Factorised Steady States in Mass Transport Models

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Abstract. We study a class of mass transport models where mass is transported in a preferred direction around a one-dimensional periodic lattice and is globally conserved. The model encompasses both discrete and continuous masses and parallel and random sequential dynamics and includes models such as the Zero-range process and Asymmetric random average process as special cases. We derive a necessary and sufficient condition for the steady state to factorise, which takes a rather simple form.

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Mass transport models form a general class of lattice models defined by dynamics in which mass is transferred (without loss) stochastically from site to site. They have attracted much recent attention, especially in connection with "condensation transitions" [1, 2, 3, 4, 5, 6]. Examples include the Zero-Range Process (ZRP) [1] and Asymmetric Random Average Process (ARAP) [7, 8], which have been used to model such diverse situations as traffic flow, clustering of buses [4], phase separation dynamics [9] and force propagation through granular media [10]. In general, it is difficult to determine the steady state distribution of such models. Thus, it is remarkable that, not only are the steady states of many models found, they often share a very convenient property, namely, a factorised steady state (also referred to as a product measure). Of course, such a property greatly facilitates the analysis of interesting behaviour, e.g., condensation.

In this letter we determine the requirement for a factorised steady state in a very broad class of mass transport models. The form of this necessary and sufficient condition, stated in (15), turns out to be appealingly simple. Encompassing both random sequential and parallel dynamics, this class includes both the ZRP and ARAP. We discuss the salient features of the approach leading to (15) and recover some previously known cases.

We consider a one-dimensional lattice of L sites with periodic boundary conditions (site $L+1=$ site 1): associated with each site is a mass m_i, $i = 1 \ldots L$. The total mass is given by $M = \sum_{i=1}^{L} m_i$. We shall most generally consider m_i as continuous variables. The dynamics is as follows: from time t to $t+1$, at each site i, mass μ_i drawn from a distribution $\varphi(\mu_i|m_i)$ 'chips off' the mass m_i, and moves to site $i+1$. Thus the master equation for the weights (unnormalised probabilities) $F_t(\underline{m})$ is

$$F_{t+1}(\underline{m}) = \prod_{i=1}^{L} \int_0^\infty \mathrm{d}m_i' \int_0^{m_i'} \mathrm{d}\mu_i \, \varphi(\mu_i|m_i') \prod_{j=1}^{L} \delta(m_j - m_j' + \mu_j - \mu_{j-1}) \, F_t(\underline{m}') \,, \quad (1)$$

where $\underline{m} \equiv \{m_1, m_2, \ldots, m_L\}$. Note that this dynamics conserves the total mass, M, so that $F_t(\underline{m})$ may be considered as a function of only $L-1$ variables. The integral of the weights, subject to the constraint of globally conserved mass,

$$Z\,(M,L) \equiv \prod_{i=1}^{L} \int_0^\infty \mathrm{d}m_i \, \delta\left(M - \sum_{i=1}^{L} m_i\right) F_t(\underline{m}) \,, \quad (2)$$

should be finite and serves as a "partition function," so that F/Z is a probability density (or "canonical distribution").

In the $t \to \infty$ limit, $F_t(\underline{m})$ approaches a t-independent function, which we denote simply by $F(\underline{m})$ and refer to as the steady state. Defining the Laplace transform

$$G(\underline{s}) = \left[\prod_{i=1}^{L} \int_0^\infty \mathrm{d}m_i e^{-s_i m_i}\right] F(\underline{m}) \,, \quad (3)$$

and transforming (1), we find

$$G(\underline{s}) = \left[\prod_{i=1}^{L} \int_0^\infty \mathrm{d}m_i' \int_0^{m_i'} \mathrm{d}\mu_i \, \varphi(\mu_i|m_i') e^{-s_i\left(m_i' - \mu_i + \mu_{i-1}\right)}\right] F(\underline{m}') \,. \quad (4)$$

We now assume that the steady state weight factorises

$$F(\underline{m}) = \prod_i f(m_i) \tag{5}$$

which implies

$$G(\underline{s}) = \prod_i g(s_i) \quad \text{where} \quad g(s) = \int_0^\infty dm f(m) e^{-sm} \,. \tag{6}$$

Then (4) becomes

$$\prod_i g(s_i) = \prod_i \left[\int_0^\infty dm_i' f(m_i') \int_0^{m_i'} \mathrm{d}\mu_i \, \varphi(\mu_i|m_i') e^{-s_i \left(m_i' - \mu_i + \mu_{i-1}\right)} \right] \,. \tag{7}$$

Changing variables to $\sigma \equiv m - \mu$ (the mass remaining after the move), we write

$$f(m)\varphi(\mu|m) = \mathcal{P}(\mu, \sigma) \,. \tag{8}$$

Note that no assumption on the form of $f(m)$ or $\varphi(\mu|m)$ is implied at this point. With this notation (7) becomes

$$\prod_i g(s_i) = \prod_i \left[\int_0^\infty \mathrm{d}\mu_i \int_0^\infty \mathrm{d}\sigma_i \, \mathcal{P}(\mu_i, \sigma_i) e^{-s_i\sigma_i - s_{i+1}\mu_i} \right] \,. \tag{9}$$

A necessary and sufficient condition for the solution of (9), is

$$\int_0^\infty \mathrm{d}\mu_i \int_0^\infty \mathrm{d}\sigma_i \, \mathcal{P}(\mu_i, \sigma_i) e^{-s_i\sigma_i - s_{i+1}\mu_i} = \ell(s_i) k(s_{i+1}) \tag{10}$$

where the two functions, k and ℓ, must satisfy

$$k\left(s\right) \ell\left(s\right) = g\left(s\right) . \tag{11}$$

That (10) is necessary and sufficient may be seen by taking the logarithm of (9) then taking derivatives with respect to s_i and s_{i+1}.

Condition (11) implies via the convolution theorem that

$$f\left(m\right) = \left[v * w\right]\left(m\right) \equiv \int_0^m d\mu \; v\left(\mu\right) w\left(m - \mu\right) \tag{12}$$

where

$$k\left(s\right) = \int_0^\infty d\mu \, e^{-s\mu} v\left(\mu\right) \quad ; \quad \ell\left(s\right) = \int_0^\infty d\sigma \, e^{-s\sigma} w\left(\sigma\right) . \tag{13}$$

Then, equations (10) and (13) imply

$$\mathcal{P}\left(\mu, \sigma\right) = v\left(\mu\right) w\left(\sigma\right) \,. \tag{14}$$

Finally we obtain from (8) and (12)

$$\varphi(\mu|m) = \frac{v\left(\mu\right) w\left(m - \mu\right)}{\left[v * w\right]\left(m\right)} . \tag{15}$$

Let us emphasize that the condition for a factorised stationary distribution for the whole lattice precisely reduces to the condition that $\varphi(\mu|m)$ has the form (15). Thus, equation (15) is the central result of this paper: for chipping rules of the form (15), one has a factorised steady state (5) with weights given by (12). Let us comment on several important points. Equations (11-15) allow us to define "equivalence classes" of chipping distributions—those leading to the same stationary state—by dividing $k(s)$ and multiplying $\ell(s)$ by any (well behaved) function of s. In particular, their roles can be "reversed" to form a "dual" φ, i.e., $w(\mu)\, v(m-\mu) / [w * v](m)$. Now, we can obviously interpret the factors in (14) as a function for μ, the mass which moves, and a function of σ, the mass which stays. In this sense, "duality" reverses these two portions of the mass, without changing $F(\underline{m})$. If we further perform a Galilean transformation (shifting the entire lattice by one site in a time step) and a parity transformation ($i \Leftrightarrow L+1-i$), then we recover the original system. Finally, note that both φ and the steady state (F/Z) are invariant under shifts of $\ln v$ and $\ln w$ by a linear function (i.e., there are arbitrary amplitudes or exponential factors in v and w: a^{μ}, a^{σ}).

In addition to treating models with parallel dynamics, manifest in (1), we can extend the approach outlined above to models with random sequential dynamics. Let the probability of a chipping event in a time step $\propto dt$ so that, to leading order in dt, at most one chipping event over the whole lattice occurs per update. Furthermore, we can let the duration of a time step be dt and take $dt \to 0$ to obtain a continuous time limit where chipping events occur with rates per unit time. Thus, we write

$$v(\mu) = \delta(\mu) + x(\mu)dt, \tag{16}$$

where $\delta(\mu)$ is the Dirac delta function. Then (12) and (15) yield

$$\begin{aligned} f(m) &= w(m) + dt[x * w](m) \quad \text{and} \\ \varphi(\mu|m) &= \frac{1}{w(m) + dt[x * w](m)} \{\delta(\mu)w(m) + dt\, x(\mu)w(m-\mu)\} \\ &= \delta(\mu)\left[1 - \frac{dt}{w(m)}\,[x * w]\,(m)\right] + dt\frac{x(\mu)w(m-\mu)}{w(m)} + O(dt^2)\ . \end{aligned}$$

Taking $dt \to 0$ we obtain the continuous time limit where mass μ moves from a site with mass m with rate $x(\mu)w(m-\mu)/w(m)$ and $f(m) = w(m)$.

Let us illustrate how this approach unifies two seemingly unrelated models – ARAP and ZRP. First we consider the ARAP [7, 8, 11, 12], a model in which each site contains a continuous amount of mass and at each time step a random fraction of the mass moves to the next site to the right. Its precise definition lies in $\psi(r|m) = \varphi(\mu|m)m$, the distribution for r, the fraction of mass that moves to the neighbouring site. A known family of distributions where one has a factorised steady state is $\psi(r|m) = (n-1)r^{n-2}$ [10, 11] which becomes

$$\varphi(\mu|m) = (n-1)\frac{\mu^{n-2}}{m^{n-1}}\ . \tag{17}$$

In our approach, the results are particularly simple:

$$v(\mu) = \mu^{n-2}, \quad w(\sigma) = 1, \tag{18}$$

$$f(m) = m^{n-1}/\,(n-1)\ . \tag{19}$$

Note that, to relate this $f(m)$ to relevant quantities in the literature (e.g., [11]), the single site mass distribution, defined as the full distribution integrated over the rest of the mass variables, is $p(m) = f(m) Z(M-m, L-1) / Z(M, L)$. In this case, $Z(M,L) = M^{nL-1} [\Gamma(n-1)]^L / \Gamma(nL)$, so that our $p(m)$ reduces, e.g., to equation (37) of [11] in the thermodynamic limit.

Another well-known case is the Zero-Range Process, reviewed in [1]. A focus of major interest (for recent developments see for example [13, 14, 15, 16]), it is a mass transport model where m_i takes integervalues and a unit mass moves from site i to site $i+1$ with probability $u(m_i)$. Within our approach, this model appears as a very special case, with Dirac delta distributions for both v and w. Since the moved mass can take only two values while the one remaining can be of any integer, the most general forms are

$$v(\mu) = \delta(\mu) + a\delta(\mu - 1), \quad w(\sigma) = \sum_{k=0}^{\infty} w_k \delta(\sigma - k) \ , \tag{20}$$

where a and w_k are arbitrary weights. As overall amplitudes are irrelevant, we have chosen the coefficient of $\delta(\mu)$ to be unity and will set $w_0 = 1$. From $f = v * w$, we see that

$$f(m) = w(m) + aw(m-1) \tag{21}$$

$$= \delta(m) + \sum_{k=1}^{\infty} [aw_{m-1} + w_m] \, \delta(m-k) \ . \tag{22}$$

With a little care, we obtain

$$\varphi(\mu|m) = \frac{w_m \delta(\mu) + aw_{m-1}\delta(\mu - 1)}{w_m + aw_{m-1}}. \tag{23}$$

The coefficient of $\delta(\mu-1)$ is precisely the chipping probability, denoted by $u(m)$ above. From here, we easily find the w_m in terms of the u:

$$w_m = a^m \prod_{n=1}^{m} \frac{1-u(n)}{u(n)} \ . \tag{24}$$

Substituting this expression into (22) yields for the weights,

$$f(m) = \sum_{k=0}^{\infty} f_k \delta(m-k) \tag{25}$$

where

$$f_k = \frac{a^k}{1-u(k)} \prod_{n=1}^{k} \frac{1-u(n)}{u(n)} \quad \text{for} \quad k \geq 1 \ , \tag{26}$$

$$= 1 \quad \text{for} \quad k = 0 \ . \tag{27}$$

This result was previously obtained by a more complicated approach [17]. Note that the factors a^k will drop out when we consider the probability density itself: $F(\underline{m})/Z$. We close this paragraph by noting the case with random sequential dynamics, which is obtained by letting $a = \tilde{a}\mathrm{d}t$ and $u(m) = x(m)\mathrm{d}t$ where $\mathrm{d}t \to 0$ yielding

$$f_k = \tilde{a}^k \prod_{n=1}^{k} \frac{1}{x(k)} \quad k \geq 1 \ . \tag{28}$$

Finally, the results presented here may be generalised to the case of heterogeneous mass transfer where $\varphi_i(\mu|m)$ depends on the site i. A necessary and sufficient condition for a factorised steady state is that

$$\varphi_i(\mu|m) = \frac{v\,(\mu)\,w_i\,(m-\mu)}{[v * w_i]\,(m)} \tag{29}$$

where v and w_i are arbitrary functions but v must be the same for each site. The weight functions are given by

$$f_i(m) = [v * w_i]\,(m) \ . \tag{30}$$

To conclude, we have determined the condition for the steady state in a general class of mass transport models to factorise. This class encompasses both continuous and discrete mass, as well as parallel and random sequential dynamics. Not only does this approach provide a unified perspective of all previously known models, it opens avenues to construct new models with this property (e.g., binomial chipping process and generalized Zero-Range Processes). In addition, we believe this approach would facilitate a deeper understanding of the existence and nature of condensates and possibly reveal novel forms of phase transitions. Implications of the gauge-like transformations should also be explored. Further work is in progress and will be published elsewhere.

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