Factorised Steady States in Mass Transport Models on an Arbitrary Graph

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Abstract.

We study a general mass transport model on an arbitrary graph consisting of L nodes each carrying a continuous mass. The graph also has a set of directed links between pairs of nodes through which a stochastic portion of mass, chosen from a site-dependent distribution, is transported between the nodes at each time step. The dynamics conserves the total mass and the system eventually reaches a steady state. This general model includes as special cases various previously studied models such as the Zero-range process and the Asymmetric random average process. We derive a general condition on the stochastic mass transport rules, valid for arbitrary graph and for both parallel and random sequential dynamics, that is sufficient to guarantee that the steady state is factorisable. We demonstrate how this condition can be achieved in several examples. We show that our generalized result contains as a special case the recent results derived by Greenblatt and Lebowitz for d-dimensional hypercubic lattices with random sequential dynamics.

PACS numbers: 05.40.-a, 02.50.Ey, 64.60.-i

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

Diverse physical phenomena such as traffic flow [1], force propagation in granular media [2, 3], clustering of buses [4], aggregation and fragmentation of clusters [5], phase separation dynamics [6], shaken granular gases [7, 8] and sandpile dynamics [9] share one common feature–their microscopic dynamics involves stochastic transport of 'mass', or some conserved quantity, from one point in space to another. To simplify analysis, continuous space is typically replaced by (or "binned" into) discrete sites. Several such lattice models of stochastic mass transport have been introduced and studied, most notably the Zero-Range Process (ZRP) [10, 11, 12], and the Asymmetric Random Average Process (ARAP) [13, 14]. These models are defined by specifying the microscopic dynamics, i.e. the basic stochastic rules for mass transport. Given the dynamics, there are two principal theoretical issues: (i) to identify the steady state if there is any, i.e. to find the invariant measure and (ii) once the steady state is known, to understand various physical properties in the steady state, e.g. the phenomenon of 'condensation' that happens when a finite fraction of the total mass condenses onto a single site [15].

It turns out that the step (i) itself is often very difficult and the exact steady state is known in only very few cases [11]. In many of these known cases, the steady state is *factorisable*. This means that the steady state probability $P(\{m_i\})$ of finding the system with mass m_1 at site 1, mass m_2 at site 2 etc is given by a product of (scalar) factors $f(m_i)$ — one factor for each of the L sites of the system — e.g. for a homogeneous system where all sites i have equivalent connectivities

$$P(\{m_i\}) = Z(M,L)^{-1} \prod_{i=1}^{L} f(m_i)\ \delta\left(\sum_{i=1}^{L} m_i - M\right)\ , \tag{1}$$

where $Z(M,L)$ is a normalisation which ensures that the integral of the probability distribution over all configurations containing total mass M is unity, hence

$$Z(M,L) = \prod_{i=1}^{L} \left[\int_0^\infty \mathrm{d}m_i f(m_i)\right]\ \delta\left(\sum_{i=1}^{L} m_i - M\right)\ . \tag{2}$$

Here, the δ-function has been introduced to guarantee that we only include those configurations containing mass M in the integral. The single-site weights, $f(m)$ are determined by the details of the mass transfer rules and for a heterogeneous system may depend on the site i.

The advantage of having a factorisable steady state is that the step (ii) mentioned above is often easier to carry out explicitly. This has been demonstrated recently by an exact analysis of the condensation phenomenon that occurs in a general class of mass transport models [15, 16]. This raises a natural question: when does the steady state in these mass transport models factorise? This issue was recently addressed in the context of a sufficiently general 'mass transport model' in one dimension, that includes, as special cases, the previously studied ZRP, ARAP and the chipping model [5]. In this model a mass m_i resides at each site i of a one dimensional lattice with periodic boundary

conditions. At each time, a stochastic portion, $\tilde{m}_i \leq m_i$ of the mass m_i at site i, chosen from a distribution $\varphi(\tilde{m}_i|m_i)$, is chipped off to site $i+1$. The distribution $\varphi(\tilde{m}|m)$ was called the chipping kernel and by choosing its form appropriately one can recover the ZRP, the ARAP and the chipping model of [5]. Even though the model above is defined in discrete time where all sites are updated in parallel, by appropriately choosing the chipping kernel it is easy to study the continuous time limit, which corresponds to a random sequential update sequence, as a special case [17]. Similarly, one can also recover, as a special case, the model with only discrete masses as in ZRP. Thus the discrete time dynamics generalises continuous time dynamics but a continuous mass variable generalises discrete mass.

A natural question, first addressed in [17], is what should be form of the chipping kernel $\varphi(\tilde{m}|m)$ for the the final steady state to be factorisable. In that study, it was proved that the *necessary and sufficient* condition for a factorised steady state in the one dimensional directed case defined above is that the chipping kernel is of the form

$$\varphi(\tilde{m}|m) = \frac{u(\tilde{m})\, v(m-\tilde{m})}{\int_0^m d\tilde{m}\, u(\tilde{m})\, v(m-\tilde{m})} \tag{3}$$

where $u(z)$ and $v(z)$ are arbitrary non-negative functions. Then the single-site weight in Eq. (1) is given by

$$f(m) = \int_0^m d\tilde{m}\, u(\tilde{m})\, v(m-\tilde{m}). \tag{4}$$

Furthermore, given a chipping kernel $\varphi(\tilde{m}|m)$, sometimes it is hard to verify explicitly that it is of the form (3) and thereby to identify the functions $u(m)$ and $v(m)$ in order to construct the weight $f(m)$ in Eq. (4). This problem was circumvented by devising a test [18] to check if a given explicit $\varphi(\tilde{m}|m)$ satisfies the condition (3) or not. Further, if it "passes this test," the weight $f(m)$ can be found explicitly by a simple quadrature [18]. Finally, for any desired function $f(m)$, one can construct dynamical rules (i.e., φ) that will yield $f(m)$ in a factorised steady state.

It was further demonstrated in Ref. [17] that the corresponding necessary and sufficient condition in the case of random sequential dynamics in continuous time can be easily obtained, by taking a suitable limit, from the condition for the parallel dynamics manifest in Eq. (3). This is done by choosing the chipping kernel as

$$\varphi(\tilde{m}|m) = \left[1 - dt \int_0^m \alpha(\tilde{m}|m)\, d\tilde{m}\right] \delta(\tilde{m}) + \alpha(\tilde{m}|m)\, dt \tag{5}$$

for small time increment dt and $\delta(z)$ is the Dirac delta function. The function $\alpha(\tilde{m}|m)$ denotes the 'rate' at which a mass $\tilde{m}$ is transferred from a site with mass m to its right neighbour. Note that the form in Eq. (5) ensures the normalization, $\int_0^m \varphi(\tilde{m}|m)\, d\tilde{m} = 1$. Then, the necessary and the sufficient condition for factorisable steady state, derived from the more general condition in Eq. (3), is that the rate $\alpha(\tilde{m}|m)$ must be of the following form [17]

$$\alpha(\tilde{m}|m) = \frac{x(\tilde{m})\, v(m-\tilde{m})}{v(m)}, \tag{6}$$

where $x(z)$ and $v(z)$ are arbitrary non-negative functions. The corresponding steady state weight is then simply, $f(m) = v(m)$.

The condition (3) for factorisability in the mass transport model was derived only in one dimension and also only for unidirectional mass transport (from site i to site $i+1$). A natural question, therefore, is whether one can generalise this condition to higher dimensional lattices, or to arbitrary graphs where mass transport can take place, in general, between any pair of sites i and j. Recently, Greenblatt and Lebowitz were able to derive a sufficiency condition [19] for factorisability in the mass transport model with nearest neighbour mass transport on a regular d-dimensional lattice with periodic boundary conditions, but considered only the case of random sequential dynamics. They showed that if $\alpha_q(\tilde{m}|m)\,dt$ is the probability of mass $\tilde{m} \leq m$ being chipped off a site with mass m to a nearest neighbour in the direction q (there being 2^d nearest neighbours on a hypercubic lattice in d dimensions), then the sufficient condition for factorisability is a direct generalization of the condition in Eq. (6), namely that the rate function must be of the form

$$\alpha_q(\tilde{m}|m) = \frac{x_q(\tilde{m})v(m-\tilde{m})}{v(m)} \tag{7}$$

for each q, where $x_q(z)$ for each q and $v(z)$ are arbitrary functions. The steady state weight is simple, $f(m) = v(m)$. However, it was not possible to prove that the condition (7) is also necessary [19], except in the case of generalized Zero-range processes.

The purpose of this paper is to derive a more general sufficiency condition, valid for arbitrary graphs where the mass transport takes place not necessarily between nearest neighbours and for the more general case of parallel dynamics. Our results boil down to equations (29) and (31). The former yields a sufficiency condition and the latter an additional consistency condition which must be satisfied.

In the special case of random sequential dynamics on a regular hypercubic lattice with nearest neighbour mass transport, our sufficiency condition reduces to the one derived by Greenblatt and Lebowitz. Our results, however, are considerably more general.

The paper is organized as follows. In Section 2, we define the mass transport model on an arbitrary graph. In Section 3, we derive a sufficient condition on the chipping kernels, valid for parallel dynamics on an arbitrary graph $\mathcal{G}$, that would gaurantee that the steady state is factorisable. We show that there are some additional consistency conditions that need to be satisfied in general and we demonstrate explicitly how these conditions are satisfied in several examples. In Section 4, we extend our approach to random sequential dynamics on an arbitrary graph. We conclude with a summary and discussion in Section 5.

2. The Mass Transport Model on an Arbitrary Graph

We consider a fixed arbitrary graph $\mathcal{G}$ consisting of L nodes labelled $i = 1, 2, 3, \cdots, L$ and a set of directed links or channels between certain pairs of nodes. At a given time

t, a node i has mass $m_i \geq 0$ where m_i are continuous variables. We consider discrete time dynamics where at each step the masses at all the nodes are updated in parallel according to the following rules. We first define a $(L \times L)$ mass-transfer matrix $[\mu]$ as follows. An element μ_{ij} of the matrix $[\mu]$ is identically zero at all times if there is no directed link from site i to site j on $\mathcal{G}$. If there is a directed link from i to j, then $\mu_{ij} \geq 0$ is a non-negative stochastic variable that represents the mass transferred from site i to site j during one update. In Fig. 1 we give an example which we refer to for illustrative purposes throughout this section. The diagonal element μ_{ii} represents the mass that stays at site i at the end of the single update. We assume that the dynamics of mass transport conserves the total mass. Thus if $\{m_1, m_2, \ldots, m_L\}$ represents the masses before the update, by virtue of mass conservation, the row sums of the matrix $[\mu]$ are given by, $\sum_j \mu_{ij} = m_i$ (see Fig. 1). Similarly, the column sum corresponding to a node i, $\sum_j \mu_{ji} = m'_i$ represents the mass at site i just after the update.

Note that, by definition, some elements of the matrix $[\mu]$ are permanently zero (when there is no directed channel available for mass transfer between a pair of sites). On the other hand, when there is an available channel from i to j, the matrix element μ_{ij} is a stochastic variable which is chosen in the following way. For each node i, we define a generalized 'chipping kernel' $\varphi_i\left(\{\mu_{ij}\}|m_i\right)$ that represents the joint distribution of the masses transported from site i in a single update. Here the set $\{\mu_{ij}\}$ runs over only those sites j which are connected to i via a directed link, and in addition it includes the diagonal element μ_{ii}. In other words, $\{\mu_{ij}\}$ is simply the set of non-zero elements in the i-th row of the mass-transfer matrix $[\mu]$. For example, in Fig. 1 where we have a graph of four nodes along with the directed links, we need to define four chipping kernels as follows: $\varphi_1\left(\mu_{11}, \mu_{12}, \mu_{14}|m_1\right)$, $\varphi_2\left(\mu_{21}, \mu_{22}, \mu_{23}|m_2\right)$, $\varphi_3\left(\mu_{31}, \mu_{33}|m_3\right)$ and $\varphi_4\left(\mu_{43}, \mu_{44}|m_4\right)$ respectively. The chipping kernels must be normalized to unity at all sites i, i.e.

$$\int \varphi_i\left(\{\mu_{ij}\}|m_i\right)\, \delta\left(\sum_j \mu_{ij} - m_i\right) \prod_j d\mu_{ij} = 1 \tag{8}$$

where it is implied that the index j in the sum as well as in the product runs over all the sites that i feeds into, including i itself. Thus in this model the amount of mass transported from a given site i in one update depends only on that site i, and not, e.g. on the destination sites to which the mass is transported. Also, we assume that the chipping kernels $\varphi_i\left(\{\mu_{ij}\}|m_i\right)$ do not contain the time t explicitly. Note also that this chipping kernel $\varphi_i\left(\{\mu_{ij}\}|m_i\right)$ generalises that of [17, 18] sufficiently to include hypercubic lattices[19] or more complicated graphs.

The chipping kernels thus specify the dynamics, i.e. the mass update rules. Given these kernels, we next ask what is the steady state joint probability distribution of masses $\underline{m} \equiv \{m_1, m_2, \cdots, m_L\}$, i.e. $P\left(\underline{m}, t \to \infty\right)$. In particular, our goal is to determine the properties of the chipping kernels $\varphi_i\left(\{\mu_{ij}\}|m_i\right)$ required in order to guarantee that the steady state joint probability distribution is factorisable, i.e. of the form

$$P\left(\underline{m}, t \to \infty\right) = Z(M, L)^{-1} \left[\prod_{i=1}^{L} f_i(m_i)\right] \delta\left(\sum_{i=1}^{L} m_i - M\right) \tag{9}$$

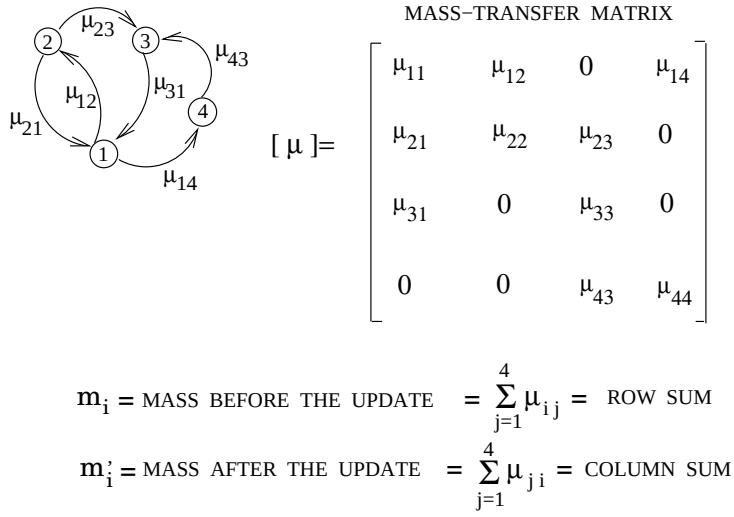

Figure 1. An example graph with four nodes labelled 1, 2, 3 and 4 and directed links between certain pairs of nodes. The associated (4×4) mass-transfer matrix $[\mu]$ is shown, whose element μ_{ij} denotes the stochastic mass transferred from site i to site j in one single update, provided there is a directed link between the two sites. If there is no directed link, the corresponding matrix element is always identically zero. The diagonal element μ_{ii} is the amount of mass that stays at site i during the update. The row sum and the column sum associated with any node i, $\sum_j \mu_{ij} = m_i$ and $\sum_j \mu_{ji} = m'_i$, represent respectively the mass at i before and after the update.

where the normalization constant is given by

$$Z(M,L) = \prod_{i=i}^{L} \left[\int_0^\infty dm_i f_i(m_i) \right] \delta \left(\sum_{i=1}^{L} m_i - M \right). \tag{10}$$

Besides, if the steady state factorises as in Eq. (9), we would also like to know the single-site weights $f_i(m_i)$ in terms of the prescribed chipping kernels $\varphi_i\left(\{\mu_{ij}\}|m_i\right)$. Note that on an arbitrary graph $\mathcal{G}$, the single-site weights $f_i(m_i)$ are, in general, different from site to site. Hence there is an additional subscript i in $f_i(m_i)$. On a homogeneous graph, where all sites have equivalent connectivities, the weight function $f(m)$ does not depend on the site i explicitly as in Eq. (1).

3. Factorisable Steady State on an Arbitrary Graph

Since the dynamics conserves the total mass, at any time t we can write $P\left(\underline{m},t\right) \propto F(\underline{m},t)\delta\left(\sum_i m_i - M\right)$ where M is the total mass and $F(\underline{m},t)$ is the unnormalized weight at time t. Below, we will first write down the general evolution equation of the weight $F(\underline{m},t)$ under the mass transport rules prescribed by the chipping kernels. While the notations that we will use for a general graph $\mathcal{G}$ may seem a bit complicated, it is instructive to keep the simple example in Fig. 1 in mind and use it as a guide to the general notations.

Let us consider a single update from time t to time $t+1$. Let $\underline{m} \equiv \{m_1, m_2, \cdots, m_L\}$ denote the masses at time t before the update and $\underline{m'} \equiv \{m'_1, m'_2, \cdots, m'_L\}$ denote the masses at time $t+1$ after the update. In terms of the elements of the mass-transfer matrix $[\mu]$, we thus have, $m_i = \sum_k \mu_{ik}$ and $m'_i = \sum_k \mu_{ki}$ for all i. The master equation for the evolution of the weight then reads

$$F(\underline{m'}, t+1) = \tag{11}$$
$$\prod_{i=1}^{L} \left[\int_0^\infty \mathrm{d}m_i \prod_j \int \mathrm{d}\mu_{ij} \varphi_i\left(\{\mu_{ij}\}|m_i\right) \delta(m'_i - \sum_k \mu_{ki}) \delta(m_i - \sum_k \mu_{ik}) \right] F(\underline{m}, t)$$

where the product over j runs over only the sites to which the site i feeds into, i.e., when there is a directed link between sites i and j (note that this set includes the site i itself). For other sites that are not connected to i by a directed link, the corresponding matrix element $\mu_{ij} = 0$ identically (see Fig. 1) and hence they are not integration variables in Eq. (11).

Our strategy now is to assume that the steady state factorises and to determine a sufficent condition for this assumption to hold. First, we take the $t \to \infty$ limit on both sides of Eq. (11) and assume that the steady state weight factorises

$$F(\underline{m}) = \prod_i f_i(m_i) \tag{12}$$

and write

$$f_i(m_i)\varphi_i\left(\{\mu_{ij}\}|m_i\right) = \mathcal{P}_i(\{\mu_{ij}\}) \delta(m_i - \sum_j \mu_{ij}) \ , \tag{13}$$

thus

$$\prod_{i=1}^{L} f_i(m'_i) = \prod_{i=1}^{L} \left[\int_0^\infty \mathrm{d}m_i \prod_j \int \mathrm{d}\mu_{ij} \mathcal{P}_i(\{\mu_{ij}\}) \delta(m'_i - \sum_k \mu_{ki}) \delta(m_i - \sum_k \mu_{ik}) \right] . \tag{14}$$

Now we Laplace transform this equation to obtain

$$\prod_{i=1}^{L} g_i(s_i) = \prod_{i=1}^{L} \left[\int_0^\infty \mathrm{d}m_i \prod_j \int \mathrm{d}\mu_{ij} \mathcal{P}_i(\{\mu_{ij}\}) \delta(m_i - \sum_k \mu_{ik}) \, \mathrm{e}^{-s_i \sum_k \mu_{ki}} \right] . \tag{15}$$

where we have defined $g_i(s_i) = \int_0^\infty f_i(m) e^{-s_i m} dm$. Next we trivially perform the integration over the m_i variables on the rhs of Eq. (15) and rearrange the s_i to get

$$\prod_{i=1}^{L} g_i(s_i) = \prod_{i=1}^{L} \left[\prod_j \int \mathrm{d}\mu_{ij} \mathcal{P}_i(\{\mu_{ij}\}) \mathrm{e}^{-\sum_k s_k \mu_{ik}} \right] \equiv \prod_i X_i(\{s_k\}_i) \ . \tag{16}$$

where $\{s_k\}_i$ indicates that k runs over the sites to which i is connected by a directed link, including the site i itself. For example, for the graph in Fig. 1, we have

$$g_1(s_1) g_2(s_2) g_3(s_3) g_4(s_4) = X_1(s_1, s_2, s_4)\, X_2(s_1, s_2, s_3)\, X_3(s_1, s_3)\, X_4(s_3, s_4). \tag{17}$$

One solution (not the most general as we shall discuss in section 5) to (16) is

$$X_i(\{s_k\}_i) = \prod_j K_{ij}(s_j) \tag{18}$$

so that

$$g_i(s_i) = \prod_j K_{ji}(s_i) \ . \tag{19}$$

As before, the product in Eq. (18) runs over only those sites j that the site i feeds into (including itself), whereas in Eq. (19) the product runs over only those sites j that feed onto i (including the site i). For example, for the graph in Fig. 1, we will have

$$\begin{aligned} X_1(s_1, s_2, s_4) &= K_{11}(s_1)\, K_{12}(s_2)\, K_{14}(s_4) \\ X_2(s_1, s_2, s_3) &= K_{21}(s_1)\, K_{22}(s_2)\, K_{23}(s_3) \\ X_3(s_1, s_3) \quad &= K_{31}(s_1)\, K_{33}(s_3) \\ X_4(s_3, s_4) \quad &= K_{43}(s_3)\, K_{44}(s_4) \end{aligned} \tag{20}$$

and correspondingly

$$\begin{aligned} g_1(s_1) &= K_{11}(s_1)\, K_{21}(s_1)\, K_{31}(s_1) \\ g_2(s_2) &= K_{12}(s_2)\, K_{22}(s_2) \\ g_3(s_3) &= K_{23}(s_3)\, K_{33}(s_3)\, K_{43}(s_3) \\ g_4(s_4) &= K_{14}(s_4) K_{44}(s_4). \end{aligned} \tag{21}$$

Since the rhs of Eq. (19) is a product of Laplace transforms, its inverse Laplace transform $f_i(m_i)$ therefore has the convolution form

$$f_i(m) = \left[\prod_{*j} v_{ji} \right] (m) \tag{22}$$

where the notation indicates a product over sites j feeding into i which is a multiple convolution and the function $v_{ji}(m)$ is the inverse Laplace transform of $K_{ji}(s_i)$, i.e.

$$K_{ji}(s_i) = \int_0^\infty v_{ji}(\sigma)\, \mathrm{e}^{-s_i \sigma} d\sigma \tag{23}$$

Again, going back to Fig. 1, we will have, for example, by inverting Eq. (21) the following convolution for the site labelled 1

$$f_1(m_1) = \int v_{11}(m_1 - \sigma_{21} - \sigma_{31})\, v_{21}(\sigma_{21})\, v_{31}(\sigma_{31}) d\sigma_{21}\, d\sigma_{31} \tag{24}$$

and similarly for the sites labelled 2, 3 and 4.

Having obtained the form of $f_i(m)$, let us ask what does this imply for the chipping kernels. To see that, we go back to the definitions of X_i in Eq. (16). Substituting the ansatz for X_i in Eq. (18) on the rhs of Eq. (16) we see that for each i

$$\prod_j \int \mathrm{d}\mu_{ij} \mathcal{P}_i(\{\mu_{ij}\}) \mathrm{e}^{-\sum_k s_k \mu_{ik}} = \prod_j K_{ij}(s_j) \tag{25}$$

For example, for the site 1 in Fig. 1, we have

$$\int d\mu_{11}\, d\mu_{12}\, d\mu_{14}\, P_1(\mu_{11}, \mu_{12}, \mu_{14})\, \mathrm{e}^{-s_1 \mu_{11} - s_2 \mu_{12} - s_4 \mu_{14}} = K_{11}(s_1)\, K_{12}(s_2)\, K_{14}(s_4), \tag{26}$$

and similarly for the sites 2, 3 and 4. Eq. (25) immediately implies for the general case

$$\mathcal{P}_i(\{\mu_{ij}\}) = \prod_j v_{ij}(\mu_{ij}) \tag{27}$$

where j runs over the sites which site i feeds into and v_{ij} is the Laplace inversion of K_{ij} as in Eq. (23). For example, for the site 1 in Fig. 1 we have

$$P_1(\mu_{11}, \mu_{12}, \mu_{14}) = v_{11}(\mu_{11})\, v_{12}(\mu_{12})\, v_{14}(\mu_{14}). \tag{28}$$

Substituting Eq. (27) in Eq. (13) we get the following expression for the chipping kernel

$$\varphi_i\left(\{\mu_{ij}\}|m_i\right) = \frac{\prod_j v_{ij}(\mu_{ij})}{\left[\prod_{*k} v_{ki}\right](m_i)}\ . \tag{29}$$

In the numerator the product is over sites j which site i feeds into via a directed link connecting i to j (this includes site i itself), and one has the constraint $\sum_j \mu_{ij} = m_i$. In the denominator, on the other hand, the product is over sites k feeding into site i (including once again the site i itself). Going back again to the example in Fig. 1, we have for the site 1 the following chipping kernel

$$\varphi_1(\mu_{11}, \mu_{12}, \mu_{14}|m_1) = \frac{v_{11}(\mu_{11})\, v_{12}(\mu_{12})\, v_{14}(\mu_{14})}{\int v_{11}(m_1 - \sigma_{21} - \sigma_{31})\, v_{21}(\sigma_{21})\, v_{31}(\sigma_{31}) d\sigma_{21}\, d\sigma_{31}} \tag{30}$$

where $\mu_{11} + \mu_{12} + \mu_{14} = m_1$. Similarly one can easily express the chipping kernels for the other sites 2, 3 and 4 in terms of the functions v_{ij}.

However this is not quite the end of the story since the chipping kernel $\varphi_i(\{\mu_{ij}\}|m_i)$ must obey a key consistency condition, namely its normalization: Eq. (8). This implies from Eq. (29) that for each site i

$$\left[\prod_{*j} v_{ji}\right](m) = \left[\prod_{*j} v_{ij}\right](m) \tag{31}$$

Again, the convolution on the lhs is over sites j feeding into site i but the convolution on the rhs is over sites j which i feeds into. For example, for the site 1 in Fig. 1, this consistency condition reads

$$\begin{aligned} &\int v_{11}(m_1 - \sigma_{21} - \sigma_{31})\, v_{21}(\sigma_{21})\, v_{31}(\sigma_{31})\, d\sigma_{21}\, d\sigma_{31} = \\ &\int v_{11}(m_1 - \mu_{12} - \mu_{14})\, v_{12}(\mu_{12})\, v_{14}(\mu_{14})\, d\mu_{12}\, d\mu_{14}. \end{aligned} \tag{32}$$

One can write down similar consistency conditions for sites 2, 3 and 4 also.

In summary, a sufficient condition for factorisation is that the chipping kernels are product form as in Eq. (29) where v_{ij} are non-negative functions that satisfy the consistency conditions in Eq. (31), but otherwise are arbitrary. If this condition holds, then the steady state will factorise with single-site weight $f_i(m)$ given by the convolution formula in Eq. (22). These conditions form the central result of this paper. The conditions arise from making the ansatz in Eq. (18) which immediately implies Eq. (22) for $f_i(m_i)$ and Eq.(27) for $\mathcal{P}_i(\{\mu_{ij}\})$. These two equations, when substituted in Eq. (13), gives Eq. (29) for the chipping kernel. Finally, Eq. (31) arises from the requirement that the chipping kernel in Eq. (29) is normalized to unity.

Condition (29,31) is certainly a sufficient condition. Whether it is also a necessary condition remains a non-trivial issue. In the special case of one dimensional graph with unidirectional transport, it was explicitly proved in Ref. [17] that this is also a necessary condition. Similarly, this is true (see below) in the case of a complete graph where every site is connected to every other site. However, it is unlikely, in general, that (29) is also necessary. In section 5, we provide an explicit example showing solutions to (16) which are not of the form (29). Although we are unable, for a technical reason to be discussed later, to prove that this class of solutions are also "legitimate," we feel that valid solutions different from (29) should exist. Beyond this simple example, finding the most general valid solution to Eq. (16) for the arbitrary graph seems to pose significant challenges.

3.1. Specific Examples

Let us now discuss several special cases where one can directly verify that the consistency conditions in Eq. (31) are satisfied automatically by the functions v_{ij} due to the nature of the underlying graph $\mathcal{G}$, thereby guaranteeing factorisability provided the chipping kernels are of the product form as in Eq. (29).

Example 1: Mass transport in one dimension over extended range: Consider a one dimensional latice with periodic boundary conditions where mass from a given site can get transported up to a range l to the left or to the right. In this case

$$\varphi_i\left(\{\mu_{ij}\}|m_i\right)\varphi_i\left(\mu_{i,j-l}\ldots\mu_{i,j+l}|m_i\right) \tag{33}$$

and we have a translationally invariant chipping kernel i.e. $\varphi_i(\cdot|m)$ is the same function of $(2l+1)$ arguments for each site. Then the consistency condition (31) reads

$$[v_{i,i-l}{*}v_{i,i-l+1}{*}\cdots{*}v_{i,i+l-1}{*}v_{i,i+l}](m_i) = [v_{i-l,i}{*}v_{i-l+1,i}{*}\cdots{*}v_{i+l-1,i}{*}v_{i+l,i}](m_i) \tag{34}$$

which is automatically satisfied due to the translational invariance $v_{i,i+d}(\mu) = v_{i-d,i}(\mu)$.

Example 2: Symmetric chipping rule: It is easy to see that if the chipping rules are symmetric, i.e. $v_{ij}(\mu) = v_{ji}(\mu)$, then the consistency condition (31) holds automatically.

Example 3: A more general chipping rule: A more general way to meet the condition (31) is to assume that the chipping rules are such that at each site i, for every directed link $(i \to j)$ out of site i, there is an incoming directed link $(k \to i)$ to i from some other site k with $v_{ij}(\mu) = v_{ki}(\mu)$. For example, in the 1-d example above, this is achieved through $v_{i,i+d}(\mu) = v_{i-d,i}(\mu)$. Similarly, one can achieve this on a hypercubic lattice in arbitrary dimensions and on any graph where the number of links out of a site equals the number of links into it.

Example 4: Complete graph with permutationally invariant chipping kernel: So far we have proved that Eq. (29) is a sufficient condition for factorisability on any arbitrary graph provided we can satisfy the consistency condition (31) for each

site. As discussed before, to prove that this condition is also necessary seems hard for a general graph. However, one can prove this for a complete graph where every site is connected to every other site via a directed link *and* the chipping kernel is permutationally invariant. This can be proved as follows. On a complete homogeneous graph the functions $g_i(s) = g(s)$ in Eq. (16) do not depend explicitly on the site index i. Moreover, since every site is connected to every other sites and using the fact that the chipping kernel is permutationally invariant Eq. (16) simply becomes

$$g(s_1)g(s_2)\cdots g(s_L) = [X(s_1, s_2, \cdots, s_L)]^L. \tag{35}$$

Clearly $X(s_1, s_2, \cdots, s_L) = K(s_1)K(s_2)\cdots K(s_L)$ with $K(s)$ arbitrary, is *one* solution to Eq. (35) and thereby $g(s) = [K(s)]^L$. As in the general case, this will then lead to the sufficiency condition (29). To prove that $X(s_1, s_2, \cdots, s_L) = K(s_1)K(s_2)\cdots K(s_L)$ is also the most general form of the solution that one can write down for Eq. (35), we take logarithm on both sides of Eq. (35) and then take derivatives with respect to s_i and s_j with $i \neq j$. This gives

$$\frac{\partial^2 \ln X}{\partial s_i \partial s_j} = 0, \tag{36}$$

for any $i \neq j$. It is then easy to see that the most general solution of the partial differential equation (36) is of the form, $X(s_1, s_2, \cdots, s_L) = K_1(s_1)\, K_2(s_2)\cdots K_L(s_L)$ where $K_i(s)$ are arbitrary functions. Since the graph is homogeneous, we also have $K_i(s) = K(s)$ independent on the site index i. Thus $X(s_1, s_2, \cdots, s_L) = K(s_1)\, K(s_2)\cdots K(s_L)$ with $K(s)$ being an arbitrary function, is the only solution of Eq. (35) that respects homogeneity. Since this solution finally leads to the sufficiency condition, the uniqueness of its form guarantees then that Eq. (29) is both necessary and sufficient. Note that the consistency condition (31) is automatically satisfied in this case.

4. Random Sequential Dynamics

The sufficiency condition in Eq. (29) and the associated consistency condition in Eq. (31), derived above for parallel update dynamics in discrete time, can be easily extended to the case of random sequential dynamics. This can be achieved by letting the probability of the chipping event in a time step $\propto dt$ so that, to leading order in dt for small dt, at most one chipping event can occur in the whole graph $\mathcal{G}$ per update, i.e. the chipping events occur sequentially one per update. In addition, taking $dt \to 0$ one can obtain the continuous time limit where chipping events occur with 'rates' per unit time. Thus, the corresponding sufficiency condition for the random sequential dynamics will be specified in terms of the chipping 'rate' kernels, rather than the chipping probablity kernels φ_i in parallel dynamics. To translate the sufficiency condition in Eq. (29) valid for the probability kernels into one that is valid for 'rate' kernels, we first redefine the functions $v_{ij}(\mu_{ij})$, for all $i \neq j$, in the following way

$$v_{ij}(\mu_{ij}) = \delta(\mu_{ij}) + x_{ij}(\mu_{ij})dt \tag{37}$$

where $x_{ij}(\mu_{ij})$ are arbitrary functions. The diagonal functions $v_{ii}(\mu_{ii})$ are left unchanged.

With this re-definition of v_{ij}, the steady state weight in Eq. (22) becomes

$$f_i(m) = v_{ii}(m) + dt\,\left[\sum_{j\neq i}\int_0^m v_{ii}(m-\mu_{ji})\,x_{ji}(\mu_{ji})\,d\mu_{ji}\right] + O(dt^2) \qquad (38)$$

where the sum over j in the second term runs over all sites $j \neq i$ that feed into site i on $\mathcal{G}$. Using the re-defined v_{ij} in Eq. (37) one can similarly rewrite the chipping kernels in Eq. (29). Since the diagonal elements play a special role, it is convenient to redefine the chipping kernel only in terms of non-diagonal elements, i.e., $\varphi_i\left(\{\mu_{ij}\}|m_i\right) \equiv \varphi_i\left(\{\mu_{ij}\}'|m_i\right)$ where $\{\mu_{ij}\}'$ denotes the set of matrix elements in the row i without the diagonal element μ_{ii}. We are allowed to get rid of the diagonal element using the row sum, $\sum_j \mu_{ij} = m_i$. For example, for the graph in Fig. 1, we will rewrite, $\varphi_1(\mu_{11},\mu_{12},\mu_{14}|m_1) \equiv \varphi_1(\mu_{12},\mu_{14}|m_1)$. Substituting Eq. (37) in Eq. (29) and taking the limit $dt \to 0$, one gets

$$\begin{aligned}\varphi_i\left(\{\mu_{ij}\}'|m_i\right) \;=\; &\left[1 - \frac{dt}{v_{ii}(m_i)}\sum_{j\neq i}[x_{ji} * v_{ii}](m_i)\right]\prod_{j\neq i}\delta(\mu_{ij}) + \\ &+\, dt\,\left[\sum_{j\neq i}\frac{x_{ij}(\mu_{ij})\,v_{ii}(m_i-\mu_{ij})}{v_{ii}(m_i)}\prod_{k\neq j}\delta(\mu_{ik})\right] + O(dt^2) \qquad (39)\end{aligned}$$

where $[x * y](m) = \int_0^m x(\sigma)y(m-\sigma)d\sigma$ denotes the convolution integral. The notation in Eq. (39) may look a bit complicated, but actually it's rather simple. For example, for the graph in Fig. 1, the kernel in Eq. (39) for the site labelled 1 reads,

$$\begin{aligned}&\varphi_1(\mu_{12},\mu_{14}|m_1) = \qquad (40)\\ &\quad\left[1 - \frac{dt}{v_{11}(m_1)}\left([v_{11} * x_{31}](m_1) + [v_{11} * x_{21}](m_1)\right)\right]\delta(\mu_{12})\,\delta(\mu_{14}) + \\ &+\, dt\,\left[\frac{x_{12}(\mu_{12})v_{11}(m_1-\mu_{12})}{v_{11}(m_1)}\,\delta(\mu_{14}) + \frac{x_{14}(\mu_{14})v_{11}(m_1-\mu_{14})}{v_{11}(m_1)}\,\delta(\mu_{12})\right].\end{aligned}$$

Taking $dt \to 0$ limit in Eq. (39) we then obtain the continuous time limit where a portion μ_{ij} gets chipped off the site i with mass m_i, to another site j which is connected to i by a directed link from i to j on $\mathcal{G}$, with a rate $x_{ij}(\mu_{ij})v_{ii}(m_i-\mu_{ij})/v_{ii}(m_i)$. Also, we notice that in the limit $dt \to 0$, the steady state weight $f_i(m_i)$ is given simply from Eq. (38)

$$f_i(m_i) = v_{ii}(m_i). \qquad (41)$$

However, as in the case of parallel update, we need the functions x_{ij} to satisfy certain additional consistency conditions. This is because the chipping kernel $\varphi_i\left(\{\mu_{ij}\}'|m_i\right)$, when integrated over its arguments $\{\mu_{ij}\}'$ must give unity. Integrating Eq. (39) then gives the required consistency condition that must be satisfied for each node i,

$$\sum_{j\neq i}[x_{ij} * v_{ii}](m_i) = \sum_{j\neq i}[x_{ji} * v_{ii}](m_i), \qquad (42)$$

where the sum on the lhs runs over all sites j that the site i feeds into (excluding the site i itself), but on the rhs the sum runs over all sites j that feed into site i (excluding the site i itself). As an example of this condition for the site labelled 1 in Fig. 1, we have

$$[x_{12} * v_{11}](m_1) + [x_{14} * v_{11}](m_1) = [x_{21} * v_{11}](m_1) + [x_{31} * v_{11}](m_1). \quad (43)$$

Similar consistency conditions can be wriiten down for sites 2, 3 and 4 in Fig. 1.

Thus, in summary, for the mass transport model on an arbitrary graph $\mathcal{G}$ with random sequential dynamics specified by the rates $\alpha_{ij}(\mu|m_i)$ of mass μ to be transferred from site i to site j (provided there is a directed link between i and j on $\mathcal{G}$), a sufficiency condition for the steady state to be factorisable with weights $f_i(m_i)$ is that $\alpha_{ij}(\mu|m_i)$ is of the form

$$\alpha_{ij}(\mu|m_i) = \frac{x_{ij}(\mu)v_{ii}(m_i - \mu)}{v_{ii}(m_i)} \quad (44)$$

for all i, where x_{ij} and v_{ii} are functions that must satisfy the consistency conditions in Eq. (42), but otherwise arbitrary. Also, the corresponding steady state weight is then given simply by $f_i(m_i) = v_{ii}(m_i)$. Note that though formally Eq. (44) looks like a direct generalization of the one dimensional condition in Eq. (6), the additional consistency conditions in Eq. (42) are nontrivial to satisfy. For the one dimensional example in Eq. (6), the corresponding consistency condition, $[x_{i,i+1} * v_i](m_i) = [x_{i-1,i} * v_i](m_i)$ is automatically satisfied due to the translational invariance, $x_{i-1,i}(\mu) = x_{i,i+1}(\mu)$.

4.1. Specific example of hypercubic lattice

It is easy to verify that the consistency condition in Eq. (42) is automatically satisfied in the four example cases of section 3.1 (as it should be since random sequential dynamics is just a limit of the discrete time case). Let us do this explicitly for the case where the graph $\mathcal{G}$ is a homogeneous hypercubic lattice with periodic boundary conditions and mass transfer takes place only between nearest neighbours. For a hypercubic lattice, from each site i there are 2^d outgoing links to the 2^d nearest neighbours of i. Similarly, there are exactly 2^d incoming links to site i from its nearest neighbours. Then the sufficiency condition in Eq. (44) can be written in a simplified notation

$$\alpha_{i,q}(\mu|m_i) = \frac{x_{i,q}(\mu)v_{ii}(m_i - \mu)}{v_{ii}(m_i)} \quad (45)$$

where the index $q = \pm 1$ runs over the 2^d directions $\{\pm e_1, \pm e_2, \cdots, \pm e_d\}$, $\alpha_{i,q}(\mu|m_i)$ denotes the rate of transfer of mass μ from site i with mass m_i in the direction q and $x_{i,q}$ denotes the function associated with the link $(i, i+q)$. Similarly, the the consistency condition in Eq. (42) can be rewritten as

$$\sum_q [x_{i,q} * v_{ii}](m_i) = \sum_{j \in \text{neighbours of } i} [x_{j,-q} * v_{ii}](m_i). \quad (46)$$

We next use the fact that the lattice is homogeneous, i.e. it is translationally invariant in all directions. Clearly then $x_{i-q,q}(z) = x_{i,i+q} = g_q(z)$ due to translational

invariance in the q-th direction. In that case the condition in Eq. (46) is clearly satisfied at all i. Also, due to the translational invariance, the rate function $\alpha_{i,q}(\mu|m_i) = \alpha_q(\mu|m_i)$ does not depend on the site index i. By the same requirement, $v_{ii}(m_i) = v(m_i)$. Thus, the sufficiency condition in Eq. (44) simply reads,

$$\alpha_q(\mu|m_i) = \frac{g_q(\mu)v(m_i - \mu)}{v(m_i)} \tag{47}$$

where $g_q(z)$ and $v(z)$ are arbitrary non-negative functions. The consistency conditions are automatically satisfied as proved above. The steady state single site weight $f(m_i)$ is simply, $f(m_i) = v(m_i)$, and naturally it does not depend on the site index i explicitly. The condition in Eq. (47) is precisely that derived by Greenblatt and Lebowitz. Thus, our general sufficiency condition, valid for an aribtrary graph $\mathcal{G}$, recovers this special case when $\mathcal{G}$ is a homogeneous hypercubic lattice with nearest neighbour mass transport.

4.2. ZRP on arbitrary graph

We can also check that the steady state for the continuous time zero-range process on an arbitrary graph is recovered. The zero-range process involves discrete masses and is specified by the rates for a a unit of mass to hop from site i to j. In the case where these rates are given by

$$\alpha_{ij}(1|m_i) = y_{ij}w_i(m_i) \ , \tag{48}$$

where $w_i(m_i)$ is the total rate for the unit mass leaving site i and y_{ij} is the probability that the random destination for a hop from site i is j, the steady state factorises with single-site weight

$$f_i(m_i) = \frac{p_i^{m_i}}{\prod_{n=1}^{m_i} w_i(n)} \tag{49}$$

where $p_i = \sum_{j\neq i} y_{ji}p_j$ is the steady state probability of a single random walker moving from site i to j with probability y_{ij} [11, 12].

To make the connection between the forms (48) and (44) we identify

$$x_{ij}(1) = y_{ij}p_i \quad \text{and} \quad w_i(m_i) = \frac{p_i v_{ii}(m_i - 1)}{v_{ii}(m_i)} \ . \tag{50}$$

Inverting the latter equality to express $v_{ii}(m_i)$ in terms of w_i yields, by virtue of (41), the single-site weight (49). Finally the consistency condition (42) becomes

$$\sum_{j\neq i} x_{ij}(1)v_{ii}(m-1) = \sum_{j\neq i} x_{ji}(1)v_{ii}(m-1), \tag{51}$$

and is satisfied with $x_{ij}(1) = y_{ij}p_i$ since we have

$$\sum_{j\neq i} x_{ij}(1) = \sum_{j\neq i} x_{ji}(1) \ . \tag{52}$$

5. Conclusion

In this work we have derived the sufficent condition (29) along with a consistency condition (31) for factorisation of the general (discrete time, continuous mass) mass transport model on an arbitrary graph. In this case the single-site weight is given by (22). We gave in section 3.1 specific examples of geometries where the additional consistency condition associated with the sufficient condition is automatically fulfilled. Moreover on a complete graph with permutationally invariant chipping functions we showed that the sufficent condition is, in fact, also necessary.

Of course a significant improvement would be to generalize condition (29,31) to a necessary and sufficient condition. To illustrate the challenges involved in accomplishing this task, we offer another simple example where we can derive a condition that is both necessary and sufficient.

This example, a seemingly trivial generalization of the one in [17, 18], involves a one-dimensional lattice with chipping to both nearest neighbour sites and we also assume that the chipping kernel is translationally invariant ($l = 1$ in Example 1 of section 3.1). Then, due to the translational invariance, (16) becomes

$$\prod_i g(s_i) = \prod_i X(s_{i-1}, s_i, s_{i+1}) \, . \tag{53}$$

Using a similar procedure to that employed in Example 4 of section 3.1 — taking the logarithm of (53) then successive derivatives with respect to s_i and s_j — we find that the general solution to (53) is

$$X(s_{i-1}, s_i, s_{i+1}) = K_{ii-1}(s_{i-1}) K_{ii}(s_i) K_{ii+1}(s_{i+1}) \frac{H(s_{i-1}, s_i)}{H(s_i, s_{i+1})} \, . \tag{54}$$

where K_{ii-1}, K_{ii}, K_{ii+1} and H are arbitrary functions, independent of i. Inverting the Laplace transforms to give $\mathcal{P}(\mu_{ii-1}, \mu_{ii}, \mu_{ii+1})$ would yield a considerably more complicated form for the chipping kernel $\varphi(\mu_{ii-1}, \mu_{ii}, \mu_{ii+1})$ than (29). In principle, therefore, there could be a whole family of chipping kernels, generated by the choice of H, which give rise to the same steady state (i.e. the same single-site weight $f(m)$) as that for $H = 1$. Thus, H could be thought of as a "gauge function".

In addition, for each choice of H, one has to ensure the consistency condition, namely that

$$\varphi(\mu_{ii-1}, \mu_{ii}, \mu_{ii+1}) = \frac{\mathcal{P}(\mu_{ii-1}, \mu_{ii}, \mu_{ii+1})}{f(m_i)} \delta(m_i - (\mu_{ii-1} + \mu_{ii} + \mu_{ii+1})) \tag{55}$$

is correctly normalized to unity. On the face of it, it may appear that this imposes a formidable constraint on the choice of H. However, we now show that this consistency condition does not impose any additional constraint on H. In other words, if the consistency condition is ensured for $H = 1$, then it is automatically satisfied for all other choices of H. To see this, consider the expression of the chipping kernel (55). Now, the denominator $f(m_i)$ is, of course, independent of the choice of H. So, to prove

that the consistency condition of normalization of φ does not impose any additional constraint on H, one has to prove that the integral

$$I(m) \equiv \int \mathrm{d}\mu_{ii-1}\mathrm{d}\mu_{ii}\mathrm{d}\mu_{ii+1}\mathcal{P}(\mu_{ii-1},\mu_{ii},\mu_{ii+1})\delta\left(m-\mu_{ii-1}-\mu_{ii}-\mu_{ii+1}\right) \quad (56)$$

is independent of the choice of H. Now, taking the Laplace transform of Eq. (56) one obtains

$$\begin{aligned}\int \mathrm{d}m e^{-sm} I(m) &= \int \mathrm{d}\mu_{ii-1}\mathrm{d}\mu_{ii}\mathrm{d}\mu_{ii+1} e^{-s(\mu_{ii-1}+\mu_{ii}+\mu_{ii+1})}\mathcal{P}(\mu_{ii-1},\mu_{ii},\mu_{ii+1}) \\ &= X(s,s,s) \\ &= K_{ii-1}(s)K_{ii}(s)K_{ii+1}(s) \end{aligned} \quad (57)$$

where we have used definition (16) in going from the first line to the second and Eq. (54) in going from the second line to the third. But this expression on the rhs does not contain the "gauge function" H thereby proving that the integral of $\mathcal{P}$ is independent of the choice of H. Thus the consistency condition is automatically fulfilled for arbitrary choices of H as long as it is ensured for $H = 1$, which is indeed the case as shown in Sec. 3.1 Example 1.

A more serious constraint on H is that the inverse Laplace transform of (54) must be *non-negative*. The case $H = 1$ in (54) obviously imposes the trivial constraint that the inverse Laplace transform of the K's be non-negative. It remains to be shown whether there is also a class of $H \neq 1$ which lead to valid chipping kernels. If it could be shown that this class is non-empty, the one-dimensional $l = 1$ example we have discussed would show explicitly that it is *not* necessary for φ to be of the form (29). Mapping the extent of this class of H (if indeed it is non-empty) remains a daunting task.

For an arbitrary graph, we have made some in-roads with similar considerations. Unfortunately, the most general solution to (16) remains difficult to formulate.

Finally we note that having a factorised steady state opens the door for the study of condensation as in [15, 16]. Thus one should be able to analyse condensation in various geometries or even on scale-free networks [20].

Acknowledgments

RKPZ acknowledges the support of the US National Science Foundation through DMR-0414122.

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