribution of the area enclosed by a hyperbolic Brownian motion. This connection between B_3 and quantum mechanics in hyperbolic geometry is the main new result of this paper; it is established by proposition 2 and theorem 1. The mathematical ingredients of this theorem, such as group cohomology and central extensions of groups, as well as the concept of generalized magnetic translation operators of definition 3, are introduced in a general way, in order to show that this interpretation of a central extension as a magnetic field is a general result that can be adapted to other extensions of lattice groups. This paper focuses on the case of B_3, because of its physical importance; in particular, using the relation between B_3 and $PSL(2, \mathbb{Z})$ and between the corresponding $C^\star$–algebras enlightened by proposition 3, we obtain theorem 2 and theorem 3, which provide rigorous derivations of the return probability and the drift for B_3, results which were more intuitively given in [10]. Here we use the more systematic method of studying the transition operator and the algebra it belongs to, which allows a possible generalization of these results to other Abelian extensions of hyperbolic groups.

Consider the 3D model of N directed polymer chains, labelled by $\vec{r}_i(t) = (x_i(t), y_i(t))$, attached at one end (on the plane $t = 0$), growing along the direction $t > 0$ and fluctuating in the plane (x, y). We assume periodic boundary condition, that is $\{\vec{r}_i(T)\} = \{\vec{r}_{p(i)}(0)\}$), p being a permutation over N elements. We are interested in the complexity of this randomly generated entanglement, which can play a crucial role in macroscopic physical properties, such as polymer elasticity. Each topological configuration can be exactly encoded by an element of B_N, that is a word whose letters are the generators of the group, associated to elementary moves as shown in figure 1. Looking at the projection onto the plane (t, x), elementary moves correspond to positive/negative crossings. We restrict our study to the simplest non commutative case $N = 3$, whose set of generators is $S = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$. A randomly generated configuration corresponds to an element w_n, random product of n generators; in other terms we focus on the following Markov process

$$
\begin{aligned}
w_0 &= \text{Id} \\
w_n &= w_{n-1}\zeta_n \text{ for } n \geq 1
\end{aligned}
\tag{1}
$$

where ζ_n are i.i.d. random variables in S. Average over this random variable will be denoted by $\langle . \rangle$. A more thorough study would require to establish a law $n(T)$, giving the time scaling of the number of elementary moves, which is not discussed here. We study the probability distribution of $L(w_n)$, the irreducible length in the metric of words, or in other terms the minimal number

of generators necessary to build w_n. $L(w_n)$ is exactly the minimal number of crossings necessary to represent the entanglement, and can therefore be seen as an indicator of the braid complexity. In particular, if $L(w_n) = 0$ the braid is trivial (or unlinked).
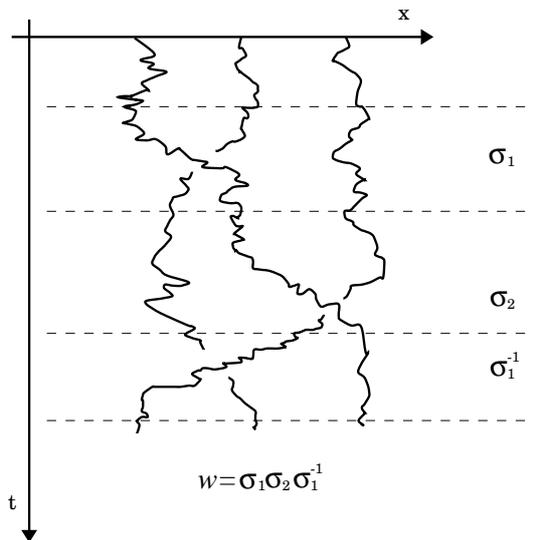

Figure 1: Braiding of directed polymers. Elementary moves σ_i generate B_3.

We now introduce B_3 in a more algebraic way, useful to study random processes on this group.

Definition 1. B_3 *is defined by the set of generators* $S = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ *together with the commutation relation*

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \tag{2}$$

or equivalently by the generators $S' = \{a, b, a^{-1}, b^{-1}\}$ *together with the relation*

$$a^2 = b^3 \tag{3}$$

The equivalence of these (so-called Artin) presentations is explicit if one sets:

$$\begin{aligned} a &= \sigma_1 \sigma_2 \sigma_1 \\ b &= \sigma_1 \sigma_2 \end{aligned} \tag{4}$$

It is known that B_3 is a central extension of $PSL(2, \mathbb{Z})$[1], i–e that B_3 factored by its center Z is isomorphic to $PSL(2, \mathbb{Z})$. Let us recall that the center Z

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[1] the group of 2×2 unimodular matrices of integer entries, factored by $\pm \mathrm{Id}$

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of B_3, which is the subgroup of elements commuting with any element of B_3, is generated by $a^2 = b^3 = \Delta^2$. Δ^2 is the usual notation for the generator of the center in Artin groups theory. For B_3, one has simply $\Delta = a$. It is useful to introduce

$$i : \begin{cases} \mathbb{Z} & \longrightarrow & Z \\ n & \longrightarrow & i(n) = \Delta^{2n} \end{cases} \tag{5}$$

which shows that Z is isomorphic to $\mathbb{Z}$. We then denote by π the canonical projection onto $PSL(2, \mathbb{Z})$ and by s a cross–section of π, that is an application such that $\pi \circ s = \mathrm{id}$. It is convenient to summarize these elements by writing the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;i\;} B_3 \xrightarrow{\;\pi\;} PSL(2, \mathbb{Z}) \longrightarrow 1 \tag{6}$$

Note that $PSL(2, \mathbb{Z})$ is a free product $\mathbb{Z}_2 \star \mathbb{Z}_3$, $\pi(a)$ and $\pi(b)$ being the corresponding generators of order 2 and 3 respectively. The generators of the group B_3 can be represented by $GL(2, \mathbb{Z}[u^{\pm 1}])$–matrices. To be more precise:

Definition 2. (see [8]) *The Burau representation of B_3 is given by*

$$\sigma_1 = \begin{pmatrix} u^2 & 1 \\ 0 & 1 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 1 & 0 \\ -u^2 & u^2 \end{pmatrix} \tag{7}$$

where u is a free parameter. This representation is faithful.

We conveniently introduce an auxiliary group, $PSL(2, \mathbb{Z})_u$, built on generators (4) of this representation of B_3 normalized by the determinant:

$$a_u = \begin{pmatrix} 0 & 1/u \\ -u & 0 \end{pmatrix}; \qquad b_u = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{8}$$

This group is a "deformation" of $PSL(2, \mathbb{Z})$, which preserves all commutation relations. For $u = 1$, one has $PSL(2, \mathbb{Z})_u = PSL(2, \mathbb{Z})$. For a more detailed discussion of braid groups the reader is referred to [7, 8].

2 Random processes on B_3: magnetic random walk formalism

We now show that a free random walk on B_3 can be viewed as a random walk on $PSL(2, \mathbb{Z})$ with magnetic field (we will say a magnetic random walk).

The key point is that B_3 is a central extension of $PSL(2, \mathbb{Z})$. We will use systematically the set of generators $S' = \{a, a^{-1}, b, b^{-1}\}$, to avoid technical complexity, keeping in mind that the formalism can be adapted to generators $S = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ introduced previously. Results of physical interest will be given in terms of these generators S. First note that the Cayley graph[2] of B_3 is three dimensional. As shown in Fig.2, the map π can then be viewed as a projection from 3D to 2D; the vertical axis represents the "center" coordinate. Intuitively, we will show that this center term counts the magnetic flux through the random path.
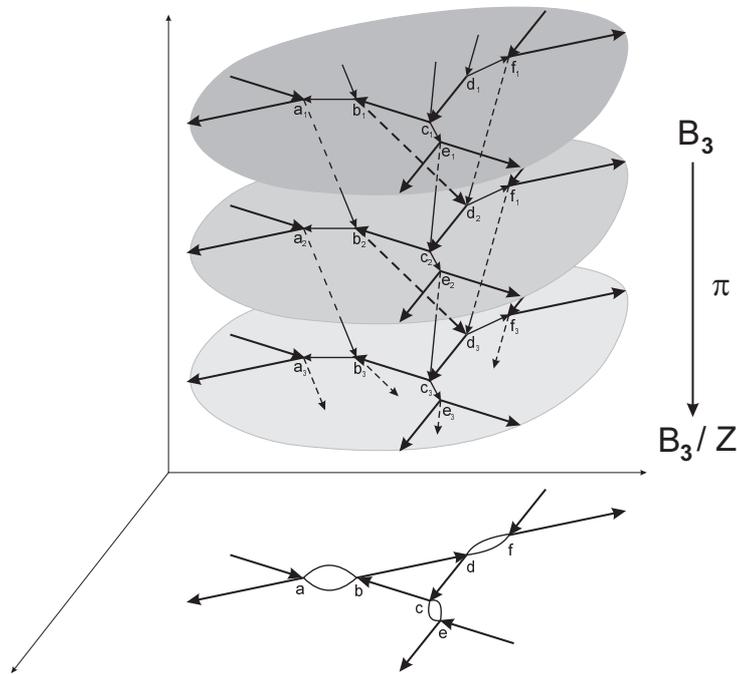

Figure 2: B_3 Cayley graph and its projection ($PSL(2, \mathbb{Z})$). Thin arrows correspond to a, thick ones to b. Note that $\pi(\alpha_i) = \alpha$, $\pi(\beta_i) = \beta$ and so on. Recall that $\pi(a)$ has to be identified with $\pi(a)^{-1}$

Namely, if one inserts in each elementary cell of the hyperbolic lattice a unit "magnetic flux" (see Fig.2) and denotes by f the total flux through a closed path on $PSL(2, \mathbb{Z})$, then any word w_n^Z corresponding to a walk ending

[2]The graph with set B_3 of vertices, and set $\{\{x, xs\} : x \in B_3, s \in S'\}$ of edges

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in $Z(B_3)$ (note that $\pi(w_n^Z)$ is a closed path on $PSL(2,\mathbb{Z})$) can be written as

$$w_n^Z = \Delta^{2f}$$

Hence the function $p_n(f)$ giving the joint probability to obtain a closed n–step loop on the graph $PSL(2,\mathbb{Z})$ carrying a flux f is of great interest, especially because $p_n(0)$ is the probability to get a *trivial braid* (i.e. completely reducible word).

Let us establish more rigorously this relation between B_3 and what could be interpreted as a hyperbolic version of the Harper-Hofstadter problem [11]. To introduce the usual setting of this problem, we define the transition operator, or discrete Laplace operator,

$$h = \frac{1}{4}(a + a^{-1} + b + b^{-1}) \tag{9}$$

viewed as an element of $C^\star(B_3)^3$. We use the following decomposition for any $X \in C^\star(B_3)$

$$X = \sum_{w \in B_3} c_w(X)w \tag{10}$$

with $c_w \in \mathbb{C}$ and the elements w of B_3 are identified with the operators acting on $l^2(B_3)$. Different quantities characterizing random walks on B_3 are obtained from h. One is the return probability

$$p_n(0) = c_{\mathrm{Id}}(h^n) \equiv \tau(h^n) \tag{11}$$

where Id denotes the identity operator. This scalar component τ of an operator is called a tracial state on $C^\star(B_3)$. Another quantity of interest is the drift

$$l_{B_3} = \lim_{n \to \infty} \frac{\langle L^{B_3}(w_n) \rangle}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{w \in B_3} c_w(h^n) L^{B_3}(w) \tag{12}$$

Note that these quantities depend only on the algebraic structure of $C^\star(B_3)$; they are purely combinatorial objects which in particular do not depend on the representation of the algebra. We are therefore allowed to use a more convenient isomorphic algebra, what is the aim of the following construction. Following the definition of rotation algebras in [16, 17], we build a $C^\star$–algebra R_θ, generated by A_θ, B_θ defined formally by the following $\star$–homomorphism

$$\Psi_\theta : \begin{cases} a & \longrightarrow & A_\theta \\ b & \longrightarrow & B_\theta \\ \Delta^2 & \longrightarrow & e^{i\theta} \end{cases} \tag{13}$$

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[3] That is the algebra generated by the left regular representation of B_3 on $l^2(B_3)$

The aim of the following is to construct explicitly a representation of R_θ on $l^2(PSL(2,\mathbb{Z}))$, and to show that $\Psi_\theta(h)$ is exactly the Hamiltonian of a particle moving on the graph $PSL(2,\mathbb{Z})$ in a magnetic field of elementary flux θ. This point accounts more rigorously for the geometrical picture given above. In a more mathematical language, we use general results of group cohomology (see [19] for review). Namely, we use the following proposition:

Proposition 1. *There exists an integer–valued map Θ satisfying*

$$\Theta(x, yz) + \Theta(y, z) = \Theta(xy, z) + \Theta(x, y), \qquad (14)$$

such that B_3 is isomorphic to the Cartesian product $\mathbb{Z} \times PSL(2,\mathbb{Z})$ with the following multiplication

$$(m, x).(n, y) = (m + n + \Theta(x, y), xy) \qquad (15)$$

Proof. This is a direct application of a general theorem of classification of extensions with Abelian kernel (see [19]) to the decomposition (6) of B_3 . Θ is an element of $H^2(PSL(2,\mathbb{Z}), \mathbb{Z})$, the set of so–called group 2–cocycles, and is unique up to cohomology equivalence. Associativity of the product (15) is ensured by (14). $\square$

We now trivially map $\mathbb{Z} \times PSL(2,\mathbb{Z})$ into $U(1) \times PSL(2,\mathbb{Z})$ via $(n, w) \rightarrow e^{in\theta}w$. Now the identification

$$\begin{aligned} A_\theta &\longleftrightarrow (0, \pi(a)) \\ B_\theta &\longleftrightarrow (0, \pi(b)) \end{aligned} \qquad (16)$$

clarifies our point: B_3 (via R_θ) is identified with $U(1) \times PSL(2,\mathbb{Z})$, which is a "magnetic" equivalent of $PSL(2,\mathbb{Z})$. Indeed the 2–cocycle Θ is the discrete analogue of a 2–form, or a magnetic field in physical terms. Cohomology equivalence is then interpreted as gauge invariance. We believe that this point, put forward in details in [15], is very important and that this magnetic interpretation of central extensions can be fruitfully generalized to other groups.

3   Quantum mechanics in hyperbolic geometry

We now come to the explicit construction of a representation of R_θ on $l^2(PSL(2,\mathbb{Z}))$, showing in particular that $\Psi_\theta(h)$ is exactly the Hamiltonian

of a particle moving on the graph $PSL(2, \mathbb{Z})$ in a magnetic field of elementary flux θ. Sunada has given in [13] a general definition of discrete magnetic Schrödinger operators (*generalized Harper operators*). We here wish to give a construction compatible with this definition, starting from the continuous problem.

3.1 Continuous case

We consider the quantum mechanics of a charged particle moving on the hyperbolic plane in a constant magnetic field B. This problem has been extensively studied in [12]. Let us recall some results of [12]. We denote by $\mathcal{H}$ the upper half-plane of the complex variable $z = x + iy$ $(y > 0)$ endowed with a metric g such that

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{17}$$

The Lagrangian of the particle, of mass $1/2$ and charge 1, reads

$$\mathcal{L} = \frac{\dot{x}^2 + \dot{y}^2}{y^2} - B\frac{\dot{x}}{y} \tag{18}$$

A thorough study of the problem shows that at a classical level, $PSL(2, \mathbb{R})$ acts as dynamical symmetry: it changes $\mathcal{L}$ by total derivative. We choose the following basis of $\mathfrak{sl}(2, \mathbb{R})$, the Lie algebra of $PSL(2, \mathbb{R})$:

$$j_0 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \ j_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \ j_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{19}$$

The corresponding constants of motion are obtained through Noether theorem. Following the canonical quantization procedure one gets

$$\begin{aligned} J_0 &= p_x(x^2 - y^2 + 1)/2 + xyp_y - By \\ J_1 &= p_x(y^2 - x^2 + 1)/2 - xyp_y + By \\ J_2 &= xp_x + yp_y \end{aligned} \tag{20}$$

where the momentum operators are given by

$$p_x = -i\frac{\partial}{\partial x}, \ p_y = -i\frac{\partial}{\partial y} + \frac{i}{y} \tag{21}$$

These generators satisfy the canonical form of the $SO(2, 1)$ (or $SU(1, 1)$ for the covering group) algebra. This symmetry is of great interest since the Hamiltonian is straightforwardly expressed in terms of the Casimir (note

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that we use the standard convention for the $SO(2,1)$ norm) $C = ||J||^2 = J_i J^i = J_0^2 - J_1^2 - J_2^2$:

$$H = \frac{B^2}{4} - \frac{C}{4} \tag{22}$$

which allows to obtain the spectrum. One can explicitly build a standard basis $|jm>$ of the Hilbert space $\mathfrak{H}$ such that

$$\begin{array}{rcl} < jm|jm' > & = & \delta_{mm'} \\ C|jm > & = & j(j+1)|jm > \\ J_0|jm > & = & m|jm > \end{array} \tag{23}$$

One distinguishes a continuous and discrete part of the spectrum, depending on the value of j. In any case, one has

$$m = B + n, \ n \in \mathbb{Z} \tag{24}$$

3.2 Discretization: B_3 generators as generalized magnetic translation operators

We now investigate further the $PSL(2, \mathbb{R})$ symmetry of this problem, and in particular we wish to define generalized magnetic translation operators, the same way as magnetic translation operators are defined in Euclidean geometry (see [18, 13, 14] for review). We are looking for a representation of the symmetry group of the problem on the Hilbert space $\mathfrak{H}$ of wave functions. For sake of simplicity, we propose a definition in the case of the hyperbolic plane, keeping in mind that a more general formulation follows straightforwardly. The same construction can be performed in particular as soon as the symmetry group is a Lie group; then discretization requires a lattice subgroup of this Lie group.

Definition 3. *We call generalized magnetic translation operator (GMTO), denoted by $\mathcal{O}_\tau$, the unitary operator corresponding to an isometry τ acting on a wave function ψ as follows:*

$$\mathcal{O}_\tau[\psi](z) = \psi(\tau^{-1} z) \tag{25}$$

such that $[\mathcal{O}_\tau, H] = 0$.

The Wigner theorem (see [20]) shows that such a representation (*projective unitary representation*) exists up to a phase:

$$\mathcal{O}_{\tau\tau'} = e^{i\Phi(\tau,\tau')} \mathcal{O}_\tau \mathcal{O}_{\tau'} \tag{26}$$

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In particular for a closed sequence of isometries $\gamma = \prod_i \tau_i = \text{Id}$, the corresponding product of operators acts on a wave function ψ as follows:

$$[\prod_i \mathcal{O}_{\tau_i}](\psi)(z) = \exp(i\Phi_\gamma)\psi(z) \tag{27}$$

Where Φ_γ is the magnetic flux through the loop γ.

We obtain explicitly GMTO by exponentiating infinitesimal generators (20). Namely we consider the following diagram

$$
\begin{array}{ccccccc}
PSL(2,\mathbb{R}) & \xrightarrow{\log} & \mathfrak{sl}(2,\mathbb{R}) & \xrightarrow{\phi} & L(\mathfrak{H}) & \xrightarrow{\exp} & U(\mathfrak{H}) \\
x & \longrightarrow & \alpha^i j_i & \longrightarrow & \alpha^i J_i & \longrightarrow & e^{i\alpha^i J_i}
\end{array} \tag{28}
$$

where $L(\mathfrak{H})$ is the set of linear operators on $\mathfrak{H}$ and $U(\mathfrak{H})$ the group of unitary operators. Then

Proposition 2. *The following homomorphism*

$$\chi = \exp \circ \phi \circ \log : PSL(2,\mathbb{R}) \longrightarrow U(\mathfrak{H}) \tag{29}$$

defines a projective unitary representation of $PSL(2,\mathbb{R})$.

Proof. The first ingredients are the usual tools employed to reduce representation problems for connected Lie groups to their Lie algebra: this uses the exponential map and the Campell-Hausdorff formula. Now at the algebra level, note that ϕ maps $\mathfrak{sl}(2,\mathbb{R})$ to a precise representation of the $SU(1,1)$ algebra in $L(\mathfrak{H})$. Irreducible representations of $SU(1,1)$ have been classified in [22, 23], and the one we consider here is explicitly defined in [12]. In particular for B non integer, it is a proper representation of the algebra, but a projective representation of the group. This comes from the fact that $SU(1,1)$ is multi-connected. $\square$

We now discretize the problem, considering transformations on hyperbolic lattices of $\mathcal{H}$: we study the representation of $PSL(2,\mathbb{Z})_u$ (which is a lattice subgroup of $PSL(2,\mathbb{R})$) induced by χ. Using the explicit form (8), we compute the operators associated to elementary steps on $PSL(2,\mathbb{Z})_u$:

$$\chi(a_u) = \exp(i\pi\alpha^i_u J_i) \qquad \text{and} \qquad \chi(b_u) = \exp(\frac{2i\pi}{3}\beta^i_u J_i) \tag{30}$$

with $\alpha_u = \frac{1}{2}(u + 1/u, 1/u - u, 0)$; $\beta_u = (2/\sqrt{3}, 0, -1/\sqrt{3})$. Note that J transforms naturally as a vector of $SO(2,1)$. This is due to the following

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identity:

$$[J^i u_i, J^i v_i] = i\epsilon_{ijk} J^i u_j v_k \tag{31}$$

which is just a generalization of the maybe more familiar relations between Euclidean rotations generators ($SU(2)$ case). α_u (resp. β_u) being of norm 1, $\alpha_u^i J_i$ (resp. $\beta_u^i J_i$) has the same spectrum as the compact generator J_0 given in (23), if one consider a basis obtained by a nonlinear rotation (depending on α_u or β_u) of the standard basis. Using the spectrum (24), this allows to compute the phase factors for the elementary loops of $PSL(2, \mathbb{Z})_u$, predicted by the Wigner theorem:

$$\chi(a_u)^2 = \chi(b_u)^3 = e^{2i\pi B}\text{Id} \tag{32}$$

The key point is that this extra relation, introduced by the representation itself, restores the complete algebraic structure of B_3. The subgroup of $U(\mathfrak{H})$ generated by $\chi(a_u)$ and $\chi(b_u)$ is then isomorphic to B_3. In other words, we claim that

Theorem 1. *The homomorphism*

$$\Psi = \chi \circ \pi \, : \, B_3 \longrightarrow U(\mathfrak{H}) \tag{33}$$

defines a representation of the braid group B_3.

Proof. It is known that any projective representation of a group is a proper representation of a covering group. Comparing (32) and (2) shows that in our case this covering group is B_3. $\square$

Note that π and therefore Ψ depend implicitly on the Burau parameter u. If $\theta = 2\pi B$, Ψ coincides with Ψ_θ introduced previously and defines by extension a representation of R_θ (rigorously, the action defined above is on $\mathfrak{H}$; we assume that this induces naturally an action on $l^2(PSL(2, \mathbb{Z}))$). $\Psi_\theta(a)$ and $\Psi_\theta(b)$ being by construction GMTO, $\Psi_\theta(h)$ is the discrete Hamiltonian of the problem.

Remark. The 2–cocycle Θ is not needed explicitly in this construction. It can nevertheless be obtained naturally as a holonomy factor as in [15].

4   Derivation of the drift and return probability within this formalism

The point is that the representation above (we now identify X and $\Psi_\theta(X)$), in agreement with proposition 3, allows to define a new decomposition in

$C^\star(B_3)$ parametrized by $PSL(2,\mathbb{Z})$. We need to introduce an irreducible form on $PSL(2,\mathbb{Z})$, namely we write each element $\gamma = P_\gamma(\pi(a), \pi(b))$, where P_γ is a polynomial of minimal order. Then

Proposition 3. *The following decomposition holds in $C^\star(B_3)$*

$$X = \sum_{\gamma \in PSL(2,\mathbb{Z})} \sum_{n \in \mathbb{Z}} c_{n,\gamma}(X) e^{in\theta} P_\gamma(A_\theta, B_\theta) \tag{34}$$

This decomposition naturally generates a map $m_\theta : C^\star(B_3) \longrightarrow C^\star(PSL(2,\mathbb{Z}))$ by setting

$$
\begin{aligned}
m_\theta(X) &= \sum_{\gamma \in PSL(2,\mathbb{Z})} \sum_{n \in \mathbb{Z}} c_{n,\gamma}(X) e^{in\theta} \gamma \\
&= \sum_{\gamma \in PSL(2,\mathbb{Z})} \tilde{c}_\gamma(m_\theta(X)) \gamma
\end{aligned}
\tag{35}
$$

Note that this map is not a homomorphism, it is built for combinatorial purposes only. We denote by tr the tracial state on $C^\star(PSL(2,\mathbb{Z}))$, that is the scalar component of an operator: $\mathrm{tr}(Y) = \tilde{c}_{Id}(Y)$. This parameterization leads, after some algebra, to the following:

$$p_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}(m_\theta(h^n)) e^{if\theta} d\theta \tag{36}$$

We now use this framework to compute the desired quantities for a random walk on B_3. The main results were originally exposed in [10], with a more intuitive approach. We here propose a more rigorous proof relying on the connection between B_3 and $PSL(2,\mathbb{Z})$ established in theorem 1 and proposition 3. This new approach is also more general since it implies a better understanding of the transition operator h and the algebra $C^\star(B_3)$, objects very useful in various contexts.

4.1   Return probability and hyperbolic area problem

Theorem 2. *The joint probability distribution for a simple random walk on $PSL(2,\mathbb{Z})$ to return to the origin after n steps enclosing an area f is given by:*

$$p_n(f) \propto \frac{\lambda^n}{n^2} \exp\left(-\frac{f^2}{2n\sigma^2}\right) \tag{37}$$

λ *and* σ *depending on the system of generators. For* $S = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\},$
$\sigma_S = 1/6$ *and* $\lambda_S = (1 + 2\sqrt{2})/4.$

Proof. Defining the auxiliary operator

$$\tilde{h}_\theta = \frac{1}{4}(e^{i\theta/2}\pi(a) + e^{-i\theta/2}\pi(a)^{-1} + e^{i\theta/3}\pi(b) + e^{-i\theta/3}\pi(b)^{-1}) \qquad (38)$$

which satisfies $\mathrm{tr}(\tilde{h}_\theta^n) = \mathrm{tr}(m_\theta(h^n))$, allows to derive asymptotics of $p_n(f)$. We define

$$\tilde{c}_k^n(\theta) = \sum_{\gamma_k,\, L(\gamma_k)=k} \tilde{c}_{\gamma_k}(\tilde{h}_\theta^n) \qquad (39)$$

which satisfies the following recursion

$$\tilde{c}_k^n(\theta) \;\; = \sum_{\gamma_{k-1},\, L(\gamma_{k-1})=k-1} \tilde{c}_{\gamma_{k-1}}(m_\theta(h^n))g_1(\gamma_{k-1}) \qquad (40)$$

$$+ \sum_{\gamma_k,\, L(\gamma_k)=k} \tilde{c}_{\gamma_k}(m_\theta(h^n)g_0(\gamma_k) \qquad (41)$$

$$+ \sum_{\gamma_{k+1},\, L(\gamma_{k+1})=k+1} \tilde{c}_{\gamma_{k+1}}(m_\theta(h^n))g_{-1}(\gamma_{k+1}) \qquad (42)$$

where $g_i(\gamma)$ is the factor corresponding to all possible steps x such that $L(\gamma x) = L(\gamma) + i$. A careful counting of the different possible transitions (such a counting is allowed since we know exactly the irreducible form P_γ for $PSL(2, \mathbb{Z})$, see [10]; only last generator of the irreducible form is important) gives

$$\begin{cases} \tilde{c}_k^{n+1}(\theta) \;\; = \;\; (\dfrac{\alpha}{2}\cos(\theta/2) + \dfrac{1-\alpha}{2}\cos(\theta/3))\,\tilde{c}_{k-1}^n(\theta) \\[2mm] \qquad\qquad + \qquad\qquad \dfrac{\alpha}{4}\cos(\theta/3)\,\tilde{c}_k^n(\theta) \\[2mm] \qquad\qquad + \;\; (\dfrac{1-\alpha}{2}\cos(\theta/2) + \dfrac{\alpha}{4}\cos(\theta/3))\,\tilde{c}_{k+1}^n(\theta) \qquad k \geq 1 \quad (43) \\[4mm] \tilde{c}_0^{n+1}(\theta) \;\; = \;\; \dfrac{1-\alpha}{2}\cos(\theta/2)\,\tilde{c}_1^n(\theta) + \dfrac{1}{2}\cos(\theta/3)\,\tilde{c}_0^n(\theta) \end{cases}$$

where α is a constant to determine, giving the ratio of random words whose irreducible form ends with $b^{\pm 1}$. For $\theta = 0$ one recovers the master equation governing a random walk on $PSL(2, \mathbb{Z})$, studied throughout [10]. This allows in particular to compute $\alpha = 3/5$. We are interested in the asymptotics $n \gg 1$. In this regime, the solution is assumed to be of the form

$$\tilde{c}_k^n(\theta) = q_k^n \mu^n(\theta) \qquad (44)$$

This factorization is implicitly accounted for in [10]: the process generating the phase of the coefficient $\tilde{c}_{\gamma_k}(m_\theta(h^n))$, described by $\mu(\theta)$ is independent of the word length k in $PSL(2,\mathbb{Z})$ (q_k^n is, up to normalization, the distribution function of a random walk on $PSL(2,\mathbb{Z})$, computed in [10]). The system (43) then gives asymptotically

$$\mu^n(\theta) = \frac{1}{2^n}(\cos(\theta/2) + \cos(\theta/3))^n \tag{45}$$

and we have estimated in particular $\tilde{c}_0^n(\theta) = \text{tr}(m_\theta(h^n))$. We then apply (36) to conclude the proof. □

Remark 2. We could have used instead of m_θ a $\star$–homomorphism:

$$M_\theta : \left\{ \begin{array}{ccc} A_\theta & \longrightarrow & e^{i\theta/2}\pi(a) \\ B_\theta & \longrightarrow & e^{i\theta/3}\pi(b) \end{array} \right. \tag{46}$$

which gives $M_\theta(h) = \tilde{h}_\theta$. This formalism is equivalent for trace computations, but for other combinatorial matters it can be confusing. In particular the length function is easier to define using m_θ.

Recall that f counts the area enclosed by the random path; it is then interesting to compare this discrete model with the exact solution of the area problem in the continuous case, derived in [9]. The probability distribution in this case reads:

$$p_t(A) \propto \frac{\exp(-A^2/2t)}{\cosh^2(\pi A/t)} \tag{47}$$

where A is the hyperbolic area and t the time. For long trajectories ($A \gg 1$), one can check that the Gaussian behavior (37) of our discrete model is recovered. For short trajectories ($A \ll 1$) one recovers the flat space limit, unreachable in our model because the length scale is fixed by the curvature. This asymptotic agreement between the continuous and discrete model is an important fact, though intuitively expectable. Indeed the definition of hyperbolic lattices is not as straightforward as in the Euclidean case (isometric lattices of $\mathbb{R}^n$ are exactly $\mathbb{Z}^n$). In particular the elementary cell of a hyperbolic lattice (or fundamental domain) is neither necessarily compact nor of finite area (see [24, 25]); therefore the agreement between the continuous an discrete approach of the area problem is not trivial, and supports the idea that $PSL(2,\mathbb{Z})$ is a "good" discretization of $\mathcal{H}$.

4.2 The drift, or average topological complexity

Theorem 3. *The drift of a simple random walk on B_3 is given by:*

$$l_{B_3} = \lim_{n \to \infty} \frac{\langle L^{B_3}(w_n) \rangle}{n} = \frac{1}{4} \tag{48}$$

Proof. We use (12) and the decomposition (34) to write

$$l_{B_3} = \lim_{n \to \infty} \frac{1}{n} \sum_{\gamma \in PSL(2,\mathbb{Z})} \sum_{k \in \mathbb{Z}} c_{k,\gamma}(h^n) L^{B_3}(e^{ik\theta} P_\gamma(A_\theta, B_\theta)) \tag{49}$$

Using $L^{PSL(2,\mathbb{Z})}(P_\gamma(A_\theta, B_\theta)) \leq L^{B_3}(e^{ik\theta} P_\gamma(A_\theta, B_\theta)) \leq L^{PSL(2,\mathbb{Z})}(P_\gamma(A_\theta, B_\theta)) + |k|$, we first show that

$$\frac{L(\pi(w_n))}{n} \leq \frac{L^{B_3}(w_n)}{n} \leq \frac{L(\pi(w_n))}{n} + \frac{1}{n} \sum_{k \in \mathbb{Z}} |k| c_{k,w_n}(h^n) \tag{50}$$

The last ingredient is the following inequality:

$$\sum_{k \in \mathbb{Z}} |k| c_{k,\gamma}(h^n) \leq |\frac{d}{d\theta} \tilde{c}_\gamma(\tilde{h}_\theta)|. \tag{51}$$

Using (45), the following holds

$$\frac{\langle L(\pi(w_n)) \rangle}{n} \leq \frac{\langle L^{B_3}(w_n) \rangle}{n} \leq \frac{\langle L(\pi(w_n)) \rangle}{n} + O\left(\frac{1}{\sqrt{n}}\right) \tag{52}$$

We use the result $l_{PSL(2,\mathbb{Z})} = 1/4$ obtained in [10] for symmetric random walk on $PSL(2,\mathbb{Z})$ to conclude the proof. $\square$

A more intuitive approach of this result is proposed in [10]. The development carried out above shows that this result could be generalized to many other groups: if E is an Abelian extension of a hyperbolic group G, then the drift on E equals the drift on G. We here give this result without proof.

5 Conclusion and Remarks

To conclude, we wish to comment the new physical content of these results. So far the only topological invariant studied in the context of entanglements of fluctuating linear objects was the so-called winding number [1, 2, 3, 4], which is known to be incomplete for more than 2 linear objects.

We here propose a conceptually simple model (in particular fluctuations are not precisely taken into account), which describes exactly the underlying non-Abelian topology of the problem. The resulting behavior of the topological complexity, the irreducible braid length, differs strongly from the Abelian case. Indeed, for n elementary moves, the winding number W for our three strands problem would scale like $\sqrt{\langle W^2 \rangle} \propto \sqrt{n}$ (compare with $\langle L(w_n) \rangle \propto n$), whereas the probability of $W = 0$ would scale like $n^{-3/2}$ (compare with $p_n(0) \propto \lambda^n/n^2$).

Beside these direct motivations of the physical model, this work allowed to study the link, through the example of B_3, between random walks on a discrete group and a projective unitary representation of this group in a Hilbert space of a quantum system in a magnetic field. This is in our case the hyperbolic counter part of the well–known Harper-Hofstadter problem in Euclidean geometry. We could in particular reformulate and asymptotically compare the distribution of the hyperbolic area enclosed by a random path, in the case of a lattice random walk and in the case of a continuous Brownian motion. This correspondence requires to define generalized magnetic translation operators, and in particular leads to a general algebraic set up for discrete problems with magnetic fields, which enabled us to rigorously prove different results concerning B_3 already obtained in [10]. We hope that this extension of the problem to hyperbolic geometry is not restrictive and could be adapted to other spaces.

Acknowledgements

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