

# Maximum of $N$ Independent Brownian Walkers till the First Exit From the Half Space

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We consider the one-dimensional target search process that involves an immobile target located at the origin and  $N$  searchers performing independent Brownian motions starting at the initial positions  $\vec{x} = (x_1, x_2, \dots, x_N)$  all on the positive half space. The process stops when the target is first found by one of the searchers. We compute the probability distribution of the maximum distance  $m$  visited by the searchers till the stopping time and show that it has a power law tail:  $P_N(m|\vec{x}) \simeq B_N(x_1 x_2 \dots x_N)/m^{N+1}$  for large  $m$ . Thus all moments of  $m$  up to the order  $(N - 1)$  are finite, while the higher moments diverge. The prefactor  $B_N$  increases with  $N$  faster than exponentially. Our solution gives the exit probability of a set of  $N$  particles from a box  $[0, L]$  through the left boundary. Incidentally, it also provides an exact solution of the Laplace's equation in an  $N$ -dimensional hypercube with some prescribed boundary conditions. The analytical results are in excellent agreement with Monte Carlo simulations.

## I. INTRODUCTION

The probability distribution of the maximum of a single one-dimensional Brownian motion (and its variants such as a Brownian bridge or an excursion) over a fixed interval of time  $[0, t]$ , has a long history in the probability literature [1–7]. The statistics of the maximum has diverse applications. One example is the Kolmogorov-Smirnov test in statistics that is used to compare, in a nonparametric way, two different probability distributions [8, 9]. Similarly, the distribution of the global maximum of a discrete-time random flights (including Lévy flights) has also been studied in the probability literature [10, 11], with more recent applications in computer science [12], physics [13] and chemistry [14].

In statistical physics, there has been a recent revival of interest in related problems in the context of the distribution of the maximal height, measured with respect to a reference point, of  $(1 + 1)$ -dimensional fluctuating interfaces [15–17]. In the stationary state of a finite sample of size  $L$ , such fluctuating interfaces are often described by a Brownian bridge in space over an interval  $[0, L]$ , albeit with certain global constraints [16]. The statistics of maximum has also been computed for continuous-time subdiffusive processes [18, 19] and has been used to analyse single particle trajectories [20].

The distribution of the maximum for a single Brownian motion (or its variants such as bridge, excursion etc.) has been extended to many Brownian motions, including certain strongly interacting random walkers, e.g. non-intersecting, so-called vicious random walkers [21–24]. (The latter problem has an intriguing connection to the Gaussian ensembles of the random matrix theory [21, 24, 25].) For independent walkers, the results on the distribution of the maximum have recently been used to compute the mean perimeter and the mean area of the convex hull of  $N$  independent planar Brownian motions [26, 27].

These results on Brownian motion and its variants represent rare exact analytical results for the extreme value statistics of *correlated* random variables, a subject of increasing current interest [28]. However, all these results about the distribution of the maximum, for a single or multiple walkers, have been derived in the case when one considers the walkers over a fixed interval of time  $[0, t]$ . An interesting variation of this problem, with several applications, arises when the interval  $[0, t]$  is not fixed, but itself varies from realization to realization, i.e., one observes the walker (or walkers) over a time interval  $[0, t_s]$  where the stopping time  $t_s$  of the process itself is a random variable. For example,  $t_s$  may represent the first-passage time (through the origin) of a walker.

To be more precise, consider first a single Brownian walker that starts at time  $t = 0$  at position  $x_1 > 0$ . The position  $x_1(t)$  of the walker evolves via the continuous-time stochastic equation,  $dx_1/dt = \eta_1(t)$  where  $\eta_1(t)$  is a Gaussian white noise with mean  $\langle \eta_1(t) \rangle = 0$  and a correlator  $\langle \eta_1(t)\eta_1(t') \rangle = \delta(t - t')$ . The process stops at the stopping time  $t_s$  when the walker hits the origin for the first time (see Fig. 1).

Let  $m$  be the maximum displacement of the particle till the stopping time  $t_s$ . The statistics of the random variable  $m$  is interesting and it represents an example of the so called first-passage Brownian functional [29]. The problem is a toy model of ‘random search’, where the origin represents a fixed ‘target’ and the Brownian walker represents a random searcher. The search is called off when the searcher finds its target and  $m$  represents the maximum distance travelled by the searcher before it finds its target. For concreteness, we shall mostly use terminology related to random search, although there are several applications of this problem. For example, in the context of trapping [30–32] or predator-prey [33] models the origin may represent an immobile target (prey) and the Brownian walker may represent a diffusing chemical trap (predator). The stopping time  $t_s$  is then the reaction time or the survival time of the prey and  $m$  denotes the maximum distance the predator travels before finding its prey. In the context of the directed

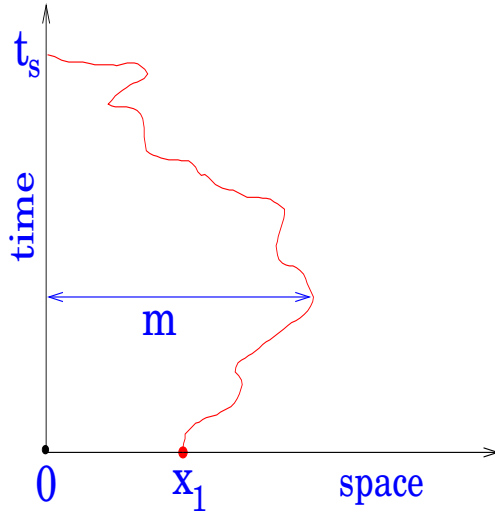


FIG. 1: The trajectory (red line) of a single Brownian walker starting initially at  $x_1$  till the stopping time  $t_s$  at which it hits the origin for the first time. The maximum distance travelled by the particle till  $t_s$  is denoted by  $m$ .

Abelian sandpile model in  $(1 + 1)$ -dimensions [34],  $m$  represents the maximum lateral size of an avalanche [35]. The random variable  $m$  also plays an important role in characterizing the so called staircase polygons [36]. In the context of queueing theory, where the position of the walker represents the length of a queue,  $m$  represents the maximum length of a queue during the so called busy period [35, 36].

The probability density function (pdf)  $P_1(m|x_1)$  of  $m$  (for fixed  $x_1$ ) can be easily computed [35] and it turns out to be a pure power law:

$$P_1(m|x_1) = \frac{x_1}{m^2}; \quad m \geq x_1. \quad (1)$$

While this pdf is evidently normalized to unity, the average  $\langle m \rangle$  and higher integer moments are infinite! The cumulative distribution of the maximum is given by

$$Q_1(L|x_1) = \text{Prob} [m \leq L|x_1] = \int_{x_1}^L P_1(m|x_1) dm = 1 - \frac{x_1}{L}. \quad (2)$$

This distribution has a very simple interpretation: it just represents the exit probability of a Brownian particle [37], starting at  $0 \leq x_1 \leq L$ , from a box  $[0, L]$  through its left boundary at 0.

In this paper, we study a generalization of this search problem where there is still one fixed target at the origin, but there are  $N$  searchers who perform independent Brownian motions on the  $x > 0$  axis, starting at the initial positions  $\vec{x} \equiv (x_1, x_2, \dots, x_N)$ . The position  $x_i(t)$  of the  $i$ -th walker evolves with time  $t$  via the Brownian evolution

$$\frac{dx_i}{dt} = \eta_i(t) \quad (3)$$

where  $\eta_i(t)$  is a Gaussian white noise satisfying  $\langle \eta_i(t) \rangle = 0$  and  $\langle \eta_i(t)\eta_j(t') \rangle = \delta(t - t')\delta_{i,j}$ . Since the walkers are independent they can cross each other. The process stops at a stopping time  $t_s$  when the origin is hit for the first time by any one of the walkers (e.g., the second walker (red) in Fig. 2). Note that  $t_s$  varies from one history of the process to another.

In the context of chemical kinetics [31] where the problem is generally referred to as the ‘target annihilation’ problem, various generalizations of this problem have been investigated including e.g. the situation where the target itself diffuses [38–42]. In the following, we shall limit ourselves to the case of immobile target and focus on the statistics of the maximum distance  $m$  (from the target) travelled by any of the walkers till the stopping time  $t_s$  when the target is found. Thus  $m$  denotes the distance of the farthest point on the  $x$  axis visited by any one of the walkers till  $t_s$ . Clearly  $m$  is a random variable fluctuating from one realization of the process to another. Our object of interest is the

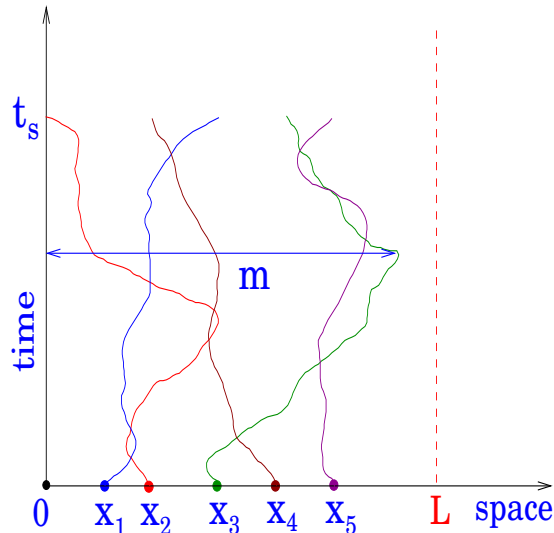


FIG. 2: The trajectories of  $N = 5$  independent Brownian walkers starting at initial positions  $x_1, x_2, x_3, x_4$  and  $x_5$  till the stopping time  $t_s$  when one of the walkers (the second one (red) in this figure) hits the origin. The maximum displacement along the  $x$  direction till  $t_s$  (undergone, e.g., by the third particle (green)) is denoted by  $m$ . The cumulative probability  $Q_N(L|\vec{x}) = \text{Prob}[m \leq L|\vec{x}]$  also represents the exit probability of the first particle from a box  $[0, L]$  through its left boundary.

probability density  $P_N(m|\vec{x})$  of this maximum distance  $m$ , given the number  $N$  of walkers and their initial positions  $\vec{x}$ . Thus  $m$  provides an estimate (worst-case) of the distance that needs to be covered by a team of  $N$  walkers to find a fixed target.

As in the single searcher case, let  $Q_N(L|\vec{x}) = \text{Prob}[m \leq L|\vec{x}] = \int_0^L P_N(m|\vec{x}) dm$  be the cumulative probability that the maximum  $m$  till  $t_s$  is less than or equal to  $L$ . This cumulative distribution of the maximum can be interpreted as the solution of a different problem as in the  $N = 1$  case. Consider, for instance, a slightly different problem where again we have a set of  $N$  independent walkers, but now inside a box  $[0, L]$ , starting at the initial positions  $\vec{x}$ . Let us define the exit probability as the probability that the first particle that exits the box  $[0, L]$  does so through 0 (and not through the upper boundary at  $L$ ), see Fig. 2. As in the  $N = 1$  case, this exit probability is precisely the cumulative distribution  $Q_N(L|\vec{x})$  of the maximum  $m$  till the stopping time in the semi-infinite system, as it counts all those events where one of the trajectories hits the lower boundary 0 before hitting the upper boundary at  $L$  while all the others stay inside the box  $[0, L]$  till this event of first-hitting the origin.

We will see that for this seemingly simple one-dimensional model of independent walkers, the statistics of  $m$  has a rich and nontrivial dependence on the number  $N$  of walkers. This is partly due to the fact that the same stopping time  $t_s$  for all the walkers effectively introduces a correlation between the trajectories of the walkers, even though each executes an independent Brownian motion. While for  $N = 1$  the solution is simple, it becomes rather nontrivial even for  $N = 2$ !

Let us first summarize our main results. We compute the pdf  $P_N(m|\vec{x})$  exactly for all  $N \geq 1$  by a path counting (or path integral) method. We show that, for arbitrary  $N \geq 1$ , the pdf of the maximum has an asymptotic power-law tail

$$P_N(m|\vec{x}) \simeq B_N \frac{x_1 x_2 \dots x_N}{m^{N+1}} \quad \text{as } m \rightarrow \infty \quad (4)$$

where the prefactor  $B_N$  has a nontrivial  $N$  dependence which we compute explicitly. For  $N = 1$ , we have  $B_1 = 1$  and the asymptotic result in (4) is actually valid exactly for all  $m \geq x_1$ . For  $N = 2$ , we will see that

$$B_2 = \frac{1}{4\pi^2} \left[ \Gamma\left(\frac{1}{4}\right) \right]^4 = 4.37688\dots \quad (5)$$

In particular, for large  $N$ , the prefactor  $B_N$  grows faster than exponentially

$$B_N \simeq N \left[ \frac{4}{\pi} \ln(N) \right]^{N/2}. \quad (6)$$

Our asymptotic result (4) indicates that for  $N$  walkers, integer moments of  $m$  up to order  $(N - 1)$  are finite, while higher integer moments are infinite. Evidently, as  $N$  increases, the distribution becomes narrower and narrower as expected but it does so in a nontrivial fashion.

Recently, the cumulative distribution of the maximum  $m$  till the first-passage time  $t_s$ , or equivalently the exit probability  $Q(L|x)$  from the box  $[0, L]$  through the origin, was studied [43] for a generic self-affine stochastic process  $x(t)$  starting at the initial position  $x$ . The process  $x(t)$  typically grows with time as  $x(t) \sim t^H$  where  $H$  is the Hurst exponent. This power law growth of distance with time makes the process self-affine. An example is the ordinary Brownian motion where  $H = 1/2$ . For such a generic self-affine process, it was argued [43] that the cumulative distribution of the maximum  $Q(L|x) = \text{Prob}(m \leq L|x) \sim 1 - A(x/L)^\phi$  in the limit  $x/L \rightarrow 0$  where  $A$  is a constant. The exponent  $\phi$  was found to be related to the persistence exponent  $\theta$  via the scaling relation  $\phi = \theta/H$  [43]. The persistence exponent  $\theta$  characterizes the late time power law decay of the survival probability, i.e., the probability that the process stays on the positive half-axis up to time  $t$  [44]. Thus the pdf of the maximum decays for large  $m$  as,  $P(m|x) \sim m^{-\phi-1}$  with  $\phi = \theta/H$ . The exact result (4) shows that if we think of the assembly of  $N$  independent Brownian motions as a single self-affine stochastic process in the  $N$ -dimensional space, then  $\phi = N$ . We will see later that the persistence exponent for this collective process is  $\theta = N/2$  and the Hurst exponent  $H = 1/2$ . Thus our exact result for this model supports the general scaling relation  $\phi = \theta/H$  found in [43].

The paper is organized as follows. In Section 2, we provide a simple heuristic argument in favour of our main result (4). This argument is not sufficient to compute the prefactor  $B_N$  exactly for all  $N$ . However, we show that this heuristic argument becomes asymptotically exact for large  $N$  and one can extract the limiting behavior of  $B_N$  for large  $N$  using an extreme value argument. In Section 3, we set up the general method for computing the cumulative distribution  $Q_N(L|\vec{x})$  of the maximum  $m$ . This requires solving Laplace's equation in an  $N$ -dimensional space with appropriate boundary conditions. We present explicit solutions for the cases  $N = 1$  and  $N = 2$ . In Section 4, we present an alternative path counting method that is more general, physically transparent and provides explicit results for all  $N \geq 1$ . In Section 5, we present numerical results to verify our analytical predictions. Finally in Section 6, we conclude with a summary and a list of interesting open problems. Some of the details of the computations are relegated to the appendices.

## II. HEURISTIC ARGUMENT

We begin with a simple heuristic argument in favour of (4). Consider the semi-infinite geometry with the independent Brownian motions, evolving via (3), starting at initial positions  $\{x_1, x_2, \dots, x_N\}$ . Let  $t_s$  be the stopping time when one of the walkers hits the origin. The probability distribution of  $t_s$  can be computed exactly as follows. As an input to solving the  $N$ -particle problem, let us first consider a single Brownian motion starting initially at  $x_0 > 0$  and let  $p(x_0, t)$  be the survival probability, i.e., the probability that the walker does not hit the origin up to time  $t$ . This can be easily computed by various standard methods and is given by the well known formula [29, 45]

$$p(x_0, t) = \text{erf}\left(\frac{x_0}{\sqrt{2t}}\right); \quad \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du. \quad (7)$$

Turning to  $N$  walkers we note that since the walkers are independent, the probability that none of them hits 0 up to time  $t$  is simply the product  $\prod_{i=1}^N p(x_i, t)$ . This is precisely the probability that the stopping time  $t_s > t$ . Thus, the cumulative distribution of the stopping time  $t_s$ , given the initial positions  $\vec{x}$ , reads [31, 39]

$$\text{Prob}[t_s > t|\vec{x}] = \prod_{i=1}^N \text{erf}\left(\frac{x_i}{\sqrt{2t}}\right). \quad (8)$$

For large  $t$ , this cumulative distribution has a power law tail

$$\text{Prob}[t_s > t|\vec{x}] \simeq \left(\frac{2}{\pi}\right)^{N/2} \frac{x_1 x_2 \dots x_N}{t^{N/2}}. \quad (9)$$

This asymptotic for survival probability tells us that the persistence exponent is  $\theta = N/2$ .

The result (8) and its asymptotic counterpart (9) are exact. Next comes the heuristic part. We note that for large  $t_s$ , the typical maximal displacement  $m$  in time  $t_s$  must scale as:  $m \sim \sqrt{t_s}$ . Taking this relationship between the two random variables  $m$  and  $t_s$  seriously, we see that  $\text{Prob}[m > L|\vec{x}] \sim \text{Prob}[t_s > L^2|\vec{x}]$  for large  $L$ . Since  $Q_N(L|\vec{x}) = \text{Prob}[m \leq L|\vec{x}]$ , we conclude that for large  $L$

$$1 - Q_N(L|\vec{x}) = \text{Prob}[m > L|\vec{x}] \sim \text{Prob}[t_s > L^2|\vec{x}] \sim \frac{x_1 x_2 \dots x_N}{L^N} \quad (10)$$

where we used the result in Eq. (9). Taking derivatives with respect to  $L$  and putting  $L = m$  then gives an approximate behavior of the probability density  $P(m|\vec{x})$  of the maximum for large  $m$

$$P_N(m|\vec{x}) \sim \frac{x_1 x_2 \dots x_N}{m^{N+1}} \quad (11)$$

as announced in (4). This heuristic scaling argument thus provides, up to an overall  $N$  dependent prefactor  $B_N$ , the leading asymptotic power law tail of the distribution of  $m$  in (4).

To compute the prefactor  $B_N$  exactly for any  $N$  one needs to go beyond this scaling argument. This requires a more sophisticated mathematical analysis that is carried out in the rest of the paper. However, it is possible to refine this heuristic argument, as shown below, that even provides the prefactor  $B_N$  exactly for large  $N$ .

In the argument above, the main approximation was to replace  $m \sim \sqrt{t_s}$  for large  $m$  and then use the exact asymptotic distribution of  $t_s$  in Eq. (9) to compute the tail of the distribution of  $m$ . This approximation clearly ignores fluctuations of  $m$  for a fixed  $t_s$ . We now use an extreme value argument to show that this approximation actually becomes exact for large  $N$ . We consider again a group of Brownian motions starting at the initial positions  $\vec{x}$  and examine their trajectories over a *fixed* time interval  $[0, t_s]$ , with  $m$  denoting their global maximum in  $[0, t_s]$ . To compute the cumulative probability  $Q_N(L, t_s|\vec{x}) = \text{Prob}[m \leq L|\vec{x}]$ , we consider the trajectories that stay below the level  $L$  till  $t_s$  and also above the level 0. Now, for large  $L$ , the trajectories that contribute to  $Q_N(L, t_s|\vec{x})$  typically have large excursions. So, to a first approximation, one can ignore the lower boundary at 0. For the  $i$ -th walker, starting at  $x_i$ , the probability that its maximum stays below  $L$  can be computed easily: it is just the survival probability  $p(x_0, t_s)$  in Eq. (7) with the initial position  $x_0 = L - x_i$ . Thus the joint probability that all walkers stay below  $L$  till  $t_s$  (ignoring the lower boundary at 0) is just the product

$$Q_N(L, t_s|\vec{x}) \simeq \prod_{i=1}^N \text{erf}\left(\frac{L - x_i}{\sqrt{2t_s}}\right). \quad (12)$$

For large argument, the error function behaves as  $1 - \text{erf}(x) = e^{-x^2}/(x\sqrt{\pi}) \simeq e^{-x^2}$  to leading order. Hence, for large  $L$  and large  $N$ , one can write

$$Q_N(L, t_s|\vec{x}) \simeq \exp\left[-N e^{-L^2/2t_s}\right] \rightarrow f[(L - a_N)/b_N] \quad (13)$$

where the scale factors  $a_N = \sqrt{2t_s \ln(N)}$  and  $b_N = \sqrt{t_s/(2 \ln N)}$  and the scaling function  $f(x) = \exp[-e^{-x}]$  is the standard Gumbel function. The pdf of  $m$  is just the derivative of the cumulative distribution. The derivative of the scaling function  $f'(x) = \exp[-x - e^{-x}]$  has a peak at  $x = 0$ . This indicates that the random variable  $m$  has a peak at  $m = a_N = \sqrt{2t_s \ln(N)}$  and the width of  $m$  around its peak is  $b_N = \sqrt{t_s/(2 \ln N)}$  that actually decreases with increasing  $N$ . Thus, for large  $N$ , the random variable  $m$  approaches to the its mean value  $a_N$  with probability 1, i.e.,  $m = \sqrt{2t_s \ln(N)}$  with fluctuations around this value essentially negligible for large  $N$ . Using this relation in Eq. (9) provides the following tail for the cumulative distribution of  $m$  for large  $N$

$$1 - Q_N(L|\vec{x}) = \text{Prob}[m > L|\vec{x}] \simeq \text{Prob}\left[t_s > \frac{L^2}{2} (\ln N)^{-2} |\vec{x}\right] \simeq A_N \frac{x_1 x_2 \dots x_N}{L^N} \quad (14)$$

with the prefactor  $A_N \simeq [4 \ln(N)/\pi]^{N/2}$ . Taking derivative with respect to  $L$  then gives the tail of the pdf of  $m$  in Eq. (4) with the prefactor, for large  $N$ ,

$$B_N = N A_N \simeq N \left[\frac{4}{\pi} \ln(N)\right]^{N/2}. \quad (15)$$

We will see later that the same asymptotic result also follows from a more rigorous approach.

### III. A BACKWARD FOKKER-PLANCK METHOD: LAPLACE'S EQUATION

In this section we show that the cumulative distribution  $Q_N(L|\vec{x}) = \text{Prob}[m \leq L|\vec{x}]$  of the maximum  $m$  satisfies a Laplace's equation in an  $N$ -dimensional hypercube with appropriate boundary conditions. To see this, it is first useful to consider  $Q_N(L|\vec{x}) \equiv Q_N(\vec{x}|L)$  as a function of the coordinates  $\vec{x}$  for a given fixed  $L$ . Note that  $Q_N(\vec{x}|L)$  is the probability that starting at  $\vec{x}$ , the maximum of the process till  $t_s$  stays below the level  $L$ . The idea is to derive a differential equation for  $Q_N(\vec{x}|L)$  using a backward approach where one focuses on the evolution of the system via

(3) over a small time interval  $[0, dt]$  starting from the initial positions  $\vec{x}$ . According to (3), in this small time interval  $dt$ , the  $i$ -th particle moves from  $x_i$  to  $x_i' = x_i + \eta_i(0)dt$  where  $\eta_i(0)$  is the noise at  $t = 0$  that kicks the  $i$ -th particle. Now, starting from this 'new' initial coordinates  $x_i'$  the maximum of the system has to subsequently stay below  $L$  till the stopping time. Finally, one must sum over all possible values of the new coordinates  $x_i'$ . Thus one must have

$$Q_N(x_1, x_2, \dots, x_N | L) = \langle Q_N(x_1 + \eta_1(0)dt, x_2 + \eta_2(0)dt, \dots, x_N + \eta_N(0)dt | L) \rangle \quad (16)$$

where the  $\langle \cdot \rangle$  denote the averages over the initial noises  $\eta_i(0)$ . Expanding the right hand side in a Taylor series and using (i)  $\langle \eta_i(0) \rangle = 0$  and (ii)  $\langle \eta_i(0)\eta_j(0) \rangle = 1/dt$  (which follows from the delta correlator), one finds that  $Q_N(\vec{x}|L)$  satisfies the Laplace's equation in the  $N$ -dimensional hypercube,  $0 \leq x_i \leq L$ ,

$$\nabla^2 Q_N(x_1, x_2, \dots, x_N | L) = 0. \quad (17)$$

The information about the maximum is captured in the boundary conditions. For example, if  $x_i = 0$  for any  $i$ ,  $Q_N = 1$  since if the  $i$ -th particle starts at the origin, the process stops immediately ( $t_s = 0$ ) and hence the maximum is necessarily (with probability 1) less than  $L$ . On the other hand, if  $x_i = L$  for any  $i$ , one has  $Q = 0$ . This follows from the fact that if the  $i$ -th particle starts at  $L$ , it will immediately cross the level  $L$  and the probability that the maximum will stay below  $L$  till  $t_s$  is necessarily zero.

To summarize,  $Q_N(\vec{x}|L)$  satisfies Laplace's equation (17) in an  $N$ -dimensional hypercube  $0 \leq x_i \leq L$  with  $Q = 1$  for any  $x_i = 0$  and  $Q = 0$  for any  $x_i = L$ . Thus, it reduces to an electrostatic problem where one needs to find the potential  $Q_N(\vec{x}|L)$  inside the hypercube  $[0, L]^N$ , whose  $N$  faces touching the origin are held at a constant potential  $Q_N = 1$  while the rest of the  $N$  faces are earthed ( $Q_N = 0$ ). We present the solutions explicitly for  $N = 1$  and  $N = 2$  in the next two subsections.

**Scaling:** Let us remark that since the only length scales are the initial positions  $\{x_i\}$  of the particles and the size of the box  $L$ , it is evident that the exit probability  $Q_N(\vec{x}|L)$  satisfies the scaling property

$$Q_N(\vec{x}|L) = Q_N\left(\frac{x_1}{L}, \frac{x_2}{L}, \dots, \frac{x_N}{L}\right) = Q_N(z_1, z_2, \dots, z_N) \quad (18)$$

where the dimensionless scaled variables  $0 \leq z_i = x_i/L \leq 1$ .

**Special initial condition and a duality relation:** It is useful to consider a special initial condition where all the particles start from the same initial positions:  $x_i = x$  for all  $1 \leq i \leq N$ . In this case, the exit probability is a function of a single scaled variable  $0 \leq z = x/L \leq 1$ :  $Q_N(x_1 = x, x_2 = x, \dots, x_N = x | L) = q_N(x/L)$  where the scaling function  $q_N(z)$  satisfies

$$q_N(z) + q_N(1 - z) = 1. \quad (19)$$

This duality relation states that the exit probability through the right boundary at  $L$  starting from the initial position  $L - x$  (of all the particles) is exactly identical to the exit probability through the left boundary 0 starting from the initial positions  $x$  (of all the particles). The duality relation (19) in particular dictates that

$$q_N(z = 1/2) = 1/2 \quad (20)$$

for all  $N$ . The general solution of the Laplace's equation must satisfy Eq. (19) which actually provides a useful check for the validity of the solution.

### A. $N = 1$

For  $N = 1$ , we have a second order ordinary differential equation

$$\frac{d^2 Q_1}{dx_1^2} = 0 \quad \text{with } Q_1(0|L) = 1 \text{ and } Q_1(L|L) = 0, \quad (21)$$

whose solution is

$$Q_1(x_1|L) = 1 - \frac{x_1}{L}; \quad 0 \leq x_1 \leq L \quad (22)$$

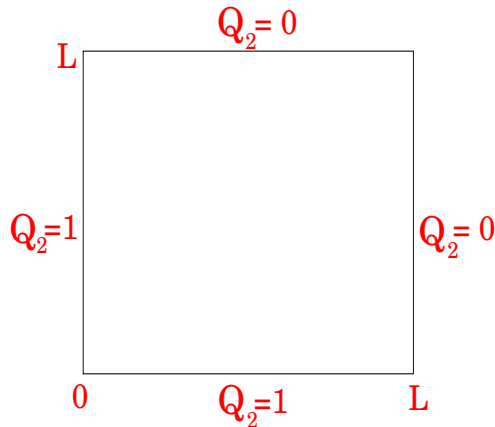


FIG. 3: Laplace's equation to be solved in a square ( $L \times L$ ) with boundary conditions as shown.

Since  $Q_1(x_1|L) = Q_1(L|x_1) = \text{Prob}[m \leq L|x_1]$ , it follows, by differentiation that the probability density of the maximum  $m$  has a strict power law form for all  $m$

$$P_1(m|x_1) = \frac{x_1}{m^2} \quad \text{for } m \geq x_1 \quad (23)$$

which is normalized to unity over  $m \in [x_1, \infty]$  and all its integer moments diverge [35]. Thus the prefactor  $B_1 = 1$  in (4).

Clearly for  $N = 1$ , the exit probability  $Q_1(x_1|L)$  is only a function of the scaled variable  $z = x_1/L$ :  $Q_1(x_1|L) = q_1(x_1/L)$  where the scaling function  $q_1(z)$  is simple

$$q_1(z) = 1 - z \quad (24)$$

and evidently it satisfies the duality relation (19).

### B. $N = 2$

The solution of the Laplace's equation becomes more involved in higher dimensions  $N > 1$ . For  $N = 2$ , one needs to solve Laplace's equation in an ( $L \times L$ ) square with boundary conditions shown in Fig. 3. The solution can be written down explicitly for this case, although the expression is rather cumbersome. Using separation of variables one finds the appropriate solution [46]

$$Q_2(x_1, x_2|L) = 1 - \frac{x_1}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{[\sinh(n\pi(1 - \frac{x_2}{L})) + (-1)^n \sinh(\frac{n\pi x_2}{L})]}{\sinh(n\pi)} \sin\left(\frac{n\pi x_1}{L}\right). \quad (25)$$

It is straightforward to verify that (25) is a solution of the Laplace's equation. Next we need to check that it satisfies the 4 boundary conditions (see Fig. 3). It is easy to check the two conditions: (i)  $Q_2(x_1 = 0, x_2|L) = 1$  for all  $x_2$  and (ii)  $Q_2(x_1 = L, x_2|L) = 0$  for all  $x_2$ . The other two conditions can also be verified. For instance, putting  $x_2 = L$  in (25) we get

$$Q_2(x_1, L|L) = 1 - \frac{x_1}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x_1}{L}\right). \quad (26)$$

Using the identity

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x_1}{L}\right) = \frac{\pi}{2} \left(1 - \frac{x_1}{L}\right) \quad (27)$$

we verify that  $Q_2(x_1, L|L) = 0$ . Similarly, putting  $x_2 = 0$  in (25) and using the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi x_1}{L}\right) = \frac{\pi x_1}{2L} \quad (28)$$

one verifies the last boundary condition  $Q_2(x_1, x_2 = 0|L) = 1$ . Note that even though the solution  $Q_2(x_1, x_2|L)$  in (25) is not manifestly symmetric under the exchange of  $x_1$  and  $x_2$ , it is actually symmetric in  $x_1$  and  $x_2$  as it should be. Later in Section 4, we will derive an alternative expression via the path integral method which is manifestly symmetric in  $x_1$  and  $x_2$ .

Let us analyse the large  $L$  behavior of (25) in the situation when  $x_1$  and  $x_2$  are kept finite. Since  $Q_2(x_1, x_2|L) = Q_2(x_1/L, x_2/L)$  is a function of only the scaled variables  $z_1 = x_1/L$  and  $z_2 = x_2/L$ , the  $L \rightarrow \infty$  limit is equivalent to taking the limits  $z_1 \rightarrow 0$  and  $z_2 \rightarrow 0$ . Clearly  $Q_2(z_1 = 0, z_2 = 0) = 1$ . It is easy to check that the first derivatives  $\partial_{z_1} Q$  and  $\partial_{z_2} Q$  vanish at the origin ( $z_1 = 0, z_2 = 0$ ). Similarly, the second derivatives  $\partial_{z_1}^2 Q_2$  and  $\partial_{z_2}^2 Q_2$  also vanish at the origin. So, in a Taylor expansion of  $Q_2(z_1, z_2)$  the first nonzero term is the cross derivative, indicating the following leading order behavior as  $z_1 \rightarrow 0$  and  $z_2 \rightarrow 0$

$$Q_2(z_1, z_2) = 1 - A_2 z_1 z_2 + \dots \quad (29)$$

The amplitude  $A_2$  is given by

$$A_2 = \frac{\partial^2 Q_2}{\partial z_1 \partial z_2} \Big|_{z_1=0, z_2=0} = 2\pi \left[ \sum_{n=1}^{\infty} (-1)^n n \coth(n\pi) - \sum_{n=1}^{\infty} \frac{n}{\sinh(n\pi)} \right] \quad (30)$$

where the last line follows from (25). Fortunately, the sums can be carried out explicitly using some known identities [47] to yield

$$A_2 = \frac{1}{8\pi^2} [\Gamma(1/4)]^4 = 2.18844\dots \quad (31)$$

Hence for large  $L$

$$\text{Prob}[m \leq L|x_1, x_2] = Q_2(x_1, x_2|L) \simeq 1 - A_2 \frac{x_1 x_2}{L^2}. \quad (32)$$

which leads to the announced power law tail for the probability density of the maximum  $m$

$$P_2(m|x_1, x_2) \simeq B_2 \frac{x_1 x_2}{m^3}; \quad \text{with } B_2 = 2A_2 = 4.37688\dots \quad (33)$$

To compare with the  $N = 1$  case, let us consider the special initial condition where both particles start from the same initial position:  $x_1 = x_2 = x$ . In this case, the exit probability  $Q_2(x_1 = x, x_2 = x|L) = q_2(x/L)$  with scaling function  $q_2(z)$  given by

$$q_2(z) = 1 - z + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{[\sinh(n\pi(1-z)) + (-1)^n \sinh(n\pi z)]}{\sinh(n\pi)} \sin(n\pi z). \quad (34)$$

One can verify that  $q_2(z)$  satisfies the duality relation (19). Near  $z = 0$  and  $z = 1$ , the scaling function has the asymptotics

$$q_2(z) \simeq \begin{cases} 1 - A_2 z^2 & \text{as } z \rightarrow 0, \\ A_2 (1-z)^2 & \text{as } z \rightarrow 1. \end{cases} \quad (35)$$

Comparing with the expression for the  $N = 1$  case, Eq. (24), one finds that  $q_2(z)$  has a much richer functional form. An analytic prediction for the function  $q_2(z)$  is in excellent agreement with results obtained from Monte Carlo simulations (Fig. 5).

#### IV. EXACT SOLUTION FOR ALL $N$ BY PATH COUNTING METHOD

The approach based on the Laplace equation (Sect. III) is difficult to extend for  $N > 2$ . The technical problem is to find the exact solution of the Laplace equation  $\nabla^2 Q_N(z_1, z_2, \dots, z_N) = 0$  in the  $N$ -dimensional hypercube of the



scaled variables  $0 \leq z_i = x_i/L \leq 1$ , with the prescribed boundary conditions that  $Q_N = 1$  for any  $z_i = 0$  and  $Q_N = 0$  for any  $z_i = 1$ . For  $N > 2$ , it is not easy to find an explicit solution to this problem. In this section, we use an alternative path counting method that is physically more explicit and, in addition, allows us to write down the exact solution  $Q_N(z_1, z_2, \dots, z_N)$  for all  $N$ . This method thus provides an alternative way to solving the Laplace's equation in a hypercube with the prescribed boundary conditions.

To set up the path counting method for general  $N$ , we need two basic ingredients from the single particle problem in a box  $[0, L]$  with absorbing boundary conditions at the two boundaries 0 and  $L$ :

(i) the survival probability  $S(x_0, t, L)$  which counts the probability that a single particle, starting at  $x_0$  at time 0, remains inside the box  $[0, L]$  during the time interval  $(0, t)$ .

(ii) the first passage probability density  $F(x_0, t, L)$  that denotes the probability density that the particle, starting initially at  $x_0$  ( $0 \leq x_0 \leq L$ ), exits for the first time the box through the boundary at 0 (and not through the other boundary at  $L$ ) at time  $t$ .

One can write the survival probability  $S(x_0, t, L) = \int_0^L G(x, x_0, t, L) dx$  where  $G(x, x_0, t, L)$  denotes the Green's function counting the probability density that the particle reaches  $x$  at time  $t$ , starting from  $x_0$  at time 0, while staying inside the box  $[0, L]$  during time  $t$ . This Green's function can be computed (a) either by solving the diffusion equation  $\partial_t G = D \partial_x^2 G$  (the diffusion constant is  $D = 1/2$  for our choice of the noise term) with absorbing boundary condition  $G(x = 0, x_0, t, L) = G(x = L, x_0, t, L) = 0$  and the initial condition  $G(x, x_0, 0, L) = \delta(x - x_0)$ , or (b) by the path integral method. The resulting Green's function admits the following representation

$$G(x, x_0, t, L) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right) e^{-\frac{n^2\pi^2}{2L^2}t}. \quad (36)$$

The survival probability, after integrating over the final position  $x$ , is

$$S(x_0, t, L) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n} \sin\left(\frac{n\pi x_0}{L}\right) e^{-\frac{n^2\pi^2}{2L^2}t}. \quad (37)$$

To calculate the first passage probability density  $F(x_0, t)$  through 0 at time  $t$ , we note that this just counts the flux of particles going out of the box through 0 at time  $t$ . The flux through a point  $x$  in the positive direction is the probability current  $-D \partial_x G$  (with  $D = 1/2$ ). Hence, the first-passage probability through the origin that counts the flux through the origin in the negative direction is simply,  $F(x_0, t) = D \partial_x G|_{x=0}$ . Using  $G$  from (36), we get

$$F(x_0, t) = \frac{\pi}{L^2} \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi x_0}{L}\right) e^{-\frac{n^2\pi^2}{2L^2}t}. \quad (38)$$

Armed with these two ingredients from the single particle problem, we are now ready to compute the exit probability  $Q_N(\vec{x}|L)$  for the  $N$ -particle problem. Consider first the event shown in Fig. 2 where one of the particles (say the  $i$ -th one) exits the box for the first time between time  $t_s$  and  $t_s + dt_s$  while the  $N - 1$  other particles stay inside  $[0, L]$  till this time  $t_s$ . Clearly, the probability for this event, using independence of walkers, is given by the product  $F(x_i, t_s) \prod_{j \neq i} S(x_j, t_s) dt_s$ . Now, the particle that hits 0 (whose label is  $i$ ) can, in fact, be any one of the  $N$  particles. Hence we have to sum over the hitting index  $i$  from 1 to  $N$ . In addition, this event can occur at any time  $t_s$ , so we need to integrate over  $t_s$ . This path counting method then gives the following net contribution to the exit probability

$$Q_N(\vec{x}|L) = \sum_{i=1}^N \int_0^{\infty} F(x_i, t_s) \prod_{j \neq i} S(x_j, t_s) dt_s. \quad (39)$$

Substituting the results for  $S(x, t)$  and  $F(x, t)$  respectively from Eqs. (37) and (38), integrating over  $t_s$  and setting  $z_i = x_i/L$  gives our main exact result, valid for all  $N$ ,

$$Q_N(\vec{x}|L) = Q_N(z_1, z_2, \dots, z_N) = \left(\frac{2}{\pi}\right)^N \sum_{\{n_i\}} \frac{a(n_1, n_2, \dots, n_N)}{n_1^2 + n_2^2 + \dots + n_N^2} \prod_{k=1}^N \frac{\sin(n_k \pi z_k)}{n_k} \quad (40)$$

where each index  $n_i = 1, 2, \dots$  for all  $1 \leq i \leq N$  and

$$a(n_1, n_2, \dots, n_N) = \sum_{i=1}^N n_i^2 \prod_{j \neq i} [1 - (-1)^{n_j}]. \quad (41)$$

Equation (40) is an exact solution of the Laplace's equation in the  $N$ -dimensional hypercube  $0 \leq z_i \leq 1$  with the prescribed boundary conditions which has a virtue of being manifestly symmetric with respect to the interchange of the coordinates  $z_i$ 's.

To extract the behavior in the limit of large  $L$ , with  $x_i$ 's fixed, we need to take the limit  $z_i \rightarrow 0$  in Eq. (40). This limiting behavior reads

$$Q_N(z_1, z_2, \dots, z_N) = 1 - A_N z_1 z_2 \dots z_N + O(z_i^2) \quad (42)$$

The coefficient  $A_N$  can be computed by taking the derivatives of Eq. (40) with respect to each  $z_i$  and then putting  $z_i = 0$  for all  $i$ . This gives

$$A_N = -N 2^N \lim_{z_i \rightarrow 0} \sum_{\{n_i\}} \frac{n_1^2 \cos(n_1 \pi z_1)}{n_1^2 + n_2^2 + \dots + n_N^2} \prod_{j=2}^N [1 - (-1)^{n_j}] \cos(n_j \pi z_j). \quad (43)$$

Noting that  $Q_N(\vec{x}|L) = Q_N(z_1, z_2, \dots, z_N)$  is precisely the cumulative distribution of the maximum  $m$ , one thus gets

$$\text{Prob}[m \leq L|\vec{x}] \simeq 1 - A_N \frac{x_1 x_2 \dots x_N}{L^N} \quad \text{as } L \rightarrow \infty. \quad (44)$$

Differentiating with respect to  $L$  and putting  $L = m$  gives the exact power-law tail of the pdf of the maximum  $m$  in Eq. (4) with the prefactor  $B_N$  is given by the formal sum

$$B_N = N A_N = -N^2 2^N \lim_{z_i \rightarrow 0} \sum_{\{n_i\}} \frac{n_1^2 \cos(n_1 \pi z_1)}{n_1^2 + n_2^2 + \dots + n_N^2} \prod_{j=2}^N [1 - (-1)^{n_j}] \cos(n_j \pi z_j) \quad (45)$$

where each index  $n_i$  runs over all positive integers.

The formal sum in Eq. (45) can be explicitly evaluated for  $N = 1$  and  $N = 2$ . For  $N = 1$

$$B_1 = -2 \lim_{z_1 \rightarrow 0} \sum_{n_1=1,2,\dots} \cos(n_1 \pi z_1) = 1 \quad (46)$$

in agreement with already known results. For  $N = 2$ , equation (45) gives

$$B_2 = -16 \lim_{z_i \rightarrow 0} \sum_{n_1, n_2} \frac{n_1^2 \cos(n_1 \pi z_1)}{n_1^2 + n_2^2} [1 - (-1)^{n_2}] \cos(n_2 \pi z_2) = \frac{1}{4\pi^2} \left[ \Gamma\left(\frac{1}{4}\right) \right]^4 = 4.37688 \dots \quad (47)$$

In Appendix A, we show how to compute the above sum explicitly.

For  $N > 2$ , we haven't deduced explicit expressions for the sum in (45). However, one can reduce it to a simpler form where the sum is rapidly convergent and can then be evaluated by Mathematica. As an example, for  $N = 3$ ,

$$B_3 = -6 + \frac{9}{8\pi^2} \Gamma^4(1/4) + 72\pi \sum_{n_1, n_2} \frac{n_1^2 + n_2^2 (-1)^{n_2}}{\sqrt{n_1^2 + n_2^2} \sinh\left(\pi \sqrt{n_1^2 + n_2^2}\right)} = 15.3369 \dots \quad (48)$$

In the limit of large  $N$ , one can evaluate the formal sum (see Appendix B) to obtain the limiting behavior

$$B_N \simeq N \left[ \frac{4}{\pi} \ln(N) \right]^{N/2} \quad (49)$$

in perfect agreement with the heuristic result in Eq. (15).

**Special initial condition:** Finally, let us consider the special initial condition when all the particles start from the same point:  $x_1 = x_2 = \dots = x$  where  $0 \leq x \leq L$ . In this case, the exit probability in Eq. (40) reduces to a function of one scaled variable  $z = x/L$ :  $Q_N(x, x, \dots, x|L) = q_N(z)$  given by

$$q_N(z) = N \left( \frac{2}{\pi} \right)^N \sum_{\{n_i\}} \frac{n_1 \sin(n_1 \pi z)}{(n_1^2 + n_2^2 + \dots + n_N^2)} \prod_{j \neq 1} [1 - (-1)^{n_j}] \frac{\sin(n_j \pi z)}{n_j}. \quad (50)$$

One can check that  $q_N(z)$  satisfies the duality relation  $q_N(z) + q_N(1-z) = 1$  and thus  $q_N(1/2) = 1/2$  for all  $N$ .

Performing this multiple sum directly by Mathematica is difficult as it converges slowly. To circumvent this problem, we first perform the sum over  $n_1$  in Eq. (50) using the following identity [47]

$$\sum_{k=1}^{\infty} \frac{k \sin(k\pi x)}{k^2 + a^2} = \frac{\pi}{2} \frac{\sinh(\pi(x-a))}{\sinh(\pi a)}. \quad (51)$$

This gives

$$q_N(z) = N \left(\frac{4}{\pi}\right)^{N-1} \sum_{n_2, n_3, \dots, n_N \rightarrow \text{odd}} \frac{\sinh\left[\pi(1-z)\sqrt{n_2^2 + n_3^2 + \dots + n_N^2}\right]}{\sinh\left[\pi\sqrt{n_2^2 + n_3^2 + \dots + n_N^2}\right]} \prod_{j=2}^N \frac{\sin(n_j \pi z)}{n_j}. \quad (52)$$

The multiple sum in Eq. (52) is now rapidly convergent and can be easily evaluated by Mathematica. In Fig. 2 we plot this function  $q_N(z)$  for  $N = 2$  and  $N = 3$ . For  $N = 2$ , it of course coincides with the earlier expression (34) obtained via the Laplace's method in the previous section.

**Average Maximum:** For fixed identical initial position of all particles  $x_i = x$ , another interesting question is how does the average maximum (till the stopping time  $t_s$ ) depend on  $N$ ? For  $N = 1$  the average is infinite but for all  $N > 1$  it is finite. However, does the average maximum for  $N > 1$  increase or decrease as the number of walkers  $N$  increases? The answer to this question is not intuitively obvious. However, knowing the function  $q_N(z)$ , one can compute the average maximum in the following way. We have,  $\text{Prob}[m \leq L|x] = q_N(x/L)$  where  $q_N(z)$  is given in Eq. (52). Therefore the probability density function of  $m$  reads

$$P_N(m|x) = -\frac{x}{m^2} q'_N\left(\frac{x}{m}\right); \quad m \geq x \quad (53)$$

where  $q'_N(z) = dq_N(z)/dz$ . The first moment is then given by, for all  $x \geq 0$  and  $N > 1$

$$\langle m \rangle = -\int_x^{\infty} \frac{x}{m} q'_N\left(\frac{x}{m}\right) dm = C_N x \quad (54)$$

with prefactor

$$C_N = -\int_0^1 \frac{q'_N(z)}{z} dz. \quad (55)$$

Thus the average maximum, for  $N > 1$ , is proportional to  $x$  for all  $x$  and the proportionality constant  $C_N$  is given by Eq. (55). We were unable to carry out the integral in Eq. (55) in closed form. However, it is clear that as  $N \rightarrow \infty$ , using  $q_N(z) = \theta(1/2 - z)$ , one gets  $C_N \rightarrow 2$ . On the other hand  $C_N$  diverges as  $N \rightarrow 1$ . Thus,  $C_N$  decreases when  $N$  increases. These results are supported by Monte Carlo simulations (Fig. 8).

## V. NUMERICAL SIMULATIONS

Our analytical result for the exit probability through the origin  $q_N(z)$ , when all particles start at the same scaled position  $0 \leq z = x/L \leq 1$ , is tested by Monte Carlo simulations. For a fixed box of size  $L$ , the exit probability for the  $N$ -particle problem can be efficiently computed using a method proposed in Ref. [43]. Naively, to compute  $q_N(z)$ , one would first fix the starting point  $0 \leq z \leq 1$  of all the Brownian motions and then generate different realizations of the process and compute the fraction of realizations where the first exit happens through the origin. One would then repeat the procedure for each value of  $0 \leq z \leq 1$  (with an appropriate bin size) in order to compute the full function  $q_N(z)$  over the range  $z \in [0, 1]$ . Instead of repeating the simulation for each starting point  $z$ , it turns out to be more efficient to follow a different algorithm described briefly below.

For illustration, we take the example of just one Brownian motion ( $N = 1$ ). The method is easily generalized for all  $N$ . We start the Brownian motion at the origin, let it evolve in time and record the maximal ( $x_{\max}$ ) and the minimal ( $x_{\min}$ ) position reached by the walker up to time  $t$ . The process is halted when  $x_{\max} - x_{\min} \geq L$  for the first time (see Fig. 4). Keeping  $x_{\min}$  and  $x_{\max}$  fixed, we now horizontally slide the whole configuration thereby changing the starting point (see the second and the third panel of Fig. 4). Measuring all distances with respect to  $x_{\min}$ , it is then clear that this configuration contributes 1 to  $q_1(z)$  for  $z \in [0, 1 - x_{\max}/L]$  and 0 for  $z \in [1 - x_{\max}/L, 1]$ . So, for this configuration, we just record the number  $y = 1 - x_{\max}/L$  and the fact that it contributes  $\theta[y - z]$  to  $q_1(z)$ .

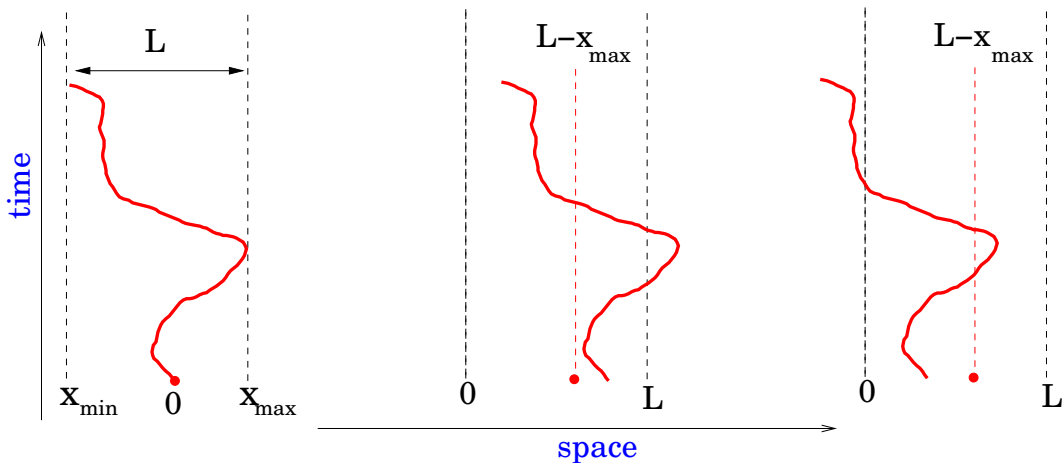


FIG. 4: The illustration of the algorithm to compute the exit probability  $q_N(z)$  for a single ( $N = 1$ ) Brownian motion.

We then repeat this procedure for another configuration starting at 0, do the sliding and record the value of  $y$  and the associated  $\theta(y - z)$ . We repeat this, say, for  $N_s$  number of samples. To sum all the contributions, we first sort the values of  $y$  associated with the configurations in increasing order. Let  $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{N_s}\}$  denote the sorted values. Then, we assign  $q_1(\tilde{y}_1) = 1$ ,  $q_1(\tilde{y}_2) = 1 - 1/N_s$  and in general  $q_1(\tilde{y}_i) = 1 - (i - 1)/N_s$ . This generates the full curve  $q_1(z)$ .

Thus this algorithm has two advantages: (i) it does not require to repeat the simulations for each value of  $z$  and (ii) it does not require any specific choice of bin sizes. This clearly makes the simulation much faster. The results of our simulations for  $N = 2$ ,  $N = 3$  and  $N = 6$  are plotted in Fig. (5). The agreement with the analytical prediction is excellent. For all  $N$ ,  $q_N(z)$  satisfies the duality relation:  $q_N(z) + q_N(1 - z) = 1$ .

In Ref. [43] we had studied the exit probability from a box  $[0, L]$  through the upper boundary  $L$  (rather than the lower boundary  $0$  considered here) for a single particle whose motion is described by a generic self-affine stochastic process. This probability was called the ‘hitting probability’ in [43]. In the present paper, we are considering the complimentary event of exiting the box  $[0, L]$  through the lower boundary  $0$ . In the notation of the present paper, the hitting probability (of the boundary  $L$ ) would correspond to  $1 - q_N(z)$  [50]. For a generic self-affine process, it was shown in Ref. [43] that the exit probability should have a power law behavior close to the origin  $1 - q(z) \sim z^\phi$  with  $\phi = \theta/H$  as mentioned in the introduction. In addition, it was observed in ref. [43] that for many processes (but not all), once we know the exponent  $\phi$ , the full function  $q(z)$  over the range  $z \in [0, 1]$  is described by a universal one-parameter (parametrized by  $\phi$ ) form [43]

$$q_\phi(z) = 1 - I_z(\phi, \phi) = 1 - \frac{\Gamma(2\phi)}{\Gamma(\phi)^2} \int_0^z [u(1-u)]^{\phi-1} du. \quad (56)$$

The function  $I_z(\phi, \phi)$  is the incomplete regularized beta function. In our present problem, we have seen in Eq. (42) that for small  $z$ ,  $q_N(z) \sim 1 - A_N z^N$  indicating  $\phi = N$ . It is then natural to investigate if our result for the full function  $q_N(z)$  can be re-expressed as the universal functional form in Eq. (56) with  $\phi = N$ . Interestingly, the answer is no, as it is clearly shown in Fig 6, thus providing us with a counterexample.

We have also computed the prefactor  $B_N$  numerically from the Monte Carlo simulations up to  $N = 7$ . The results are shown in Fig. 7 by squares. For  $N = 1$ ,  $N = 2$  and  $N = 3$ , they agree with our exact analytical predictions  $B_1 = 1$ ,  $B_2 = 4.37688\dots$  and  $B_3 = 15.3369\dots$  as discussed in Section IV. Had the  $q_N(z)$  be described by the universal function in Eq. (56) with  $\phi = N$ , one would get from the small  $z$  expansion in Eq. (56),  $q_N(z) \rightarrow 1 - A_N z^N$  with  $A_N = \Gamma(2N)/N\Gamma^2(N)$ . This would predict the prefactor  $B_N = NA_N = \Gamma(2N)/\Gamma^2(N)$ . In Fig. 7, this prediction from the universal curve is shown by the dashed line. Clearly, it does not match the simulation results, confirming once more that  $q_N(z)$  is not described by the universal function in Eq. (56). We have also plotted the exact asymptotic prediction of  $B_N$  in Eq. (49) as a solid line for comparison. While it is difficult to extract the small  $z$  behavior of  $q_N(z)$  and hence  $B_N$  for larger values of  $N$ , we notice that the asymptotic large  $N$  behavior is already approached for  $N = 7$ .

Finally we have also computed the average maximum (till the stopping time  $t_s$ ) of  $N$  Brownian motions starting from the same initial positions  $x > 0$ . We verified that for all  $N > 1$ , the average maximum exists, and it is proportional to  $x$ , as predicted analytically in Eq. (54). In Fig. 8 we plot the amplitude  $C_N$ , given analytically in Eq. (55), as a function of  $N$ . As predicted,  $C_N$  approaches 2 for large  $N$ .

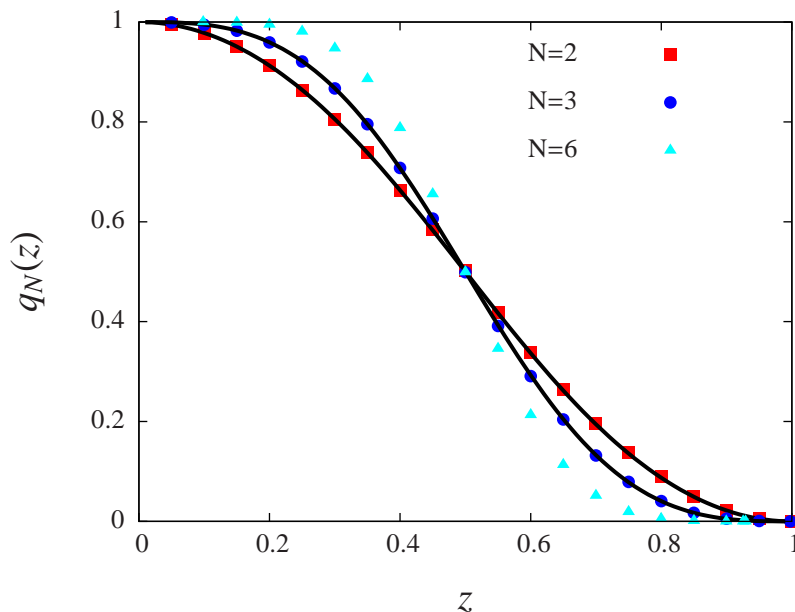


FIG. 5: The exit probability  $q_N(z)$  through the origin obtained from Monte Carlo simulations for  $N = 2$  (red squares),  $N = 3$  (deep blue circles) and  $N = 8$  (blue triangles). The numerical data are compared to analytical results (solid lines) for  $N = 2$  and  $N = 3$  (obtained from Eq. (52) respectively for  $N = 2$  and  $N = 3$ ).

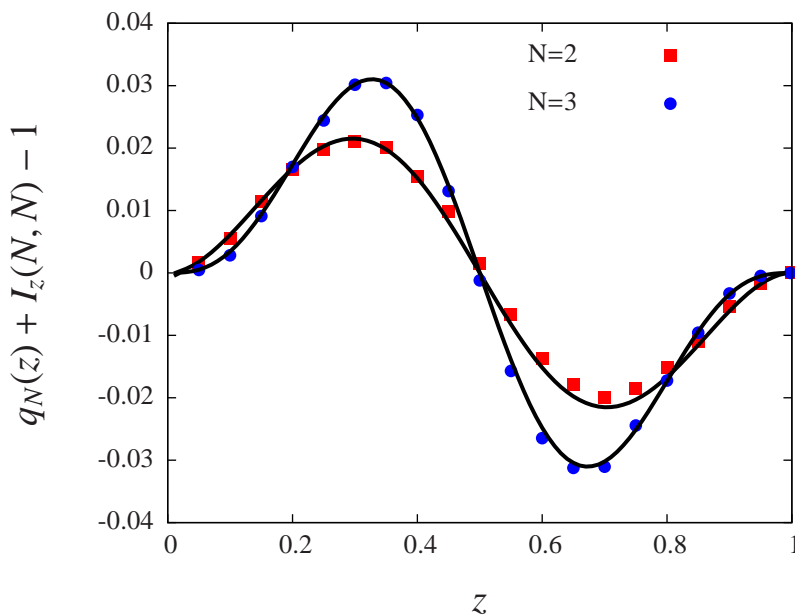


FIG. 6: Difference between the exit probability for  $N = 2$  and  $N = 3$  and the universal function in Eq. (56) with  $\phi = N$ . Solid lines are the analytical results from Eq. (52) with  $N = 2$  and  $N = 3$ . Symbols are the numerical results for  $10^5$  realizations of  $N = 2$  (squares) and  $N = 3$  (circles) Brownian motions starting from the same initial position in a box of size  $L = 1000$ .

## VI. CONCLUSION

To summarize, we have presented an exact solution for the probability distribution of the maximum  $m$  of a set of  $N$  independent Brownian motions starting at the initial positions  $\vec{x} \equiv \{x_1, x_2, \dots, x_N\}$  on the positive half-axis and the process terminating when any one of the walkers crosses the origin. We have shown that for large  $m$ , the pdf of  $m$  decays as a power law,  $P_N(m|\vec{x}) \simeq B_N(x_1 x_2 \dots x_N)/m^{N+1}$  where the prefactor  $B_N$  has an interesting  $N$  dependence. For a fixed  $N > 1$ , integer moments of  $m$  up to order  $(N - 1)$  are finite, while all higher integer moments are infinite. The cumulative distribution of this maximum also provides an exact solution to the first-exit probability

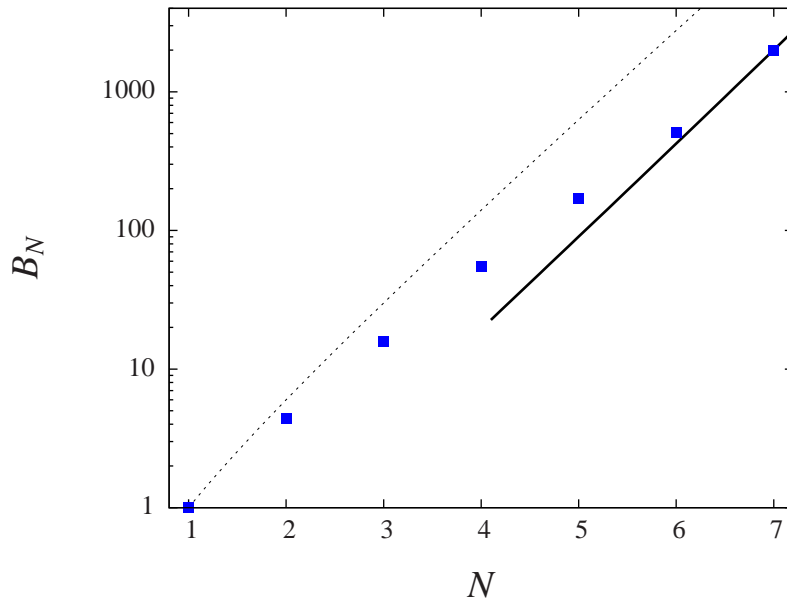


FIG. 7: Dashed line:  $B_N = \Gamma(2N)/\Gamma^2(N)$  from the universal function in Eq. (56). Solid line: asymptotic behavior for large  $N$  given in Eq.(49) and squares: direct simulations of  $N$  Brownian motions starting from the same position in a box of size  $L = 1000$ . Averages are performed over  $10^6$  samples.

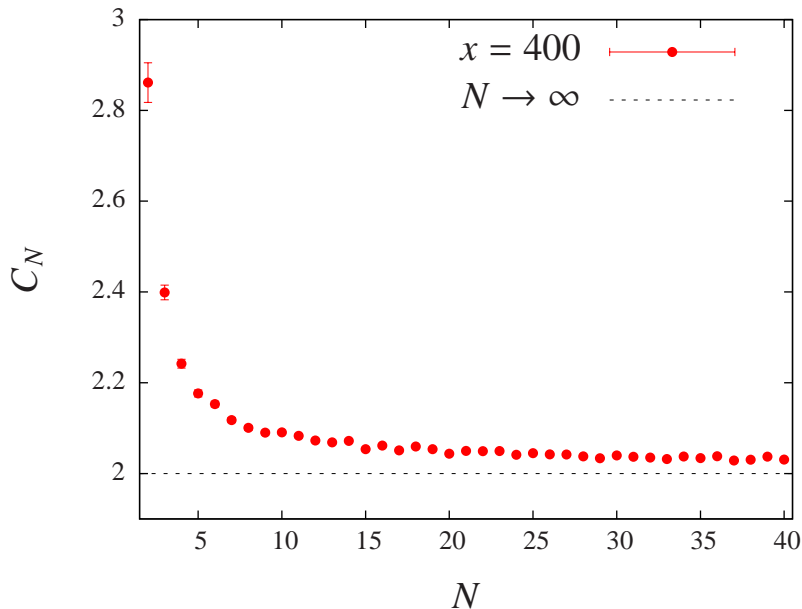


FIG. 8: Monte Carlo results for the average maximum of  $N$  Brownian motions till the first passage time through the origin, all starting at the same initial position  $x > 0$ . The coefficient  $C_N = \langle m \rangle / x$ , see Eq. (55), is plotted vs.  $N$ . Averages are performed over  $10^3$  realizations.

through the origin (rather than through  $L$ ) of  $N$  walkers from a box  $[0, L]$ . Incidentally, our path counting method also provides an exact solution to the  $N$ -dimensional Laplace's equation  $\nabla^2 Q_N = 0$  in a hypercube  $[0, L]^N$  with the boundary conditions  $Q_N = 1$  on any face of the hypercube passing through the origin and  $Q_N = 0$  on the rest of the faces. Monte Carlo simulations confirm our analytical results.

This work raises some interesting open questions. We have focused only on the maximum  $m$  till the stopping time  $t_s$  of  $N$  independent walkers. Another interesting observable is not just the actual value of the maximum, but the time  $t_m$  at which this maximum occurs before the stopping time  $t_s$ . This random variable has recently been studied in a number of contexts. For a stochastic process over a fixed time interval  $[0, t]$ , the distribution of the time  $t_m$  has been computed for a variety of Brownian paths, such as a free Brownian motion, Brownian bridges, Brownian excursions

and Brownian meanders, using path integral method [7] and also by an alternative functional renormalization group method [18]. The distribution of  $t_m$  was also computed exactly for the random acceleration process which is a non-Markov process [51]. It has also been computed both for independent Brownian walkers [26, 27] and very recently for vicious walkers [52]. On the other hand, when the process stops at a random stopping time  $t_s$  where for instance  $t_s$  is the first time a walker hits the origin, the distribution of  $t_m$  has been computed for a single Brownian motion ( $N = 1$ ) where it is already nontrivial [53]. It would be interesting to extend the results of Ref. [53] to the case of  $N > 1$  independent Brownian motions.

Another interesting challenging problem would be to compute the distribution of  $m$  as well as that of  $t_m$  for a set of vicious walkers till the stopping time  $t_s$  when the walker closest to the origin crosses the origin for the first time.

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### Appendix A: Evaluation of $B_2$

To evaluate the sum in Eq. (47) we first split the sum into two parts:  $B_2 = I_1 + I_2$  with

$$I_1 = -16 \lim_{z_1, z_2 \rightarrow 0} \sum_{n_1, n_2} \frac{n_1^2}{n_1^2 + n_2^2} \cos(n_1 \pi z_1) \cos(n_2 \pi z_2) \quad (\text{A1})$$

$$I_2 = 16 \lim_{z_1, z_2 \rightarrow 0} \sum_{n_1, n_2} \frac{(-1)^{n_2} n_1^2}{n_1^2 + n_2^2} \cos(n_1 \pi z_1) \cos(n_2 \pi z_2) \quad (\text{A2})$$

where all the sums run over positive integers.

Let us first evaluate  $I_1$ . Due to the symmetry we can re-write  $I_1$  as

$$I_1 = -8 \lim_{z_1, z_2 \rightarrow 0} \sum_{n_1, n_2} \frac{n_1^2 + n_2^2}{n_1^2 + n_2^2} \cos(n_1 \pi z_1) \cos(n_2 \pi z_2) = -8 \lim_{z_1, z_2 \rightarrow 0} \sum_{n_1, n_2} \cos(n_1 \pi z_1) \cos(n_2 \pi z_2) = -2 \quad (\text{A3})$$

In the last step we have used the identity,  $\sum_{n \geq 1} \cos(n\pi z) = -1/2$ .

To evaluate  $I_2$  we use another standard identity [47]

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos(k\pi z)}{k^2 + a^2} = \frac{\pi}{2a} \frac{\cosh(a\pi z)}{\sinh(a\pi)} - \frac{1}{2a^2}. \quad (\text{A4})$$

We now sum over  $n_2$  in Eq. (A2) using above identity. This gives

$$I_2 = -8 \lim_{z_1 \rightarrow 0} \sum_{n_1=1}^{\infty} \cos(n_1 \pi z_1) + 8\pi \lim_{z_1, z_2 \rightarrow 0} \sum_{n_1=1}^{\infty} \frac{n_1}{\sinh(n_1 \pi)} \cos(n_1 \pi z_1) \cos(n_1 \pi z_2) = 4 + 8\pi \sum_{n_1=1}^{\infty} \frac{n_1}{\sinh(n_1 \pi)}. \quad (\text{A5})$$

The remaining sum in (A5) can be explicitly evaluated using the identity [47]

$$\sum_{n=1}^{\infty} \frac{n}{\sinh(n\pi)} = \frac{1}{32\pi^3} \Gamma^4\left(\frac{1}{4}\right) - \frac{1}{4\pi}. \quad (\text{A6})$$

Adding  $I_1$  and  $I_2$  we arrive at the announced expression (5) for  $B_2$ .

### Appendix B: Evaluation of $B_N$ for large $N$

Here we show that  $B_N$  in Eq. (45) behaves, to leading order for large  $N$ , as

$$B_N \simeq N \left[ \frac{4}{\pi} \ln(N) \right]^{N/2}. \quad (\text{B1})$$

First we rewrite the denominator in Eq. (45) using the integral representation

$$\frac{1}{n_1^2 + n_2^2 + \dots + n_N^2} = \int_0^\infty dt e^{-t(n_1^2 + n_2^2 + \dots + n_N^2)}. \quad (\text{B2})$$

Using this representation, one can then decouple the sums over different indices  $n_i$  in Eq. (45) giving

$$B_N = -N^2 2^N \lim_{z_i \rightarrow 0} \int_0^\infty dt \sum_{n_1=1}^\infty n_1^2 \cos(n_1 \pi z_1) e^{-n_1^2 t} \prod_{j \neq 1} \sum_{n_j=1}^\infty [1 - (-1)^{n_j}] \cos(n_j \pi z_j) e^{-n_j^2 t} \quad (\text{B3})$$

If we now take the limits  $z_j \rightarrow 0$  inside the sums, each of the sums is convergent. However, it is easy to check that the integrand, as a function of  $t$ , diverges as  $\sim t^{1+N/2}$  as  $t \rightarrow 0$ . Thus the integral is longer convergent. To circumvent this difficulty, one can use a standard regularization scheme used often in evaluating the Madelung constant in the context of lattice sums in crystals [48] whereby we introduce a parameter  $s$  and rewrite Eq. (B3) as, upon taking the limits  $z_j \rightarrow 0$  inside the sums,

$$B_N(s) = -\frac{N^2 2^N}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_{n_1=1}^\infty n_1^2 e^{-n_1^2 t} \left[ \sum_{n_j=1}^\infty [1 - (-1)^{n_j}] e^{-n_j^2 t} \right]^{N-1}. \quad (\text{B4})$$

Note that this integral is convergent for all  $s > (1 + N/2)$ . The idea is to first evaluate  $B_N(s)$  for large  $N$  with  $s > (1 + N/2)$  and then analytically continue this result to  $s \rightarrow 1$  to evaluate  $B_N = B_N(s \rightarrow 1)$ .

Let us next define the function

$$g(t) = 4\sqrt{\frac{t}{\pi}} \sum_{k=0}^\infty e^{-(2k+1)^2 t}. \quad (\text{B5})$$

Up to the factor  $\sqrt{t/\pi}$ , this function  $g(t)$  can be expressed in terms of standard Jacobi theta functions [49]. Then one can rewrite Eq. (B4) as

$$B_N(s) = -\frac{2N^2(\sqrt{\pi})^{N-1}}{\Gamma(s)} \int_0^\infty dt t^{s-(N+1)/2} [g(t)]^{N-1} \left[ \sum_{n=1}^\infty n^2 e^{-n^2 t} \right]. \quad (\text{B6})$$

To evaluate this integral for large  $N$ , we need to know how the function  $[g(t)]^{N-1}$  behaves for large  $N$ .

Let us first focus on the function  $g(t)$  in Eq. (B5). Clearly, for large  $t$ , the dominant contribution comes from the  $k = 0$  term in the sum and hence  $g(t) \sim \sqrt{t} e^{-t}$  as  $t \rightarrow \infty$ . In contrast, the opposite limit  $t \rightarrow 0$  is more tricky. To derive its behavior as  $t \rightarrow 0$ , we first use the following Jacobi identity [49]

$$1 + 2 \sum_{k=1}^\infty (-1)^k e^{-k^2 z} = 2\sqrt{\frac{\pi}{z}} \sum_{k=0}^\infty e^{-(2k+1)^2 \pi^2 / 4z} \quad (\text{B7})$$

to rewrite the function  $g(t)$  (upon identifying  $t = \pi^2 / 4z$ ) as

$$g(t) = 1 + 2 \sum_{k=1}^\infty (-1)^k e^{-\pi^2 k^2 / 4t}. \quad (\text{B8})$$

This representation of  $g(t)$  is more amenable to the asymptotic analysis in the  $t \rightarrow 0$  limit. We obtain from Eq. (B8) the leading behavior of  $g(t)$  as  $t \rightarrow 0$

$$g(t) \simeq 1 - 2 e^{-\pi^2 / 4t} + \dots \quad (\text{B9})$$

Thus  $g(t)$  has an essential singularity at  $t = 0$  and it approaches to 1 as  $t \rightarrow 0$  in an extremely flat way. Thus the function  $g(t)$  starts at  $g(0) = 1$ , stays flat for a while and then decreases exponentially fast to 0 as  $t$  increases (see Fig. (9)).

Next consider the function  $[g(t)]^N$  that appears in the integral in Eq. (B6). As  $N$  increases, the function  $[g(t)]^N$  almost approaches a step function (see Fig. ((9))

$$[g(t)]^N \rightarrow \theta(t^*(N) - t) \quad (\text{B10})$$



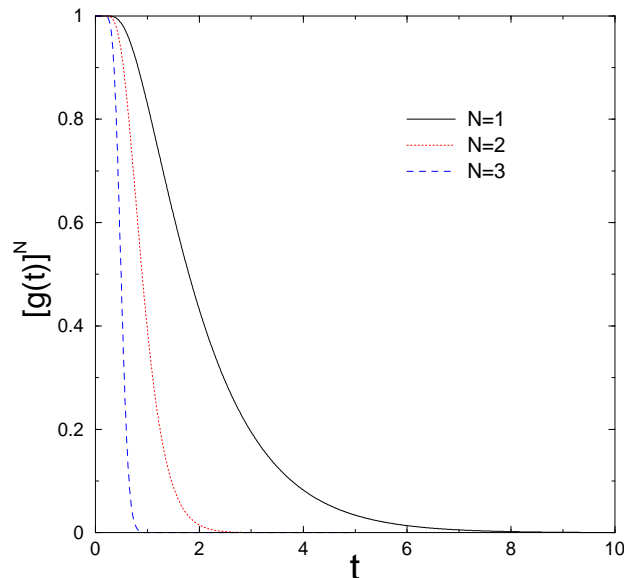


FIG. 9: The function  $[g(t)]^N$  plotted as a function of  $t$  for  $N = 1$  (black solid),  $N = 5$  (red dotted) and  $N = 50$  (blue dashed). As  $N$  becomes large,  $[g(t)]^N$  almost approaches to a step function.

where the characteristic scale  $t^*(N)$  decreases very slowly with increasing  $N$ . One can easily estimate  $t^*(N)$  for large  $N$  from the asymptotic behavior in Eq. (B9). For small  $t$ , one finds, to leading order for large  $N$

$$[g(t)]^N \sim \left[1 - 2e^{-\pi^2/4t}\right]^N \simeq \exp\left[-2Ne^{-\pi^2/4t}\right]. \quad (\text{B11})$$

Thus as  $N$  increases, it approaches to 0 rapidly for all  $t > t^*$  where  $2Ne^{-\pi^2/4t^*} \approx 1$ . This provides an estimate of  $t^*(N)$  which, to leading order for large  $N$ , reads

$$t^*(N) \simeq \frac{\pi^2}{4\ln(N)}. \quad (\text{B12})$$

Therefore, for large  $N$ , using Eq. (B10), we can cut off the upper limit of the integral in Eq. (B6) at  $t = t^*(N)$  and replace  $[g(t)]^{N-1}$  by 1 over the interval  $t \in [0, t^*(N)]$ . Furthermore, over this small interval  $t \in [0, t^*(N)]$ , one can replace the function  $\sum_{n=1}^{\infty} n^2 e^{-n^2 t}$  by its small  $t$  behavior  $\simeq \sqrt{\pi}/4t^{3/2}$ . Substituting these results in Eq. (B6) then yields, for large  $N$ ,

$$B_N(s) \simeq -\frac{N^2 \pi^{N/2}}{2\Gamma(s)} \int_0^{t^*(N)} dt t^{s-(N+4)/2} \simeq -\frac{N^2 \pi^{N/2}}{2\Gamma(s)} \frac{[t^*(N)]^{s-N/2-1}}{(s-N/2-1)}. \quad (\text{B13})$$

In deriving this result, we have assumed  $s > (1 + N/2)$ . After obtaining this large  $N$  formula for  $B_N(s)$ , we can now analytically continue it to  $s \rightarrow 1$  which finally yields

$$B_N = B_N(s \rightarrow 1) = N \pi^{N/2} [t^*(N)]^{-N/2}. \quad (\text{B14})$$

Upon using the expression for  $t^*(N)$  from Eq. (B12) gives the final large  $N$  expression for  $B_N$  in Eq. (B1).

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