Partial Survival and Crossing Statistics for a Diffusing Particle in a Transverse Shear Flow

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We consider a non-Gaussian stochastic process where a particle diffuses in the y-direction, $dy/dt = \eta(t)$, subject to a transverse shear flow in the x-direction, $dx/dt = f(y)$. Absorption with probability p occurs at each crossing of the line $x = 0$. We treat the class of models defined by $f(y) = \pm v_\pm(\pm y)^\alpha$ where the upper (lower) sign refers to $y > 0$ ($y < 0$). We show that the particle survives up to time t with probability $Q(t) \sim t^{-\theta(p)}$ and we derive an explicit expression for $\theta(p)$ in terms of α and the ratio v_+/v_-. From $\theta(p)$ we deduce the mean and variance of the density of crossings of the line $x = 0$ for this class of non-Gaussian processes.

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There has been a resurgence of interest in first-passage problems in recent years [1], particularly in the context of systems with many degrees of freedom [2]. There are, however, relatively few exactly solved models involving *non-Gaussian* processes. This paper is devoted a class of such models, involving just two degrees of freedom, for which exact results can be obtained. These models are most simply described in terms of a particle moving in the two-dimensional plane (x, y), with stochastic motion in the y-direction and 'deterministic' motion in the x-direction (deterministic in the sense that the velocity in the x-direction depends only on the y-coordinate, which is itself however a stochastic variable).

The class of models is defined by the following equations:

$$\dot{y} = \eta(t) , \quad (1)$$
$$\dot{x} = f(y) = v_\pm \mathrm{sgn}(y)\, |y|^\alpha , \quad (2)$$

where the upper (lower) sign refers to $y > 0$ ($y < 0$), dots indicate time derivatives, and $\eta(t)$ is Gaussian white noise with mean zero and correlator $\langle \eta(t)\eta(t')\rangle = 2D\delta(t - t')$. These models are non-Gaussian except for the case $\alpha = 1$, $v_+ = v_- = v$, which reduces to the random acceleration process $\ddot{x} = v\eta(t)$.

Previous work on this class of models has addressed the first-passage problem in which the line $x = 0$ is an absorbing boundary and the particle starts in the half-plane $x > 0$. It was shown [3] that the probability, $Q(t)$, that the particle survives until time t decays as $Q(t) \sim t^{-\theta}$ for large t, with

$$\theta = \frac{1}{4} - \frac{1}{2\pi\beta} \tan^{-1}\left[\frac{\gamma - 1}{\gamma + 1} \tan\left(\frac{\pi\beta}{2}\right)\right] \quad (3)$$

where

$$\beta = \frac{1}{2 + \alpha}, \qquad \gamma = \left(\frac{v_+}{v_-}\right)^\beta . \quad (4)$$

The exponent θ takes the value $1/4$ for all α for the 'antisymmetric' case $\gamma = 1$ (for which $f(y)$ is an odd function) which has a simple explanation [4, 5] in terms the Sparre Anderson theorem [6].

In the present paper we consider the case where the line $x = 0$ is a *partially* absorbing boundary, at which the particle is absorbed with probability p at each crossing. We have shown elsewhere [7] that for such models the survival probability decays at late times as a power law with an exponent θ that depends, in general, on p: $Q(t) \sim t^{-\theta(p)}$. Explicit analytical results for $\theta(p)$ have been obtained for a number of simple (and not-so-simple) models [7]. We also showed that the function $\theta(p)$ determines the moments of the number of crossings of the boundary $x = 0$ for the case where there is no absorption. In the present work we use the methods of ref. [3] to determine the function $\theta(p)$ and the method outlined in ref. [7] to investigate the crossing statistics for the problem without absorption, obtaining closed form results for the mean and variance of the number of crossings in a given (large) time interval. These results for the mean and variance are, to our knowledge, the first such results for non-Gaussian processes.

The derivation of $\theta(p)$ follows closely that of $\theta(0)$ in Ref. [3], so we only give the main steps. The principal difference from the former work lies in the boundary conditions imposed by the partially absorbing boundary.

From Eqs. (1) and (2) we can write down the backward Fokker-Planck equation

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial y^2} \pm v_\pm(\pm y)^\alpha \frac{\partial Q}{\partial x} , \quad (5)$$

where $Q(x, y, t)$ is the probability that the particle still survives at time t given that it started at position (x, y). The partially absorbing boundary at $x = 0$ implies the boundary conditions

$$Q(0+, -y, t) = p\,\tilde{Q}(0+, y, t), \quad y > 0 \quad (6)$$
$$\tilde{Q}(0+, -y, t) = p\,Q(0+, y, t), \quad y > 0, \quad (7)$$

where $\tilde{Q}(x,y,t)$ is the survival probability for a model in which v_+ and v_- are interchanged. It is clear that Q and $\tilde{Q}$ are described by the same value of θ. In fact $\tilde{Q}(x,y,t) = Q(-x,-y,t)$, since after interchanging v_+ and v_- the system is restored to its original configuration after a rotation by π about an axis perpendicular to the xy plane. The initial condition is $Q(x,y,0) = 1 = \tilde{Q}(x,y,0)$.

Solving the full initial value problem is difficult so we will follow the approach used in ref. [3] by specializing to the late-time scaling regime where $Q(x,y,t) \sim t^{-\theta(p)}$. This approach exploits a generalization of the method introduced by Burkhardt [8] for $\alpha = 1 = \gamma$ (the random acceleration problem). The idea is to extract explicitly the time-dependence $t^{-\theta}$ expected at large t. This gives, asymptotically,

$$Q(x,y,t) \sim \left(\frac{x^{2\beta}}{t}\right)^{\theta} F_{\pm}\left(\pm\frac{v_{\pm}(\pm y)^{1/\beta}}{Dx}\right) \ , \qquad (8)$$

and a similar expression for $\tilde{Q}(x,y,t)$, with scaling functions $\tilde{F}_{\pm}$. In Eq. (8), the functions $F_{\pm}(z)$ are the scaling function for $y > 0$ (+) and $y < 0$ (−) respectively. The (dimensional) prefactors (for $y > 0$ and $y < 0$) in Eq. (8) have been omitted since Eq. (5) is linear. The functions $F_+(z)$ and $F_-(z)$ are defined such that the prefactor is the same for $y > 0$ and $y < 0$.

Inserting the form (8) into the backward Fokker-Planck equation (5), we see immediately that the term $\partial Q/\partial t$ leads to a term of order $t^{-(\theta+1)}$, which is subdominant for large t and can therefore be dropped. The remaining terms give

$$zF''_{\pm}(z) + (1-\beta-\beta^2 z)F'_{\pm}(z) + 2\beta^3\theta F_{\pm}(z) = 0 \ . \qquad (9)$$

Expressed in terms of the variable $u = \beta^2 z$, this equation becomes Kummer's equation. Independent solutions are the confluent hypergeometric functions $M(-2\beta\theta, 1-\beta, \beta^2 z)$ and $U(-2\beta\theta, 1-\beta, \beta^2 z)$ [9]. The function $M(-2\beta\theta, 1-\beta, \beta^2 z)$ diverges exponentially for $z \to \infty$, so must be rejected for $y > 0$. Thus we write

$$F_+(z) = A\,U\left(-2\beta\theta, 1-\beta, \frac{v_+\beta^2 y^{1/\beta}}{Dx}\right) \qquad (10)$$

$$F_-(z) = B\,U\left(-2\beta\theta, 1-\beta, \frac{-v_-\beta^2(-y)^{1/\beta}}{Dx}\right) + C\,M\left(-2\beta\theta, 1-\beta, \frac{-v_-\beta^2(-y)^{1/\beta}}{Dx}\right) \qquad (11)$$

with similar equations for $\tilde{F}_{\pm}(z)$ involving amplitudes $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$.

Relations between the coefficients A, B, C, and the corresponding tilded variables can be obtained by imposing the boundary conditions (6) and (7), and requiring continuity of Q, $\partial Q/\partial y$, $\tilde{Q}$ and $\partial\tilde{Q}/\partial y$ at $y = 0$. After some straightforward but lengthy algebra, these boundary and continuity conditions eventually lead to a consistency condition on θ, which determines $\theta = \theta(p)$ as

$$\theta(p) = \frac{1}{4} - \frac{1}{2\pi\beta}\sin^{-1}\left(\sqrt{\delta}\sin\left(\frac{\pi\beta}{2}\right)\right), \qquad (12)$$

where

$$\delta = \frac{2p^2\cos^2\left(\frac{\pi\beta}{2}\right) + 2\sinh^2\left(\frac{1}{2}\ln\gamma\right)}{\cos(\pi\beta) + \cosh(\ln\gamma)} \qquad (13)$$

and we recall that β, γ are defined by Eq. (4).

The general result, Eq. (12) can be checked in a number of special cases:

(i) $p = 1$. In this case $\delta = 1$ and $\theta = 0$. This is clearly correct since for $p = 1$ there is no absorbtion and $Q(x,y,t) = 1$.

(ii) $p = 0$. For this case, $\delta = (\sqrt{\gamma} - 1/\sqrt{\gamma})^2/[\gamma + 1/\gamma + 2\cos(\pi\beta)]$. Inserting this into (12), and carrying out some elementary manipulations, one recovers Eq. (3), as required.

(iii) $\gamma = 1$. This gives $\delta = p^2$, and

$$\theta = \frac{1}{4} - \frac{1}{2\pi\beta}\sin^{-1}\left(p\sin\left(\frac{\pi\beta}{2}\right)\right). \qquad (14)$$

The special case, $\beta = 1/3$, of this result corresponds to the random acceleration problem ($\alpha = 1$) with partial absorption, and gives $\theta = 1/4 - (3/2\pi)\sin^{-1}(p/2)$, recovering the results of Burkhardt [8] and De Smedt et al.[10] for this special case. In the following, we exploit the general result (12) to obtain results for the crossing statistics of the process defined by Eqs. (1) and (2).

In order to place the discussion of crossing statistics for *non-Gaussian* processes in the proper perspective, we begin with a short discussion of crossing statistics for *Gaussian* processes. Within the class of models discussed in this paper, only the case $\alpha = 1 = \gamma$, corresponding to the random acceleration process, is Gaussian.

For illustrative purposes, we begin with the random acceleration process, $\dot{x} = y$, $\dot{y} = \eta(t)$ or, equivalently, $\ddot{x} = \eta(t)$, with $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$. For convenience we take the initial condition $x(0) = 0 = \dot{x}(0)$. Then $x(t) = \int_0^t dt' \int_0^{t'} dt''\ \eta(t'')$, and the two-time correlator is $C(t_1,t_2) =< x(t_1)(x(t_2) >= t_2 t_1^2/2 - t_1^3/6$, where we have taken $t_2 \geq t_1$ without loss of generality. The normalized two-time correlator is

$$\tilde{C}(t_1,t_2) = \frac{C(t_1,t_2)}{\sqrt{C(t_1,t_1)C(t_2,t_2)}} = \frac{3}{2}\left(\frac{t_1}{t_2}\right)^{1/2} - \frac{1}{2}\left(\frac{t_1}{t_2}\right)^{3/2}, \qquad (15)$$

for $t_2 \geq t_1$. Notice that $\tilde{C}(t_1, t_2)$ depends on t_1, t_2 only through the *ratio* t_1/t_2. This is an immediate consequence of the fact that, with the initial condition $x(0) = 0 = \dot{x}(0)$, there is no timescale in the problem, so dimensionless correlation functions can only depend on ratios of times. If the initial position and velocity are non-zero, the scaling form (15) is recovered in the limit that t_1 and t_2 are both taken large with the ratio held fixed. It should be noted that this scaling property of temporal correlations is not restricted to Gaussian processes, but holds for all the models discussed here.

This scaling property implies that, if one introduces a logarithmic time variable [11, 12], $T = \ln t$, normalized correlation functions can only depend on differences of T-variables, i.e. the processes become *stationary* in logarithmic time. Again, this applies to both the Gaussian and the non-Gaussian processes that we consider. For the random acceleration process, it follows immediately from (15) that the normalized correlator in logarithmic time, $f(T) = \langle X(T)\, X(0)\rangle$ where $X(T) = x(t)/\sqrt{\langle x^2(t)}$, is

$$f(T) = (3/2)\exp(-T/2) - (1/2)\exp(-3T/2) \qquad (16)$$

for $T \geq 0$ (and $f(-T) = f(T)$).

For Gaussian processes there is a major simplification: the two-time correlator implicitly determines all properties of the system, including the first-passage exponent θ and the crossing statistics. As an example, consider the mean density, ρ, of zero crossings in logarithmic time. The mean density ρ is finite only for *smooth* Gaussian processes whose correlator $f(T)$ has the short time behavior, $f(T) = 1 - aT^2$ as $T \to 0$ where $a = -f''(0)/2$ is finite. For such processes, the mean number of zero-crossings of the process $X(T)$ in time interval T is

$$\langle n\rangle = \int_0^T dT' \langle \delta(X(T'))|\dot{X}(T')|\rangle = T\langle \delta(X(T'))\rangle\, \langle|\dot{X}(T')|\rangle \qquad (17)$$

since $X(T)$ and $\dot{X}(T)$ are uncorrelated for smooth processes, $\langle X(T)\dot{X}(T)\rangle = f'(0) = 0$. Furthermore, X and $\dot{X}$ are normally distributed with variances $f(0) = 1$ and $-f''(0)$ respectively. It follows that the mean crossing density is given by

$$\rho = \langle n\rangle/T = \frac{1}{\pi}\sqrt{-f''(0)}, \qquad (18)$$

a result first derived by Rice [13].

In a similar way it is possible to obtain an (albeit rather more complicated) expression for the variance, $\langle n^2\rangle - \langle n\rangle^2$, of the number of crossings in time T, again expressed in terms of the correlator $f(T)$, a result due to Bendat [14].

For non-Gaussian processes, this approach no longer works. The first line, $\langle n\rangle = T\langle \delta(X)\,|\dot{X}|\rangle$ still holds, but since the stationary distribution of $(X, \dot{X})$ is in general not known, further progress seems to be impossible. We will show, however, that exact results can be obtained for the class of non-Gaussian models discussed in this paper by exploiting our knowledge of the function $\theta(p)$. The connection between the two was first noted in ref.[7].

Working in logarithmic time, as discussed above, we can write the survival probability $Q(T)$ for the partial survival problem in the form

$$Q(T) = \sum_{n=0}^{\infty} p^n P_n(T) \qquad (19)$$

where $P_n(T)$ is the probability of the process crossing the line $x = 0$ n times in the (logarithmic) time interval T, and p^n is the probability of surviving all n crossings. Now $Q(T)$ decays asymptotically as $Q(T) \sim \exp(-\theta(p)T)$, and the right-hand side of Eq. (19) can be written in terms of the cumulants of n, to give

$$\exp\left(\sum_{r=0}^{\infty} \frac{(\ln p)^r}{r!}\langle n^r\rangle_c\right) \sim \exp\left(-\theta(p)\, T\right), \qquad (20)$$

for large T, where $\langle n^r\rangle_c$ is the $r^{\rm th}$ cumulant of n, and therefore

$$\sum_{r=0}^{\infty} \frac{(\ln p)^r}{r!}\langle n^r\rangle_c \sim -\theta(p)\, T, \qquad (21)$$

for large T.

The cumulants of n can now be determined by expanding both sides of Eq. (21) around $p = 1$, by writing $p = 1 - \epsilon$ and equating coefficients of powers of ϵ, to obtain the cumulants of n in terms of the derivatives of $\theta(p)$ evaluated at $p = 1$. In this way one obtains

$$\langle n\rangle = -\theta'(1)T \qquad (22)$$

$$\langle n^2\rangle_c - \langle n\rangle = -\theta''(1)T \qquad (23)$$

etc. This approach gives, for the mean crossing density,

$$\rho = \frac{\langle n\rangle}{T} = \frac{1}{2\pi\beta}\,\frac{\sin(\pi\beta)}{\cos(\pi\beta) + \cosh(\ln\gamma)}. \qquad (24)$$

For the special case $\beta = 1/3$, $\gamma = 1$ that corresponds to the (Gaussian) random acceleration process, we can check Eq. (24) against the general result (18) that holds for any Gaussian stationary process. We find that both expressions reduce to $\rho = \sqrt{3}/2\pi$ for this case.

In a similar way we can calculate the second cumulant of the number of crossings by expanding to second order in $1 - p$. The result for $\langle n^2\rangle_c \equiv \langle n^2\rangle - \langle n\rangle^2$ is

$$\frac{\langle n^2\rangle_c}{T} = \frac{1}{2\pi\beta}\,[2F\sin(\pi\beta) - F^2\sin(2\pi\beta)], \qquad (25)$$

where

$$F = 1/[\cos(\pi\beta) + \cosh(\ln\gamma)]. \qquad (26)$$

Again, this result can be checked for the random acceleration process $\beta = 1/3$, $\gamma = 1$, which is Gaussian. For this case, Eq. (25) gives $\langle n^2 \rangle_c / T = 2/\pi\sqrt{3}$. On the other hand, the second cumulant can be calculated exactly for any Gaussian process using Bendat's formula. This gives the result $2/\pi\sqrt{3}$, as expected.

It is noteworthy that for the class of models discussed here, arbitrary cumulants of the crossing number are readily obtained by simply computing the appropriate derivatives of $\theta(p)$, evaluated at $p = 1$. Thus a knowledge of $\theta(p)$ provides a powerful tool. Even for Gaussian processes, a general expression for the moments of n above the second is not available for models where $\theta(p)$ is not known, which is essentially all but a few special models [7].

It is also interesting to compute the mean time intervals (logarithmic scale) $l_\pm$ between crossings that the particle spends on the positive (negative) side of the X axis. For the special case $\gamma = 1$ (where $v_+ = v_-$), it is clear that $l_+ = l_- = 1/\rho$ with ρ being the mean density of crossings given by Eq. (24). However, for arbitrary $\gamma \neq 1$, l_+ is, in general, different from l_-. Let $\rho_\pm$ denote the mean number of crossings of $X = 0$ from the right (left) of the Y axis. Clearly $\rho_+ = \rho_- = \rho/2$. To calculate the mean intervals $l_\pm$ in the general case ($\gamma \neq 1$) one can proceed as follows. We consider the normalized process $X(\tau)$ in the logarithmic time scale τ so that it is stationary. Let us first define a new variable, the 'occupation time', that measures the fraction of time spent by the process $X(\tau)$ above (below) the X axis, $L_\pm = \frac{1}{T}\int_0^T \theta(\pm X(\tau))\, d\tau$. Taking the average, and using the stationarity, one gets $\langle L_\pm \rangle = \langle \theta(\pm X) \rangle$. However, it is evident that $\langle L_\pm \rangle = \rho_\pm l_\pm$. Hence we get an expression for the mean intervals

$$l_\pm = \frac{1}{\rho_\pm} \langle \theta(\pm X) \rangle. \tag{27}$$

For the case where one has the symmetry $X \to -X$, such as the case $\gamma = 1$, one gets, using $\langle \theta(X) \rangle = 1/2$, the expected result $l_+ = l_- = 1/\rho$. However, for $\gamma \neq 1$, the exact knowledge of ρ from Eq. (24) is not enough to calculate the mean size of intervals $l_\pm$. One needs to compute, in addition, the quantity $\langle \theta(\pm X) \rangle$. To calculate this average we need to know the stationary probability density $P(X)$, since $\langle \theta(X) \rangle = \int_0^\infty P(X)\, dX$.

The calculation of the probability density $P(X)$, for $\gamma \neq 1$, is nontrivial. The only case where we have succeeded in calculating $P(X)$ exactly is the case where $\alpha = 0$. In this case, the equation of motion $x(t)$ in the original time t reads, from Eq. (2), $\dot{x} = v_+\theta(y) - v_-\theta(-y)$. Then one can write, $x(t) = \epsilon t + v T_t$ where $T_t = \int_0^t \mathrm{sign}[y(t')]\, dt'$ is the sign-time of an ordinary Brownian motion, $\epsilon = (v_+ - v_-)/2$ measures the anisotropy and $v = (v_+ + v_-)/2$. The distribution of T_t for Brownian motion is well known to have the famous arcsine form of Lévy [15], $P(T_t, t) = \frac{1}{t} f\left(\frac{T_t}{t}\right)$ where $f(x) = 1/[\pi\sqrt{1-x^2}]$ for $-1 \leq x \leq 1$ and $f(x) = 0$ outside. Using this result, one gets the exact distribution of $x(t)$, $P(x,t) = \frac{1}{vt} G\left(\frac{x}{vt}\right)$ where $G(z) = 1/[\pi\sqrt{1-(z-\epsilon)^2}]$ for $-1 + \epsilon/v \leq z \leq 1 + \epsilon/v$ and $G(z) = 0$ otherwise. Carrying out the integral of $P(x,t)$ only over the positive (negative) x axis, one gets $\langle \theta(\pm x(t)) \rangle = \frac{1}{2} \pm \frac{1}{\pi}\sin^{-1}\left(\frac{\epsilon}{v}\right)$. Using $\theta(X) = \theta(x)$ and $\rho = 1/[\pi \cosh(\ln\gamma)]$ (obtained by putting $\beta = 1/2$ in Eq. (24)) we finally get the exact mean intervals for the $\alpha = 0$ case:

$$l_\pm = 2\pi \cosh(\ln\gamma) \left[\frac{1}{2} \pm \frac{1}{\pi} \sin^{-1}\left(\frac{v_+ - v_-}{v_+ + v_-} \right) \right]. \tag{28}$$

The determination of $l_\pm$ for other values of α (including the $\alpha = 1$ case) remains an open problem.

In this paper we have calculated the partial survival exponent, $\theta(p)$, for a class of (in general non-Gaussian) stochastic processes describing a random walker moving in a transverse "shear" flow. We have then used the result for $\theta(p)$ to derive exact expressions for the first two cumulants of the crossing number (number of crossings of the line $x = 0$) working on a logarithmic timescale where the process is stationary. To our knowledge these are the first results for the statistics of crossing numbers for any non-Gaussian process. We have checked that our general result reduces, in special cases, to the known results for the random acceleration process. We have also computed exactly the mean time intervals between successive crossings of the y axis for the special case $\alpha = 0$. The calculation of the mean time intervals for $\alpha \neq 0$ remains a challenging open problem.

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