Precise Asymptotics for a Random Walker's Maximum

Alain Comtet [1,2] and Satya N. Majumdar [1]
[1]Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud. Bât. 100. 91405 Orsay Cedex. France
[2]Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75005 Paris, France

We consider a discrete time random walk in one dimension. At each time step the walker jumps by a random distance, independent from step to step, drawn from an arbitrary symmetric density function. We show that the expected positive maximum $E[M_n]$ of the walk up to n steps behaves asymptotically for large n as, $E[M_n]/\sigma = \sqrt{2n/\pi} + \gamma + O(n^{-1/2})$, where σ^2 is the variance of the step lengths. While the leading $\sqrt{n}$ behavior is universal and easy to derive, the leading correction term turns out to be a nontrivial constant γ. For the special case of uniform distribution over $[-1, 1]$, Coffmann et. al. recently computed $\gamma = -0.516068\ldots$ by exactly enumerating a lengthy double series. Here we present a closed exact formula for γ valid for arbitrary symmetric distributions. We also demonstrate how γ appears in the thermodynamic limit as the leading behavior of the difference variable $E[M_n] - E[|x_n|]$ where x_n is the position of the walker after n steps. An application of these results to the equilibrium thermodynamics of a Rouse polymer chain is pointed out. We also generalize our results to Lévy walks.

PACS numbers: 02.50.-r, 89.75.Hc, 89.20.Ff

I. INTRODUCTION

Brownian motion is perhaps one of the most widely studied subjects in classical physics. A Brownian walker moves in continuous space and time which leads to a great simplification: one can write down a simple differential equation, the famous diffusion equation, that governs the time development of the probability density of the position of the walker. Subsequently, many other involved properties of the Brownian motion, such as its first-passage probability through a given point, the distribution of the maximum displacement of the walker up to a given time etc. can be calculated analytically relatively easily [1]. In contrast, the related problem of a random walker that hops only at discrete time steps in a continuous space is not so straightforward, even in one dimension [2,3]. A classic example of such a walk can be found in bacterial chemotaxis, where a bacteria, in search of food, jumps from one position to another at discrete time steps [4]. Another famous example of such a walk occurs in the Rouse model of a polymer chain that consists of monomers or beads connected by harmonic springs [5]. Many other examples can be found in Refs. [6–8]. While the asymptotic properties of this discrete hopper, after a sufficiently large number of steps and given that the variance in step sizes is finite, are correctly described by the continuous time diffusion equation [8], there are many interesting finite size effects that can not be captured by the diffusion equation. The difficulty that arises in dealing with a finite number of steps is due to the fact that the probability density of the random hopper usually satisfies an integral equation which is technically much harder to solve than a differential equation. In this paper, we study analytically one such finite size effect, namely the behavior of the expected maximum position of a discrete time hopper in one dimension. We will show that even this relatively simple problem has rather interesting finite size behavior.

We consider a discrete time random walker moving on a continuous line. The position x_n of the walker after n steps evolves for $n \geq 1$ via,

$$x_n = x_{n-1} + \xi_n \tag{1}$$

starting at $x_0 = 0$, where the step lengths ξ_n's are independent and identically distributed (i.i.d) random variables with zero mean and each drawn from the same probability distribution, $\text{Prob}(\xi_n \leq x) = \int_{-\infty}^{x} f(y)\, dy$, $f(x)$ being the normalized symmetric probability density. Let M_n denote the positive maximum of the random walk up to n steps,

$$M_n = \max(0, x_1, x_2, \ldots, x_n). \tag{2}$$

We are interested in the asymptotic large n behavior of the expected maximum $E(M_n)$.

This question recently arose in the context of a packing problem in two dimensions where n rectangles of variable sizes are packed in a semi-infinite strip of width one [9,10]. It was shown in Ref. [10] that for the special case of the uniform jump distribution, $f(x) = 1/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$ outside, for large n,

$$E[M_n] = \sqrt{\frac{2n}{3\pi}} - 0.297952\ldots + O(n^{-1/2}). \tag{3}$$

The leading $\sqrt{n}$ behavior is easy to understand and can be derived from the corresponding behavior of a continuous time Brownian motion after a suitable rescaling [10]. However, the leading finite size correction term turns out to be a nontrivial constant $c = -0.297952\ldots$ that was computed in Ref. [10] by enumerating a somewhat awkward double series obtained after a lengthy calculation. This constant c characterizing the leading finite size behavior is nonuniversal and is expected to depend on the details of the probability density $f(x)$ of the noise. A natural question is: can one calculate this constant for arbitrary density function $f(x)$? In this paper we provide an exact formula for this constant c valid for arbitrary symmetric $f(x)$.

Our results are twofold. First we consider the class of density function $f(x)$ that has a finite second moment, $\sigma^2 = \int_{\infty}^{\infty} x^2 f(x)dx$. Then σ denotes the characteristic length of a single jump. Since, $E[M_n]$ has the dimension of length, it is preferable to consider the dimensionless variable $E[M_n]/\sigma$. We show that for large n

$$\frac{E(M_n)}{\sigma} = \sqrt{\frac{2n}{\pi}} + \gamma + O\left(\frac{1}{\sqrt{n}}\right). \tag{4}$$

The leading $\sqrt{n}$ behavior is universal (does not depend on the details of the density function $f(x)$) and easy to compute by appropriately rescaling the continuous time Brownian result. Our main new result is to obtain an exact expression for the nonuniversal constant γ. Our result is best expressed in terms of the characteristic function,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx. \tag{5}$$

For density functions with a finite second moment, i.e., when $\hat{f}(k) = 1 - \sigma^2 k^2/2 + O(k^4)$ as $k \to 0$, we show that

$$\gamma = \frac{1}{\pi\sqrt{2}} \int_0^{\infty} \frac{dk}{k^2} \ln\left[\frac{1 - \hat{f}\left(\frac{\sqrt{2}}{\sigma}k\right)}{k^2}\right]. \tag{6}$$

We prove in appendix-A that $\gamma < 0$ for arbitrary $f(x)$, a fact not apriori obvious. Let us quote a few examples where the integral in Eq. (6) can be performed explicitly,

$$f(x) = \frac{1}{2}[\delta(x+1) + \delta(x-1)] \;\Rightarrow\; \gamma = -1/2 = -0.5 \tag{7}$$

$$f(x) = \frac{1}{2}e^{-|x|} \;\Rightarrow\; \gamma = -1/\sqrt{2} = -0.70710\ldots \tag{8}$$

$$f(x) = \frac{a^2}{2}|x|\, e^{-a|x|} \;\Rightarrow\; \gamma = -(2\sqrt{3}-1)/\sqrt{6} = -1.00597\ldots \tag{9}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \;\Rightarrow\; \gamma = \zeta(1/2)/\sqrt{2\pi} = -0.58259\ldots. \tag{10}$$

where $\zeta(z)$ is the Riemann zeta function (analytically continued for $z < 1$). For the uniform distribution over $[-1, 1]$, our exact formula in Eq. (6) reproduces very simply the result obtained in Ref. [10]. In this case, using $\hat{f}(k) = \frac{\sin k}{k}$ and $\sigma = 1/\sqrt{3}$ in Eq. (6) one gets

$$\gamma = \frac{1}{\pi\sqrt{2}} \int_0^{\infty} \frac{dk}{k^2} \ln\left[\frac{1 - \frac{\sin(\sqrt{6}k)}{\sqrt{6}k}}{k^2}\right] = -0.516068\ldots \tag{11}$$

where the integral was performed using Mathematica. Note that the constant $c = -0.297952\ldots$ in Eq. (3) is simply $c = \gamma\sigma$ with $\sigma = 1/\sqrt{3}$ for the uniform distribution and $\gamma = -0.516068\ldots$ given in Eq. (11).

We have also generalized our results to the case of Lévy flights where the second moment diverges and one has $\hat{f}(k) = 1 - |ak|^{\mu} + O(k^2)$ as $k \to 0$ with $1 < \mu \leq 2$ [11]. Note that a has the dimension of length. The probability density $f(x)$ of the step lengths has an algebraic tail for large $|x|$, $f(x) \sim |x|^{-1-\mu}$. In this case we show that for large n, the dimensionless expected maximum behaves as

$$\frac{E(M_n)}{a} = \frac{\mu}{\pi}\Gamma\left(1 - \frac{1}{\mu}\right) n^{1/\mu} + \gamma + O(n^{1/\mu - 1}). \tag{12}$$

The leading term is again shown to be universal. We show that the leading finite size correction term is again a constant given by

$$\gamma = \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln \left[\frac{1 - \hat{f}\left(\frac{k}{a}\right)}{k^\mu} \right]. \tag{13}$$

For example, for the case $\hat{f}(k) = \exp[-|ak|^\mu]$ with $1 < \mu \leq 2$, we obtain

$$\gamma = \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln \left[\frac{1 - e^{-k^\mu}}{k^\mu} \right] = \frac{\zeta(1/\mu)}{(2\pi)^{1/\mu} \sin(\pi/2\mu)}. \tag{14}$$

The evaluation of the integral is presented in appendix-B.

An interesting fact about the constant γ is that even though it characterises the leading finite size correction to the expected maximum, it actually shows up even in the thermodynamic limit $n \to \infty$ provided one looks at a behavior of a suitably defined quantity as follows. Let $|x_n|$ denote the absolute value of the position of the walker after n steps. The distribution of x_n, for arbitrary density $f(x)$, can be computed relatively easily (see section-IV) and hence one can calculate $E[|x_n|]$. We focus here on the case when the variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx$ as well as the fourth moment $\mu_4 = \int_{-\infty}^{\infty} x^4 f(x) dx$ of the jump distribution are finite. In that case, one can show that for large n,

$$\frac{E[|x_n|]}{\sigma} = \sqrt{\frac{2n}{\pi}} - \frac{1}{12\sqrt{2\pi}} \left(\frac{\mu_4}{\sigma^4} - 3 \right) \frac{1}{\sqrt{n}} + O(n^{-3/2}). \tag{15}$$

Thus the leading term of $E[|x_n|]/\sigma$ for large n is exactly the same as that of the expected maximum $E[M_n]/\sigma$ in Eq. (4). However, unlike in the case of the maximum in Eq. (4), the leading finite size correction term in Eq. (15) is of $O(n^{-1/2})$ and not a constant. Using Eqs. (4) and (15) one then gets

$$\frac{E[M_n] - E[|x_n|]}{\sigma} = \gamma + O(n^{-1/2}). \tag{16}$$

Thus the difference between the expected positive maximum up to n steps and the absolute value of the expected final position of the walker after n steps, in units of σ, tends to a negative constant γ in the thermodynamic limit $n \to \infty$, a fact that is not apriori obvious.

We end this section by mentioning a simple physical application of the results above. Let us consider the simplest model of a polymer chain namely the Rouse model [5] where the monomers are connected by harmonic springs. A configuration of the chain consisting of n monomers is specified by the position vectors $\{\vec{r}_i\}$ of the monomers with $i = 0, 1, 2 \ldots, n$. We assume that one end of the chain is grafted at the origin, $\vec{r}_0 = 0$ while the other end is free. We assume that the chain is at thermal equilibrium so that the probability of any given chain configuration is given by its Boltzmann weight,

$$P[\{\vec{r}_i\}] = \frac{1}{Z_n} \exp\left[-\frac{\beta\kappa}{2} \sum_{i=1}^{n} (\vec{r}_i - \vec{r}_{i-1})^2 \right], \tag{17}$$

where Z_n is the partition function, $\beta = 1/k_B T$ is the inverse temperature and the spring constant κ characterises the harmonic coupling between neighbouring monomers. Let us now look at the components of the position vectors along any particular direction, say $\{x_i\}$, which can also be thought of as a one dimensional Rouse chain. The equilibrium weight in Eq. (17) indicates that the difference in position between the i-th and $(i-1)$-th monomer can be represented by a noise

$$x_i = x_{i-1} + \xi_i, \tag{18}$$

where ξ_i's are independent and Gaussian distributed , $f(\xi) = e^{-\xi^2/2\sigma^2}/\sigma\sqrt{2\pi}$ where $\sigma^2 = 1/\beta\kappa$. Thus M_n in Eq. (2) refers to the maximum displacement of the polymer chain along x direction and x_n denotes the x coordinate of the end point of the chain (see Fig. 1).

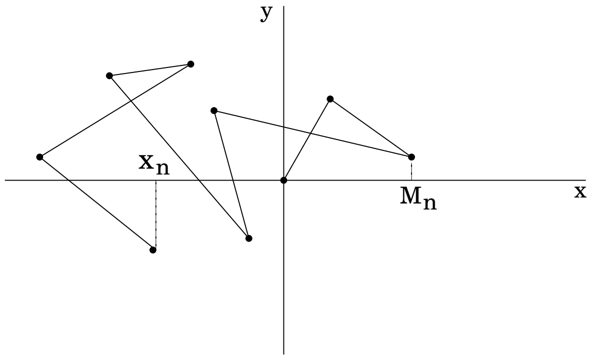

FIG. 1. A typical configuration of the Rouse chain in 2-dimensions. M_n denotes the positive maximum along the x direction and x_n denotes the x co-ordinate of the position of the end point.

Thus, using the result in Eq. (10) in Eq. (16) we find that in the limit of a very long chain ($n \to \infty$) at thermal equilibrium, the difference between the expected maximum displacement of the chain along a particular direction (say the x direction) and the absolute end to end displacement along the same direction tends to a nontrivial constant

$$\frac{E[M_n] - E[|x_n|]}{\sigma} \to \frac{\zeta(1/2)}{\sqrt{2\pi}} = -0.58259\ldots. \tag{19}$$

Note that in the context of the Rouse chain, the expectation E means a thermal equilibrium average over the Bolzmann weight in Eq. (17). The facts that (a) the difference approaches a constant and (b) that too, a negative constant, are not apriori obvious for the Rouse chain.

The rest of the paper is organized as follows. In Section II, we set up the basic integral equation for the distribution of the maximum and provide an exact solution in Section II-A for the special case when the jump density is exponential. Section III deals with the general jump distribution where we present the Pollaczek-Spitzer formula and extract the finite size correction term exactly from an asymptotic expansion of this formula. These results are generalized to Lévy processses in Section III-B. In Section-IV we calculate the expected value of the absolute position of the end point and demonstrate how the constant γ shows up in the thermodynamic limit. We conclude with a summary and open problems. Explicit derivations of some of the formulae and integrations are relegated to the two appendices.

II. AN INTEGRAL EQUATION FOR THE DISTRIBUTION OF THE MAXIMUM

In this section we set up an integral equation satisfied by the distribution of the maximum of a random walk for arbitrary symmetric jump distribution. We consider a random walk starting at $x_0 = 0$ at step $n = 0$ and evolving via Eq. (1) where the noise ξ_n's are i.i.d. variables drawn from the common symmetric distribution Prob$(\xi_n \le x) = \int_{-\infty}^{x} f(y)\, dy$. We would like to compute the distribution of the maximum $M_n = \max(0, x_1, x_2, \ldots, x_n)$ up to n steps, i.e., Prob$[M_n \le y]$. To derive this, we first define $Q_n(x, y)$ as the probability that, starting at $x_0 = x$, the maximum of the walk up to n steps is less than or equal to y. Evidently, Prob$[M_n \le y] = Q_n(0, y)$. Consider the first step where the particle jumps by an amount $x_1 - x$ which occurs with a probability density $f(x_1 - x)$ (see Fig. 2).

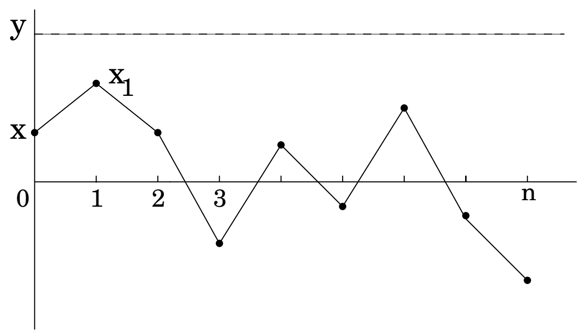

FIG. 2. A random walker, starting at x at $n = 0$, makes a flight to $x_1 \le y$ at step $n = 1$.

It follows, using the Markov property of the walk, that $Q_n(x,y)$ satisfies the following recursion relation,

$$Q_n(x,y) = \int_{-\infty}^{y} Q_{n-1}(x_1,y)\, f(x_1 - x)\, dx_1 \tag{20}$$

with the initial condition $Q_0(x,y) = \theta(y-x)$ where $\theta(z)$ is the Heaviside step function. Due to the translational invariance, it is also clear that $Q_n(x,y)$ depends only on the difference $z = y - x$, i.e., $Q_n(x,y) = q_n(z = y - x)$ where $z \geq 0$. Making the change of variable, $z' = y - x_1$ and using the translation invariance, the recursion in Eq. (20) becomes simpler,

$$q_n(z) = \int_0^{\infty} q_{n-1}(z') f(z - z')\, dz', \tag{21}$$

valid for all $z \geq 0$ and starting with $q_0(z) = \theta(z)$. Thus, if one finds the solution $q_n(z)$ of Eq. (21), then the distribution of the maximum is just $\mathrm{Prob}[M_n \leq y] = Q_n(0,y) = q_n(y)$. The density of the maximum is $q_n'(y) = dq_n/dy$. Hence, the expected maximum is $E[M_n] = \int_0^{\infty} q_n'(y) y\, dy$, the quantity we are after.

The generating function, $\tilde{q}(z,s) = \sum_{n=1}^{\infty} q_n(z) s^n$ then satisfies an integral equation

$$\tilde{q}(z,s) = s \int_0^{\infty} \tilde{q}(z',s) f(z-z') dz' + s \int_0^{\infty} f(z-z') dz', \tag{22}$$

valid for all $z \geq 0$. This integral equation is an inhomogeneous Wiener-Hopf equation [12] and in general, for arbitrary kernel $f(z-z')$, it is very hard to solve this integral equation. The main source of difficulty is the fact that the limits of the integral on the right hand side of Eq. (22) are 0 and ∞, as opposed to say $-\infty$ and ∞ [12,2]. However, when the kernel $f(z-z')$ is a normalized probability density function we may use the Pollaczek-Spitzer formula [22,21], to which we will come back to in section III. But, before that, it is instructive to solve Eq. (22) explicitly for special cases, whenever possible. In Section II-A, we solve Eq. (22) explicitly for the exponential density function.

A. Exponential density function: An exactly solvable case

For the exponential density functon, $f(z) = \frac{1}{2} e^{-|z|}$, one can obtain an exact solution of Eq. (22). We first assume that the integral Eq. (22) is valid for all $-\infty \leq z \leq \infty$, though we are interested only in the solution for $z \geq 0$. Next, we note the identity, $f''(z) = f(z) - \delta(z)$, where $f''(z) = d^2 f/dz^2$. Differenting twice Eq. (22) and using the above identity one readily converts the integral equation into the following differential equation,

$$\frac{d^2 \tilde{q}}{dz^2} = [1 - s\theta(z)]\tilde{q} - s\theta(z). \tag{23}$$

For $z \geq 0$, the general solution is readily obtained,

$$\tilde{q}(z,s) = \frac{s}{1-s} + A(s) e^{-\sqrt{1-s}\, z} + B(s) e^{\sqrt{1-s}\, z}, \tag{24}$$

where $A(s)$ and $B(s)$ are two arbitrary constants (independent of z). Now, in the limit $z \to \infty$, $q_n(z) \to 1$ for all n, since the probability that the particle, starting at $z \to \infty$, will not cross 0 up to any finite step n is 1. Thus, one expects that as $z \to \infty$, $\tilde{q}(z,s) \to s/(1-s)$. Using this boundary condition in Eq. (24), one gets $B(s) = 0$. The other constant $A(s)$ will be fixed by the matching conditions at $z = 0$.

Now, for $z \leq 0$, the solution of Eq. (23) is given by

$$\tilde{q}(z,s) = C(s)\, e^{z}, \tag{25}$$

where we have used the boundary condition, $\tilde{q}(z \to -\infty, s) \to 0$. Now, we are ready to match the solution in Eq. (24) for $z \geq 0$ with that in Eq. (25) for $z \leq 0$. The continuity of the solution at $z = 0$ and also the continuity of the first derivative at $z = 0$ fixes the two constants $A(s)$ and $C(s)$ uniquely. We get,

$$A(s) = -\frac{1 - \sqrt{1-s}}{1-s}; \qquad C(s) = \frac{1 - \sqrt{1-s}}{\sqrt{1-s}}. \tag{26}$$

Thus, for $z > 0$, the exact solution of Eq. (22) is given by

$$\tilde{q}(z,s) = \frac{s}{1-s} - \frac{1-\sqrt{1-s}}{1-s}\, e^{-\sqrt{1-s}\,z}. \tag{27}$$

The probability density for the maximum then has the generating function,

$$\sum_{n=1}^{\infty} q_n'(z) s^n = \frac{1-\sqrt{1-s}}{\sqrt{1-s}} e^{-\sqrt{1-s}\,z}. \tag{28}$$

Subsequently, the generating function for the expected maximum is given by

$$\sum_{n=1}^{\infty} E[M_n] s^n = \sum_{n=1}^{\infty} s^n \int_0^{\infty} q_n'(z) z dz = \frac{1}{(1-s)^{3/2}} - \frac{1}{1-s}. \tag{29}$$

Expanding the right hand side of Eq. (29) in powers of n, we get

$$E[M_n] = -1 + \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+3/2)}{\Gamma(n+1)}, \tag{30}$$

where $\Gamma(x)$ is the Gamma function. Note that the result in Eq. (30) is valid for all $n \geq 0$. The variance of the step lengths for the exponential density is given by, $\sigma^2 = \int_{-\infty}^{\infty} z^2 e^{-|z|} dz/2 = 2$. Hence, the dimensionless number $E[M_n]/\sigma$ is given by

$$\begin{aligned} \frac{E[M_n]}{\sigma} &= -\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{\pi}} \frac{\Gamma(n+3/2)}{\Gamma(n+1)} \\ &= \sqrt{\frac{2n}{\pi}} - \frac{1}{\sqrt{2}} + O(n^{-1/2}), \end{aligned} \tag{31}$$

where we have made the asymptotic expansion for large n in the second line. Thus, the result in Eq. (31) is of the general asymptotic form as in Eq. (4) with the nontrivial constant

$$\gamma = -\frac{1}{\sqrt{2}}. \tag{32}$$

III. THE GENERAL CASE: ASYMPTOTIC EXPANSION OF POLLACZEK-SPITZER FORMULA

In this section, we derive the exact asymptotic behavior in Eq. (4) for the expected maximum for an arbitrary, symmetric jump density function $f(z)$. As mentioned in the previous section, the solution of the integral equation Eq. (22) is hard to obtain analytically for an arbitrary kernel $f(z-z')$. However, when $f(z)$ is a probability density function, Pollaczek derived a general formula giving the Laplace transform of the probability density of ordered partial sums of random independent variables [22]. In the special case of the distribution of the maximum, this formula was rederived by Spitzer [21] by a combinatorial approach. In principle, this solves the problem. However, extracting the precise asymptotic behavior of the first moment of the maximum is still nontrivial and this is what is precisely achieved in this section. Consider the following Laplace transform,

$$E\left[e^{-\rho M_n}\right] = \int_0^{\infty} e^{-\rho z} q_n'(z) dz, \tag{33}$$

where $q_n'(z) = dq_n/dz$ is the probability density of the maximum and $q_n(z)$ satisfies the recursion relation in Eq. (21). The Pollaczek-Spitzer formula for the generating function of the above Laplace transform [13] reads

$$\sum_{n=0}^{\infty} s^n E\left[e^{-\rho M_n}\right] = \frac{1}{\sqrt{1-s}}\, \phi(s,\rho); \qquad \text{where} \quad \phi(s,\rho) = \exp\left[-\frac{\rho}{\pi}\int_0^{\infty} \frac{\ln\left(1-s\hat{f}(k)\right)}{\rho^2+k^2} dk\right], \tag{34}$$

where $0 \leq s \leq 1$ and $f(z)$ is an arbitrary symmetric normalized density function. In Eq. (34), $\hat{f}(k) = \int_{-\infty}^{\infty} f(z) e^{ikz}\, dz$ is the Fourier transform of $f(z)$.

The generating function for the expected maximum can then be obtained by differentiation,

$$h(s)=\sum_{n=0}^{\infty} s^n E[M_n] = -\frac{1}{\sqrt{1-s}}\frac{\partial \phi(s,\rho)}{\partial \rho}\Big|_{\rho=0}. \tag{35}$$

To determine the asymptotic behavior of $E[M_n]$ for large n, we need to know the behavior of $h(s)$ near its principal singularity $s=1$. It then follows from Eq. (35) that we need to know the precise behavior of the function $\phi(s,\rho)$ near $s=1$ and $\rho=0$. Below we analyse these asymptotic behaviors separately for two cases : (i) For density functions with a finite second moment, so that $\hat{f}(k)\to 1-\sigma^2k^2/2+O(k^4)$ as $k\to 0$ where σ^2 is the variance of the jump lengths and (ii) for Lévy flights where the jump lengths are power law distributed so that $\hat{f}(k)\to 1-|ak|^\mu+O(k^2)$ as $k\to 0$ where a is a microscopic length and $1<\mu\leq 2$.

A. Jump lengths with a finite variance

In this case, $\hat{f}(k)=1-\sigma^2k^2/2+O(k^4)$ as $k\to 0$. To analyse $\phi(s,k)$ near $s=1$ and $\rho=0$, it is first necessary to extract the most singular part of $\phi(s,k)$ near $s=1$ and $\rho=0$. To do this, we first rewrite

$$\ln\left(1-s\hat{f}(k)\right)=\ln\left(1-s\left(1-\frac{1}{2}\sigma^2k^2\right)\right)+\ln\left(\frac{1-s\hat{f}(k)}{1-s\left(1-\frac{1}{2}\sigma^2k^2\right)}\right). \tag{36}$$

We next substitute Eq. (36) in the expression for $\phi(s,\rho)$ in Eq. (34) and subsequently perform the first integral using the following identity [14]

$$\int_0^\infty \frac{\ln\left(1-s+s\sigma^2k^2/2\right)}{\rho^2+k^2}\,dk=\frac{\pi}{\rho}\ln\left(\sqrt{1-s}+\sigma\rho\sqrt{s/2}\right). \tag{37}$$

This gives

$$\phi(s,\rho)=\frac{1}{\left[\sqrt{1-s}+\sigma\rho\sqrt{s/2}\right]}\exp\left[-\frac{\rho}{\pi}\int_0^\infty \frac{dk}{\rho^2+k^2}\ln\left(\frac{1-s\hat{f}(k)}{1-s+s\sigma^2k^2/2}\right)\right]. \tag{38}$$

The extraction of the most singular part near $s=1$ and $\rho=0$ gives

$$\phi(s,\rho)\approx\frac{1}{\left[\sqrt{1-s}+\sigma\rho/\sqrt{2}\right]}\exp\left[-\frac{\rho}{\pi}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1-\hat{f}(k)}{\sigma^2k^2/2}\right)\right], \tag{39}$$

where $\approx$ in Eq. (39) means, in the strict mathematical sense, the following identity,

$$\lim_{s\to 1,\rho\to 0}\frac{1}{\rho}\ln\left[\left(\sqrt{1-s}+\sigma\rho\sqrt{s/2}\right)\phi(s,\rho)\right]=-\frac{1}{\pi}\int_0^\infty\frac{dk}{k^2}\ln\left(\frac{1-\hat{f}(k)}{\sigma^2k^2/2}\right). \tag{40}$$

Taking the derivative with respect to ρ and putting $\rho=0$ one gets from Eqs. (39) and (35), near $s=1$

$$h(s)=\frac{\sigma}{\sqrt{2}}\frac{1}{(1-s)^{3/2}}+\frac{1}{\pi(1-s)}\int_0^\infty\frac{dk}{k^2}\ln\left[\frac{2}{\sigma^2}\left(\frac{1-\hat{f}(k)}{k^2}\right)\right]+O\left(\frac{1}{\sqrt{1-s}}\right). \tag{41}$$

Noting that the singular behavior $h(s)\sim(1-s)^{-\beta}$ ($\beta>0$) of the generating function near $s=1$ translates into the estimate $E[M_n]\approx n^{\beta-1}/\Gamma(\beta)$ for large n, we get from Eq. (41) the following exact asymptotic behaviors of the expected maximum,

$$E[M_n]=\sigma\sqrt{\frac{2n}{\pi}}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k^2}\ln\left[\frac{2}{\sigma^2}\left(\frac{1-\hat{f}(k)}{k^2}\right)\right]+O(n^{-1/2}). \tag{42}$$

Dividing by σ yields our main result announced in Eq. (4) with an exact expression for γ given in Eq. (6). As a check, we find that for the exponential density, $f(z)=e^{-|z|}/2$, i.e., with $\hat{f}(k)=1/(1+k^2)$, the formula in Eq. (6)

yields $\gamma = -1/\sqrt{2}$, in agreement with the direct solution in Eq. (32). A few other cases where the integral in Eq. (6) can be performed explicitly are listed in Eqs. (7)-(10). For the case of uniform density, i.e., $f(z) = 1/2$ for $-1 \leq z \leq 1$ (and 0 otherwise), we thus obtain an exact closed form expression for γ whose numerical value is in agreement with the result obtained by Coffmann et. al. by a different method [10]. Our result is evidently more general and holds for arbitrary symmetric jump density function. In appendix-A, we will prove that γ given by Eq. (6) is always negative for arbitrary $f(z)$.

B. Lévy distributed jump lengths

We now consider the case when the second moment of the jump distribution diverges. In particular, we consider Lévy jumps such that the Fourier transform of the density function behaves as $\hat{f}(k) = 1 - |ak|^\mu + O(k^2)$ for small k with $1 < \mu \leq 2$. This indicates that in real space the steps lengths have an algebraic tail, $f(x) \sim |x|^{-1-\mu}$ for large $|x|$. Note that the Spitzer's formula in Eq. (34) is still valid for such processes. However, as we will see, the asymptotic behavior of $E[M_n]$ is quite different from the case where σ^2 is finite.

We proceed as in the previous subsection by extracting the most singular behavior of $\phi(s,\rho)$ near $s = 1$ and $\rho = 0$. We first rewrite, for $k \geq 0$,

$$\ln\left(1 - s\hat{f}(k)\right) = \ln\left(1 - s\left(1 - (ak)^\mu\right)\right) + \ln\left(\frac{1 - s\hat{f}(k)}{1 - s\left(1 - (ak)^\mu\right)}\right). \tag{43}$$

We next substitute Eq. (43) in the expression for $\phi(s,\rho)$ in Eq. (34). This gives

$$\phi(s,\rho) = \exp\left[-I_1(s,\rho) - I_2(s,\rho)\right], \tag{44}$$

where

$$\begin{aligned} I_1(s,\rho) &= \frac{\rho}{\pi}\int_0^\infty \frac{dk}{\rho^2 + k^2}\ln\left(1 - s + s(ak)^\mu\right) \\ &= \frac{1}{2}\ln(1-s) + \frac{1}{\pi}\int_0^\infty \frac{du}{1+u^2}\ln\left[1 + \frac{s}{1-s}(a\rho u)^\mu\right], \end{aligned} \tag{45}$$

and

$$I_2(s,\rho) = \frac{\rho}{\pi}\int_0^\infty \frac{dk}{\rho^2 + k^2}\ln\left[\frac{1 - s\hat{f}(k)}{1 - s + s(ak)^\mu}\right]. \tag{46}$$

Taking the derivative with respect to ρ and keeping only the leading singular terms near $s = 1$ and $\rho = 0$, we get

$$\frac{\partial\phi(s,\rho)}{\partial\rho}\Big|_{\rho\to 0} \approx -\frac{1}{\pi\sqrt{1-s}}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1 - s\hat{f}(k)}{(ak)^\mu}\right) - \frac{\mu a^\mu \rho^{\mu-1}}{\pi\sqrt{1-s}}\int_0^\infty \frac{u^\mu\, du}{(1+u^2)(1 - s + (a\rho u)^\mu)}. \tag{47}$$

The second integral on the right hand side can further be simplified by first making a change of variable $\rho u = y$ and subsequently taking the limit $\rho \to 0$. The resulting integral can be performed in closed form. Putting everything together, we find the leading singular behavior of $h(s)$ near $s = 1$ and $\rho = 0$,

$$h(s) = \sum_{n=0}^\infty s^n E[M_n] = \frac{aB(1/\mu, 1 - 1/\mu)}{\pi(1-s)^{1+1/\mu}} + \frac{1}{\pi(1-s)}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1 - s\hat{f}(k)}{(ak)^\mu}\right) + O\left(\frac{1}{(1-s)^{1/\mu}}\right). \tag{48}$$

where $B(x,y)$ is the standard Beta function. Subsequently, one obtains the following large n behavior of $E[M_n]$,

$$E[M_n] = \frac{a\mu\Gamma\left(1 - \frac{1}{\mu}\right)}{\pi} n^{1/\mu} + \frac{1}{\pi}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1 - \hat{f}(k)}{(ak)^\mu}\right) + O(n^{1/\mu - 1}). \tag{49}$$

Note that for $\mu = 2$ and $a = \sigma/\sqrt{2}$, Eq. (49) reduces to Eq. (42), as it should. The dimensionless expected maximum is obtained by dividing Eq. (49) by the microscopic length a and one arrives at the result in Eq. (12) with the constant γ given by the exact formula in Eq. (13).

IV. APPEARANCE OF THE CONSTANT γ IN THE THERMODYNAMIC LIMIT

In the previous section, we have demonstrated how the constant γ appears as the leading correction term to the asymptotic $\sqrt{n}$ behavior of the expected maximum $E[M_n]$ for large n. In this section we show how γ appears as the leading term in the large n limit if one considers the difference $\left(E[M_n] - E[|x_n|]\right)/\sigma$, where $E[|x_n|]$ is the expected absolute end to end distance of the walker after n steps.

The calculation of $E[|x_n|]$ is relatively straightforward compared to that of $E[M_n]$. We start with the random walk in Eq. (1) where the jump density $f(\xi)$ is symmetric and has a finite second and fourth moment, $\sigma^2 = \int_{-\infty}^{\infty} x^2\, f(x)dx$ and $\mu_4 = \int_{-\infty}^{\infty} x^4\, f(x)dx$. Let $P_n(x)$ be the probability density for the particle to be between x and $x+dx$ after n steps, starting from $x=0$ at $n=0$. Using the Markov property of the walk, it is easy to see that $P_n(x)$ satisfies the recursion relation

$$P_n(x) = \int_{-\infty}^{\infty} P_n(y) f(x-y) dy, \tag{50}$$

starting from the initial condition, $P_0(x) = \delta(x)$. Note that unlike the recursion relation in Eq. (21), the limits of the integral in Eq. (50) are respectively $-\infty$ and $+\infty$ and hence Eq. (50) can be easily solved by taking the Fourier transform, $\tilde{P}_n(k) = \int_{-\infty}^{\infty} P_n(x) e^{i\,k\,x}$. One gets $\tilde{P}_n(k) = \left[\hat{f}(k)\right]^n$. Hence the solution, valid for all $n \geq 0$, is obtained from the inverse Fourier transform,

$$P_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\hat{f}(k)\right]^n e^{-i\,k\,x}\, dk. \tag{51}$$

The large n behavior of $P_n(x)$ can then be obtained by a straightforward scaling analysis of the integral in Eq. (51). In the large n limit, the most important contributions to the integral will come from the neighbourhood of $k=0$ where one can expand $\hat{f}(k) = 1 - \sigma^2 k^2/2 + \mu_4 k^4/24 + O(k^6)$. This gives, near $k=0$,

$$\ln\left[\hat{f}(k)\right] = -\frac{\sigma^2}{2} k^2 + \frac{\mu_4 - 3\sigma^4}{24} k^4 + O(k^6) \tag{52}$$

Substituting in Eq. (51) and rescaling $k\sqrt{n} = q$ and $y = x/\sqrt{n}$ one gets

$$\begin{aligned} P_n(x) &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \exp\left[-\frac{\sigma^2 q^2}{2} + \frac{(\mu_4 - 3\sigma^4) q^4}{24 n} + O(q^6/n^2)\right] e^{-i\,q\,y} \\ &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-\sigma^2 q^2/2 - i\,q\,y} \left[1 + \frac{(\mu_4 - 3\sigma^4) q^4}{24 n} + O(q^6/n^2)\right], \end{aligned} \tag{53}$$

where we have expanded the exponential for large n holding $y = x/\sqrt{n}$ fixed. Using this probability density in Eq. (53), it is then easy to calculate the expectation $E[|x_n|]$ and one gets as $n \to \infty$,

$$E[|x_n|] = \sigma \sqrt{\frac{2n}{\pi}} - \frac{1}{12\sqrt{2\pi}} \left[\frac{\mu_4 - 3\sigma^4}{\sigma^3}\right] \frac{1}{\sqrt{n}} + O(n^{-3/2}), \tag{54}$$

in agreement with the result obtained by Petrov [23] using a somewhat different method.

Comparing Eq. (54) with Eq. (42), one sees that the leading $\sqrt{n}$ term in both $E[|x_n|]$ and $E[M_n]$ have the same coefficient. However, while the next subleading term in $E[M_n]$ is a constant, the one in $E[|x_n|]$ decays as $n^{-1/2}$ for large n. Taking the difference of Eq. (42) and Eq. (54) and dividing by σ, we obtain Eq. (16). Thus, as $n \to \infty$, the difference between the two quantities (scaled by σ) approaches a dimensionless constant γ given by Eq. (6). Hence it is possible to observe the constant γ even in the thermodynamic limit provided one looks at the difference between two observables.

V. CONCLUSION

In this paper, we have studied analytically the finite size corrections to the asymptotic large n behavior of the expected maximum $E[M_n]$ of a one dimensional random walk of n steps with arbitrary, symmetric jump distribution. While the leading $\sqrt{n}$ behavior is universal and easy to understand by extrapolating the result of continuous time

Brownian motion, the leading finite size correction term turns out to be a nonuniversal constant γ which is nontrivial. In this paper, we have presented an exact formula for this constant, valid for arbitrary symmetric jump distribution. We have also generalized our results to the case of Lévy processes.

We have also demonstrated how this constant appears even in the $n \to \infty$ limit as the leading behavior of the difference $E[M_n] - E[|x_n|]$, where x_n is the position of the walker after n steps. As a nice application, we considered a Rouse polymer chain consisting of n monomers at thermal equilibrium and showed that the difference between the expected positive excursion along any direction (say the x direction) and the expected absolute value of the end to end distance of the chain approaches a negative constant in the thermodynamic limit

$$\frac{E[M_n] - E|x_n|}{\sigma} \to \frac{\zeta(1/2)}{\sqrt{2\pi}} = -0.58259\ldots. \tag{55}$$

where E denotes the thermal average and $\sigma = 1/\sqrt{\beta\kappa}$ (β being the inverse temperature and κ being the spring constant of the chain). The result in Eq. (55) is nontrivial and somewhat counterintuitive.

There are several possible extensions of this work. In this paper, we have only considered the finite size behavior of the expected maximum. It would be nice to extend this finite size study to higher moments of the maximum, or even to the full distribution of the maximum. The distribution of the maximum of a set of correlated random variables is a subject of current interest and several papers have recently studied the distribution of the maximum or related objects for the continuous time Brownian motion, in the context of fluctuating interfaces [17–19] and also in the context of convex polygons and queueing theory [20]. For discrete step random walks, the continuous time results (suitably rescaled) will provide the leading asymptotic behavior of the distribution for large n. It would be interesting to see if the method presented in this paper can be used to investigate the finite size effect in the maximum distribution in these systems.

It would also be interesting to extend our results to higher dimensions. For example, is there a leading constant correction term to $E[M_n]$ for large n, where M_n is the maximal radial distance from the origin of a random walker of n steps in d-dimensions? If so, it would be interesting to calculate this constant. For this, one needs to develop a higher dimensional analogue of the Pollaczek-Spitzer formula which would be interesting in its own right.

APPENDIX A: PROOF THAT $\gamma < 0$

In this appendix we show that the constant γ that appears in Eq. (6) or Eq. (13) is negative for arbitrary symmetric jump density function $f(x)$. For convenience, we first rewrite $\gamma = C/\sigma$ where

$$C = \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln\left[\frac{1 - \hat{f}(k)}{\sigma^2 k^2/2}\right], \tag{A1}$$

where $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$. We assume that $f(x)$ is symmetric with a finite second moment σ^2. Clearly, the argument inside the logarithm in Eq. (A1) is positive. To prove that $C < 0$, it is sufficient to prove that the argument $[1 - \hat{f}(k)]/\sigma^2 k^2/2$ inside the logarithm is less than 1. For symmetric $f(x)$ one can write $\hat{f}(k) = 2\int_0^\infty f(x)\cos(kx)dx$. The next step is to write the identity

$$\frac{\left(1 - \hat{f}(k)\right)}{\sigma^2 k^2/2} = 1 - \frac{4}{k^2\sigma^2} \int_0^\infty f(x) \left[\cos(kx) - 1 + \frac{k^2 x^2}{2}\right] dx, \tag{A2}$$

which can be proved by carrying out the integration on the right hand side explicitly and using the definition $\sigma^2 = 2\int_0^\infty x^2 f(x) dx$. To prove that the left hand side of Eq. (A2) is less than 1 we need just to prove that the second term on the right hand side is positive. Using the elementary inequality, $\cos(kx) - 1 + k^2x^2/2 \geq 0$ for all x and the fact that $f(x) \geq 0$, it follows immediately that indeed the second term is positive. Thus one has

$$\frac{\left(1 - \hat{f}(k)\right)}{\sigma^2 k^2/2} < 1, \tag{A3}$$

which then proves that $C < 0$ and hence $\gamma < 0$. A similar proof can be easily constructed to show that for Lévy processes as well γ as given in Eq. (13) satisfies the inequality $\gamma < 0$.

APPENDIX B: PROOF OF AN IDENTITY

In this appendix we prove the following identity valid for all $1 < \mu \leq 2$,

$$I = \frac{1}{\pi}\int_0^\infty \frac{dk}{k^2}\ln\left[\frac{1-e^{-k^\mu}}{k^\mu}\right] = \frac{\zeta(1/\mu)}{(2\pi)^{1/\mu}\sin(\pi/2\mu)}, \tag{B1}$$

where $\zeta(z) = \sum_{m=1}^\infty m^{-z}$ is the Riemann zeta function which is usually convergent for $z > 1$. However, it is possible to analytically continue $\zeta(z)$ for $z < 1$ [15] and one defines

$$\zeta(z) = \lim_{n\to\infty}\left[\sum_{m=1}^{n} m^{-z} - \frac{n^{1-z}}{1-z}\right]. \tag{B2}$$

We first make a change of variable $k^\mu = x$ in the integral in Eq. (B1). This gives

$$I = \frac{1}{\mu\pi}\int_0^\infty dx x^{-1-1/\mu}\ln\left(\frac{1-e^{-x}}{x}\right). \tag{B3}$$

Next we make one integration by parts and use the fact that $\mu > 1$. This yields

$$I = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{e^x-1} - \frac{1}{x}\right] x^{-1/\mu}dx. \tag{B4}$$

Note that each integral on the right hand side of Eq. (B4) is separately divergent near $x = 0$, though their difference is convergent. To make progress, we introduce a small cut-off ϵ and define

$$I(\epsilon) = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{e^x-1+\epsilon} - \frac{1}{x+\epsilon}\right] x^{-1/\mu}dx = I_1(\epsilon) - I_2(\epsilon). \tag{B5}$$

Eventually we are intersted in $I = I(0)$. The reason behind introducing this additional cut-off is so that the two integrals will be separately convergent for finite ϵ and then after performing the two separate integrals, we will eventually take the $\epsilon \to 0$ limit.

The first integral can be written as

$$I_1(\epsilon) = \frac{1}{\pi}\int_0^\infty \frac{x^{-1/\mu}dx}{e^x-1+\epsilon} = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\mu}\right)\Phi\left(1-\epsilon, 1-\frac{1}{\mu}, 1\right), \tag{B6}$$

where $\Phi(z,s,v) = \sum_{n=0}^\infty (v+n)^{-s}z^n$ is the Lerch function [14]. The second integral is elementary and can also be performed exactly to give

$$I_2(\epsilon) = \frac{1}{\pi}\int_0^\infty \frac{x^{-1/\mu}\,dx}{x+\epsilon} = \frac{\epsilon^{-1/\mu}}{\pi}B\left(1-\frac{1}{\mu}, \frac{1}{\mu}\right), \tag{B7}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the standard Beta function. Putting these together, we then have

$$I(\epsilon) = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\mu}\right)\left[\Phi\left(1-\epsilon, 1-\frac{1}{\mu}, 1\right) - \Gamma\left(\frac{1}{\mu}\right)\epsilon^{-1/\mu}\right]. \tag{B8}$$

Now, the tricky part is to take the $\epsilon \to 0$ limit in Eq. (B8). To do this, we will make use of the following asymptotic behavior of $\Phi(z,s,v)$ as $z \to 1$ [16]

$$\Phi(z,s,v) = \Gamma(1-s)[-\ln z]^{s-1}z^{-\alpha} + \zeta(s,\alpha) \tag{B9}$$

where $\zeta(s,\alpha) = \sum_{n=0}^\infty (n+\alpha)^{-s}$. Putting $z = 1-\epsilon$ in Eq. (B9) and expanding for ϵ, one then finds that as $\epsilon \to 0$, the two leading order terms are given by

$$\Phi\left(1-\epsilon, 1-\frac{1}{\mu}, 1\right) \to \Gamma\left(\frac{1}{\mu}\right)\epsilon^{-1/\mu} + \zeta\left(1-\frac{1}{\mu}, 1\right) + O(\epsilon^{1-1/\mu}). \tag{B10}$$

Substituting this behavior in Eq. (B8) we get

$$I = I(\epsilon \to 0) = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\mu}\right)\zeta\left(1-\frac{1}{\mu},1\right) = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\mu}\right)\zeta\left(1-\frac{1}{\mu}\right), \tag{B11}$$

where we have used the fact that $\zeta(z,1) = \zeta(z)$. One can further rewrite Eq. (B11) by using the identity [14],

$$\zeta(z) = 2(2\pi)^{z-1}\sin(\pi z/2)\Gamma(1-z)\zeta(1-z). \tag{B12}$$

Using this identity in Eq. (B11) one readily arrives at the final result in Eq. (B1).

In particular, note that the result in Eq. (B1) is valid even for $\mu = 2$. In that case one gets, $I(\mu = 2) = \zeta(1/2)/\sqrt{\pi}$. Note that when the jump lengths are Gaussian distributed as in Eq. (10), one has $\hat{f}(k) = \exp[-k^2\sigma^2/2]$. Hence, from Eq. (6)

$$\gamma = \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{dk}{k^2}\ln\left[\frac{1-e^{-k^2}}{k^2}\right] = \frac{I(\mu=2)}{\sqrt{2}} = \frac{\zeta(1/2)}{\sqrt{2\pi}} = -0.58259\ldots, \tag{B13}$$

thus proving the result in Eq. (10). The numerical value of $\zeta(1/2) = -1.46035$ can be obtained to arbitrary precision using Mathematica and was used in Eq. (B13).

Note added in proof: We thank R.M. Ziff for pointing out that the constant $c = \gamma\sigma = -0.297952..$ for the uniform case also appeared in an apparently unrelated three dimensional trapping problem first studied in [Ziff R. M., *flux to a trap*, 1991 J. Stat. Phys. **65** 1217] where it was evaluated by numerically iterating a set of recurrence relations. Our Eq. (11) provides an exact expression of this constant.

---

[1] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, 2001).
[2] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1968), vol-II, p-381.
[3] B. Hughes, *Random Walks and Random Environments* (Clarendon Press, Oxford, 1996).
[4] D.E. Koshland, *Bacterial Chemotaxis as a Model Behavioral System* (Raven, New York, 1980).
[5] P.E. Rouse, A theory of the linear viscoelastic properties of dilute solutions of coiling polymers, J. Chem. Phys. **21**: 1272-1280 (1953).
[6] E.W. Montroll and M.F. Shlesinger, The wonderful world of random walks, in *Nonequilibrium Phenomena II: From Stochastics to Hydrodynamics, Studies in Statistical Mechanics*, vol-11, eds J.L. Lebowitz and E.W. Montroll (North Holland, Amsterdam, 1984).
[7] A. Blumen, J. Klafter and G. Zumofen, in *Optical Spectroscopy of Glasses* ed. I. Zschokke (Reidel, Dordrecht, 1986), p-199.
[8] J.P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: statistical mechanics, models and physical applications, Phys. Rep. **195**, 127-293 (1990).
[9] E.G. Coffman and P.W. Shor, Packing in two dimensions: asymptotic average-case analysis of algorithms, Algorithmica, **9**: 253-277 (1993).
[10] E.G. Coffman, P. Flajolet, L. Flato, and M. Hofri, The maximum of random walk and its application to rectangle packing, Probability in Engineering and Informational Sciences, **12**: 373-386 (1998).
[11] J. Bertoin, *Lévy Processes* (Cambridge University Press, Cambridge, 1998).
[12] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), vol-I, p-978.
[13] F. Spitzer, The Wiener-Hopf Equation whose kernel is a probability density, Duke Math. J. **24**: 327-343 (1957).
[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, series and Products* (Academic, San Diego, 1994).
[15] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1973).
[16] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, (Springer-Verlag, Berlin-Heidelberg, 1966), p-33.
[17] S. Raychaudhuri, M. Cranston, C. Przybyla, and Y. Shapir, Maximal height scaling of kinetically growing surfaces, Phys. Rev. Lett. **87**:136101 (2001).
[18] S.N. Majumdar and A. Comtet, Exact maximal height distribution of fluctuating interfaces, Phys. Rev. Lett. **92**: 225501 (2004); see also cond-mat/0409566 (to appear in J. Stat. Phys. (2005)) for a longer version.
[19] G. Gyorgyi, P.C.W. Holdsworth, B. Portelli, and Z. Racz, et. al., Statistics of extreme intensities for Gaussian interfaces, Phys. Rev. E **68**: 056116 (2003).
[20] M.J. Kearney and S.N. Majumdar, On the area under a continuous time Brownain motion till its first-passage time, J. Phys. A: Math. Gen. **38**: 4097-4104 (2005). For a discussion of the discrete perspective of queues, see M.J. Kearney, On

a random area variable arising in discrete-time queues and compact directed percolation, J. Phys. A: Math. Gen. **37**: 8421-8431 (2004).

[21] F. Spitzer, A combinatorial lemma and its application to probability theory, Trans. Am. Math. Soc. **82**: 323 (1956).

[22] F. Pollaczek, Fonctions caractéristiques de certaines répartitions définies au moyen de la notion d'ordre, Comptes rendus **234**: 2334 (1952).

[23] V.V. Petrov, *Sums of Independent Random Variables* (Springer, Berlin, 1977), chapter 8.