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LETTER TO THE EDITOR

The exact distribution of the oscillation period in the underdamped one-dimensional Sinai model

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Abstract
We consider the Newtonian dynamics of a massive particle in a one-dimensional random potential which is a Brownian motion in space. This is the zero-temperature nondamped Sinai model. As there is no dissipation the particle oscillates between two turning points where its kinetic energy becomes zero. The period of oscillation is a random variable fluctuating from sample to sample of the random potential. We compute the probability distribution of this period exactly and show that it has a power law tail for large period, $P(T) \sim T^{-5/3}$, and an essential singularity $P(T) \sim \exp(-1/T)$ as $T \to 0$. Our exact results are confirmed by numerical simulations and also via a simple scaling argument.

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Transport in random systems is an important and common physical phenomenon. Even small amounts of disorder in physical systems can dramatically modify their static and dynamical properties and lead to interesting and complex new physics. In the presence of quenched, or time-independent, disorder it is necessary to average physical quantities over different realizations of the disorder. This disorder averaging is often technically very difficult to carry out and few exact results exist, even in one dimension. In this Letter we study the zero-temperature frictionless Newtonian dynamics of a particle in a one-dimensional random potential, which is a Brownian motion in space. Due to the absence of friction, the particle oscillates back and forth between two turning points where the kinetic energy becomes zero. The period of this oscillation is a random variable, varying from sample to sample of the disorder. The main result of this Letter is to present an exact result for the distribution of this time period.

In general the dynamics of a particle in a one-dimensional force field can be represented by the Langevin equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + F(x) + \eta(t) \tag{1}$$

where γ is the friction coefficient and $\eta(t)$ represents a thermal noise with zero mean and a correlator $\langle\eta(t)\eta(t')\rangle = 2\gamma k_B T\delta(t-t')$, T being the temperature of the medium. The force field $F(x) = -\mathrm{d}\phi/\mathrm{d}x$ originates from the background potential $\phi(x)$. We shall consider the case when the potential $\phi(x)$ is a Brownian motion in x. This means the force $F(x)$ is simply a Gaussian white noise in space with zero mean and a correlator $\langle F(x)F(x')\rangle = \delta(x-x')$.

The physics of this model is rather different in the two complementary limits, namely the overdamped and the underdamped case. In the overdamped limit where $m = 0$, this model reduces to the celebrated Sinai model [1]. Since its introduction the Sinai model has been studied extensively over the past two decades [2]. It is well known that the disorder drastically alters the diffusive behaviour of the system and the mean squared displacement at any nonzero temperature grows with time as $\overline{\langle x_t^2\rangle} \sim \ln^4(t)$ for large t, where the angle brackets here indicate the thermal average (over η) and the overline indicates the disorder average (over F).

In the underdamped limit ($m > 0$) this model represents, in the absence of thermal fluctuations ($T = 0$), the deterministic motion of a massive particle in a random medium. Recently there has been much interest in understanding the dynamics of manifolds in a random medium [3] where one neglects the thermal fluctuations as a first approximation. The deterministic limit $\eta = 0$ of equation (1) with $m > 0$ then represents a toy model where one considers the motion of a zero-dimensional manifold. For nonzero friction $\gamma > 0$, the particle will eventually stop at some distance from the starting point since there is no thermal noise. Recently Jespersen and Fogedby [4] studied the distribution of this stopping distance numerically and via simple scaling analysis. In the limit of vanishing friction $\gamma = 0$, this model was earlier studied by Stepanow and Schulz [5] who showed that the particle resumes normal diffusion in dimensions $d > 1$ provided it starts with a nonzero initial velocity v_0. However, in one dimension with $\gamma = 0$, the particle does not diffuse but oscillates between two turning points as mentioned before. In this Letter we focus on this limiting case in one dimension and calculate the distribution of the oscillation period exactly.

We set $m = 1$ for convenience and study simply the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F(x) \tag{2}$$

where $F(x) = -\mathrm{d}\phi/\mathrm{d}x$ is a white noise with zero mean and a correlator $\langle F(x)F(x')\rangle = \delta(x-x')$. The potential $\phi(x) = \phi_0 + \int_0^x F(x')\,\mathrm{d}x'$ is then a Brownian motion in x which starts at the value $\phi(0) = \phi_0$ at $x = 0$. Without loss of generality we set $\phi_0 = 0$. We assume that the particle starts at $t = 0$ at $x = 0$ with an initial velocity $v > 0$. The particle will move to the right until it reaches a turning point x_c, where $\phi(x_c) = v^2/2$, and then it will move back down to the point 0, where its velocity will be $-v$ (see figure 1). We define by T the time taken to arrive at the point x_c starting from $x = 0$. The particle will then move to the left until it similarly reaches the left turning point x_c'; the time taken to do this is T'. The total period of oscillation is then given by $T_{osc} = 2(T + T')$. Since $\phi(x)$ is a Brownian motion in x, it follows from its Markov property that $\phi(x)$ for $x > 0$ and $x < 0$ are completely independent of each other. Thus T and T' are also independent and have the identical distribution which we compute below. The distribution of T_{osc} then follows simply as the convolution of the distribution of $2T$ with that of $2T'$.

It is easy to express T explicitly in terms of the potential $\phi(x)$,

$$T = \int_0^{x_c} \frac{\mathrm{d}x}{\sqrt{v^2 - 2\phi(x)}} \tag{3}$$

where we have used $\phi_0 = 0$. Since $\phi(x)$ is random, the period T is also a random variable ,whose distribution we denote by $P(T)$. Even though in the physical problem the turning

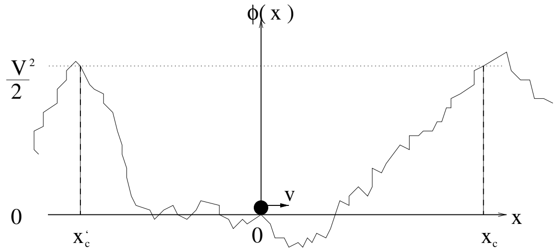

Figure 1. Particle in a Brownian potential with initial velocity v. The right/left turning points are shown as x_c and and x'_c respectively where the potential first becomes equal to $\phi = \frac{v^2}{2}$.

time T is given by the formula in equation (3) with $\phi_0 = 0$, it is useful to consider the random variable T in equation (3) as a function of an arbitrary ϕ_0 and compute the distribution $P(T, \phi_0)$ that depends on ϕ_0. Eventually the distribution of the physical time T is given by $P(T) = P(T, 0)$. We define the Laplace transform

$$Q(\mu, \phi_0) = \int_0^\infty P(T, \phi_0) \exp(-\mu T)\, \mathrm{d}T. \tag{4}$$

Substituting the explicit form of T from equation (3) in the Laplace transform one finds

$$Q(\mu, \phi_0) = E_{\phi_0}\left[\exp\left(-\mu \int_0^{x_c} V(\phi(x))\, \mathrm{d}x\right)\right] \tag{5}$$

where E_{ϕ_0} denotes the expectation value over Brownian paths starting at ϕ_0 at *time* $x = 0$ and

$$V(\phi(x)) = \frac{1}{\sqrt{v^2 - 2\phi(x)}}. \tag{6}$$

There is a nice probabilistic interpretation of $Q(\mu, \phi_0)$ in equation (5). Consider a Brownian trajectory $\{\phi(x) : 0 \leqslant x \leqslant \infty\}$ starting at ϕ_0 at $x = 0$. If the particle is killed with probability $\mu V(\phi(x))\,\mathrm{d}x$ in the *time* interval $[x, x + \mathrm{d}x]$, then $Q(\mu, \phi_0)$ is simply the probability that the particle is not killed before reaching the level $\phi = \frac{v^2}{2}$. In this interpretation $x_c = H_{\frac{v^2}{2}}$, where H_a is the first hitting time at level a, i.e. $H_a = \inf_{x>0}\{x : \phi(x) = a\}$.

To calculate $Q(\mu, \phi_0)$ we employ the standard backward Fokker–Planck technique [6, 7] in the following way. One first evolves the Brownian motion over an infinitesimal *time* interval $\mathrm{d}x$ from the initial *time* $x = 0$. The Brownian path starts at ϕ_0 at $x = 0$. In *time* $\mathrm{d}x$ it evolves to a new position $\phi_0 + \mathrm{d}\phi_0$. Thus the initial position of the trajectory that starts at *time* $\mathrm{d}x$ is $\phi_0 + \mathrm{d}\phi_0$. As a result of this infinitesimal change the function $Q(\mu, \phi_0)$ evolves as

$$Q(\mu, \phi_0) = (1 - \mu V(\phi_0)\,\mathrm{d}x)\, E_{\mathrm{d}\phi_0}[Q(\mu, \phi_0 + \mathrm{d}\phi_0)]. \tag{7}$$

The equation (7) follows from the fact that the particle survives in the interval $[0, \mathrm{d}x]$ with probability $(1 - \mu V(\phi_0)\,\mathrm{d}x)$ and then restarts its trajectory from the point $\phi_0 + \mathrm{d}\phi_0$ at *time* $\mathrm{d}x$. Of course the variable $\mathrm{d}\phi_0$ is random and one needs to sum over all histories of evolution in the interval $[0, \mathrm{d}x]$, which is denoted by the expectation $E_{\mathrm{d}\phi_0}$ in equation (7). Using the Ito prescription we average over $\mathrm{d}\phi_0$ to first order in $\mathrm{d}x$ and then equating the terms of order $\mathrm{d}x$ yields the time-independent Schrödinger equation

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi_0^2} Q(\mu, \phi_0) - \mu V(\phi_0) Q(\mu, \phi_0) = 0 \tag{8}$$

where $V(\phi)$ is given by equation (6). The function Q satisfies the following boundary conditions. (i) $Q(\mu, v^2/2) = 1$. This follows from the fact that if $\phi_0 = v^2/2$, then by definition the hitting time $H_{\frac{v^2}{2}} = 0$ and subsequently $x_c = 0$ in equation (5). (ii) $Q(\mu, -\infty) = 0$, as if $\phi(0) \to -\infty$, the hitting time $H_{\frac{v^2}{2}} \to \infty$ implying $x_c \to \infty$. Subsequently the integral in equation (5) diverges since $V(\phi(x)) > 0$ even when $x \to \infty$ and one obtains this boundary condition.

Making the change of variables $y = \sqrt{v^2 - 2\phi_0}$ in equation (8) we obtain

$$\frac{d^2 Q}{dy^2} - \frac{1}{y}\frac{dQ}{dy} - 2\mu y Q = 0. \tag{9}$$

The general solution to this equation is [8]

$$Q(y) = AyK_{2/3}\left[\sqrt{\frac{8\mu y^3}{9}}\right] + ByI_{2/3}\left[\sqrt{\frac{8\mu y^3}{9}}\right] \tag{10}$$

where A and B are constants and $K_\nu(z)$ and $I_\nu(z)$ are modified Bessel functions of index ν. As $\phi_0 \to -\infty$, $y \to \infty$. In this limit the boundary condition (ii) implies the constant $B = 0$ since $I_\nu(z)$ diverges for large argument. The boundary condition (i) implies that as $y \to 0$, $Q(y) \to 1$, which fixes the constant $A = 2^{4/3}\mu^{1/3}/\Gamma(2/3)3^{2/3}$. Changing back to the ϕ_0 variable, we then have the exact expression for $Q(\mu, \phi_0)$

$$Q(\mu, \phi_0) = \frac{2^{\frac{4}{3}}\mu^{\frac{1}{3}}}{\Gamma(\frac{2}{3})3^{\frac{2}{3}}}\sqrt{v^2 - 2\phi_0}K_{2/3}\left[\sqrt{\frac{8\mu(v^2 - 2\phi_0)^{3/2}}{9}}\right]. \tag{11}$$

The distribution $P(T, \phi_0)$ is then obtained by inverting the Laplace transform in equation (11), which fortunately can be done exactly [8]. Setting $\phi_0 = 0$ we finally obtain

$$P(T) = \frac{2^{\frac{2}{3}}v^2}{3^{\frac{4}{3}}\Gamma(\frac{2}{3})}T^{-\frac{5}{3}}\exp\left[-\frac{2v^3}{9T}\right]. \tag{12}$$

One can easily check that this distribution is normalized. In terms of the scaled variable $\tau = \frac{9T}{2v^3}$ we find that the probability density $p(\tau) = P(T)\, dT/d\tau$ of τ becomes independent of v and is given by the simple expression

$$p(\tau) = \frac{1}{\Gamma(\frac{2}{3})}\tau^{-\frac{5}{3}}\exp\left(-\frac{1}{\tau}\right) \tag{13}$$

which is valid for all $t \geqslant 0$. In particular for large τ, $p(\tau) \sim \tau^{-\alpha}$ decays as a power law with the exponent $\alpha = \frac{5}{3}$. Thus none of the integer moments of the distribution exist and there are periods of all sizes extending up to infinity. The distribution $P(T)$ has a unique maximum at $T = T_m = \frac{2}{15}v^3$, indicating the existence of a most probable turning time T_m.

The exponent $\alpha = 5/3$ can also be derived by an intuitive scaling argument. From the energy conservation it follows that $\phi(x_c) - \phi(x_0) = (dx_0/dt)^2/2$, where x_c is the turning point, x_0 is any other reference point and dx_0/dt is the velocity at x_0. For large $|x_c - x_0|$, clearly $\phi(x_c) - \phi(x_0)$ typically scales as $\sim\sqrt{|x_c - x_0|}$ since $\phi(x)$ is a Brownian motion. This indicates $dx_0/dt \sim |x_c - x_0|^{1/4}$ and hence $|x_c - x_0| \sim t^{4/3}$ for large t. The distribution of x_c behaves as $\rho(x_c) \sim x_c^{-3/2}$ for large x_c since x_c is simply the first passage *time* to the level $v^2/2$ of a Brownian walker starting at the origin. It then follows that the distribution of the turning time $t \sim |x_c - x_0|^{3/4}$ has a power law tail $t^{-\alpha}$ with $\alpha = 5/3$. In fact this scaling argument can be generalized to an arbitrary random potential with a surface roughness exponent χ, such that $\phi(x) \sim x^\chi$ ($\chi = 1/2$ for the Brownian case). The distribution of the first passage *time* in general has a power law decay, $\rho(x_c) \sim x_c^{-1-\theta_s}$, where θ_s is the spatial persistence exponent [9]. Following the same argument as in the Brownian case, one finds that the distribution of turning

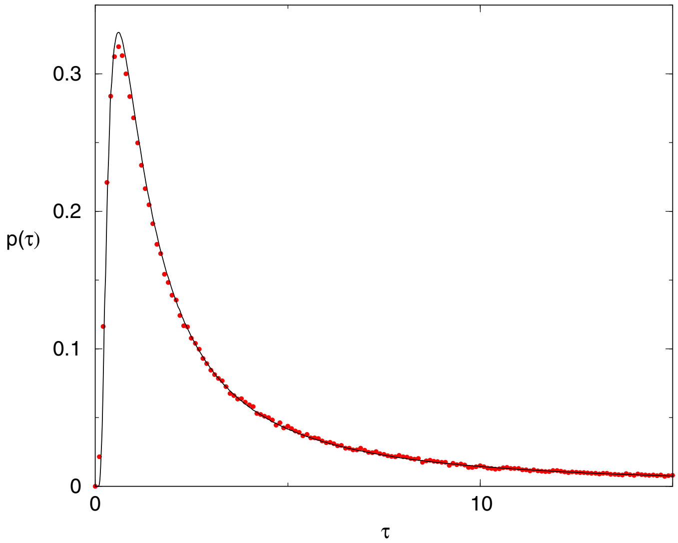

Figure 2. Histogram of τ (circles) generated from 5×10^5 samples, with initial velocity $v = 1$ and with temporal cutoff $T^* = 5000$; the bin size is 0.1. Also shown is the analytical prediction. (This figure is in colour only in the electronic version)

time has a power law tail with an exponent $\alpha = 1 + 2\theta_s/(2 - \chi)$. For the Brownian case where $\chi = 1/2$ and $\theta_s = 1/2$, one recovers $\alpha = 5/3$.

The distribution (13) has a rather remarkable link with the original (overdamped) Sinai model in the interval $[0, \infty]$, in the presence of an external force $F_{\text{ext}} = -f_0$ in the interval $[0, \infty]$. Here the partition function in the canonical ensemble is given by

$$Z = \int_0^\infty \exp\left[-\beta(\phi(x) + f_0 x)\right] \, \mathrm{d}x \tag{14}$$

where β is the inverse temperature. For $f_0 > 0$ the distribution $\rho(Z)$ of Z exists, can be computed exactly [10] and is given by

$$\rho(Z) = \frac{\beta^2}{2\Gamma(\frac{2f_0}{\beta})} \left(\frac{2}{\beta^2 Z}\right)^{1+\frac{2f_0}{\beta}} \exp\left(-\frac{2}{\beta^2 Z}\right). \tag{15}$$

Hence, when $f_0 = \beta/3$ the partition function Z (up to a constant multiplicative factor) for the statics of the Sinai model in the presence of a constant force $-f_0$ (towards the origin) has the same distribution as the turning time T in our model.

We have also verified our exact results via the numerical integration of equation (3). We chose a discretization of the integral as follows:

$$T(i+1) = T(i) + \frac{\Delta x}{\sqrt{v^2 - 2\phi(i)}} \tag{16}$$

$$\phi(i+1) = \phi(i) + \sigma_i \sqrt{\Delta x}. \tag{17}$$

Here the σ_i are independent Gaussian variables of mean zero and variance one. The Brownian limit is clearly approached when $\Delta x \to 0$. This system is integrated up until the point i where $\phi(i+1) > \frac{v^2}{2}$ and the value of $T(i)$ recorded. This is done for 5×10^5 realizations to construct the histogram of the T. The fact that T has a broad distribution implies that the number of integration steps required for a given sample can become very large. Therefore a temporal cutoff T^* is placed on the values of T measured. This means we discard those samples where T exceeds a large fixed value T^*. Thus the numerical histogram, once normalized, has the distribution

$$P(T, T^*) = \frac{P(T)\theta(T^* - T)}{\int_0^{T^*} P(S)\,\mathrm{d}S}. \tag{18}$$

In the limit when T^* is large, $P(T, T^*) \to P(T)$. We first compute $P(T, T^*)$ analytically from equation (18) using the exact form of $P(T)$ from equation (12). We then compare this analytical result with the numerically obtained $P(T, T^*)$ in figure (2) using the scaled variable $\tau = 9T/2v^3$ with the choice $v = 1$ and $T^* = 5000$. The agreement is evidently excellent.

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