Coarsening in the Presence of Kinetic Disorders: Analogy to Granular Compaction

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We study the zero temperature dynamics in an Ising chain in the presence of a dynamically induced field that favors locally the "−" phase compared to the "+" phase. At late times, while the "+" domains coarsen as $t^{1/2}$, the "−" domains coarsen as $t^{1/2}\log(t)$. Hence, at late times, the magnetization decays slowly as $m(t) = -1 + \text{const}/\log(t)$. We establish this behavior both analytically within an independent interval approximation and numerically. Our model can be viewed as a simple model for granular compaction, where the system decays into a fully compact state (with all spins "−") in a slow logarithmic manner as seen in recent experiments on granular systems.

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The effect of *quenched disorder* on the relaxation dynamics of many body systems has been studied quite extensively [1]. In systems such as structural glasses, where quenched disorder is absent, an alternative approach has been put forward that considers the slow relaxation due to *kinetic disorder*, induced by the dynamics itself [2]. Another important system where kinetic disorders give rise to slow relaxation is granular material. The density relaxation of loosely packed glass beads was studied in recent experiments and it was found that the density $\rho(t)$ compactified slowly as, $\rho(\infty) - \rho(t) \sim 1/\log(t)$ under mechanical tapping [3]. It is natural to expect that such kinetic disorders may play important roles in the dynamics of other systems as well. In this Letter we study, for the first time, the effect of a dynamically self-induced *field* in an important class of out-of-equilibrium problems, namely, the domain growth problems, and show that such systems also exhibit logarithmic relaxation, suggesting that inverse logarithmic relaxation is extremely robust.

Domain growth following a rapid quench in temperature in pure ferromagnetic spin systems is one of the better understood out-of-equilibrium phenomena [4]. For example, if an Ising system is quenched rapidly from infinite temperature to $T = 0$ without breaking the symmetry between the two ground states, domains of up and down spins form and grow with time. The average linear size of a domain grows with time as $l(t) \sim t^{1/2}$ for zero temperature nonconserved dissipative dynamics. Domain growth problems in the presence of quenched ferromagnetic disorder has also been studied extensively [4]. In such systems, quenched disorder pins the domain walls leading to a complete freezing of dynamics at $T = 0$. However, at small positive T, the system coarsens via activated dynamics, leading to an extremely slow growth law, $l(t) \sim (\log t)^{1/4}$ [4]. However, virtually nothing is known about coarsening in such systems where the disorder is not quenched but is kinetic in nature. In this paper, we take the first step towards understanding the nature of coarsening in Ising systems in the presence of a particular type of kinetic disorder, namely, a symmetry breaking kinetic field, and find novel coarsening behavior.

Following a rapid temperature quench in a pure Ising model, if one puts on a small uniform external field (say, in the down direction), then even at $T = 0$ the system quickly reaches the pure state of magnetization -1 in a finite time proportional to the initial size of the up domains. A natural question is: What happens when, instead of applying a global external bias, the symmetry between the pure states is broken locally by the dynamics itself?

In this Letter we address this question in the context of a simple Ising spin chain with spins $S_i = \pm 1$. Starting from a given initial configuration, the system evolves by single spin flip continuous time dynamics. Let $W(S_i; S_{i-1}, S_{i+1})$ denote the rate at which the flip $S_i \to -S_i$ occurs when the two neighboring spins are S_{i-1} and S_{i+1}. In our model the rates are specified as follows:

$$\begin{aligned} W(+;++) &= W(-;--) = 0\,, \\ W(+;+-) &= W(+;-+) = W(-;+-) \\ &= W(-;-+) = \tfrac{1}{2}\,, \\ W(+;--) &= 1\,, \\ W(-;++) &= \alpha\,. \end{aligned} \tag{1}$$

Here we restrict ourselves to the case $\alpha = 0$. We note that the case $\alpha = 1$ corresponds to the usual zero temperature Glauber dynamics [5]. The only difference is that, for $\alpha = 0$, the move $(+,-,+) \to (+,+,+)$ is not allowed and thereby the symmetry between "+" and "−" spins is locally dynamically broken. Thus isolated "−" spins (surrounded on both sides by a "+") block the coalescence of "+" domains and locally favor the "−" spins. As a result, we show below that the system eventually decays into the state where all spins are "−" but does so in a very slow manner, $m(t) + 1 \sim 1/\log(t)$.

Our main results can be summarized as follows. In contrast to the case $\alpha = 1$ [where the average size of both "+" and "−" domains grows as $l_{\pm}(t) \sim t^{1/2}$ at late times [6]

and the average magnetization $m(t) = (l_+ - l_-)/(l_+ + l_-)$ is a constant of motion [5]], for $\alpha = 0$ we show that at late times, while $l_+(t) \sim t^{1/2}$, $l_-(t) \sim t^{1/2}\log(bt)$ where b is a constant that depends on the initial condition. Thus due to the dynamically generated local bias, the "$-$" domains grow slightly faster than the "$+$" domains and, as a result, the magnetization decays as $m(t) = -1 + \text{const}/\log(bt)$ for large t. Notice that the average domain size grows faster for $\alpha = 0$ than for $\alpha = 1$, i.e., paradoxically coarsening is *enhanced* by putting one of the rates to zero.

Our model can alternately be viewed as a toy lattice model of granular compaction if one identifies the "$-$" spins as particles,"$+$" spins as holes, and the 1D lattice as a section of the bottom layer of a granular system. The flip of a spin in the Ising model then corresponds to deposition or evaporation of a particle from the bottom layer of a granular packing. The specific rules of spin flipping in Eq. (1) can then be directly translated into the rules for particle evaporation or deposition. The final state where all spins are negative (magnetization, $m = -1$) corresponds to the fully compact state with particle density 1. While the rules of the local dynamics studied here are not derived from a realistic model of granular compaction, the idea is to reproduce the correct nature of compaction within a simplistic Ising-type model by incorporating the basic mechanism of compaction that can be summarized as follows. There exist local kinetic "defects" and the system can gain in compaction only by relaxing such local defects. Such relaxation happens via the tapping process. However, these defects become rarer with time and it becomes harder and harder for the system to find such a local defect, relax it, and thereby gain in compaction. This is the origin of the slow logarithmic relaxation. In our model, the triplets "$+\ -\ +$" play the role of such local defects which decay only via the diffusion of kinks. Thus the diffusion effectively plays the role of tapping. The density of these triplets decays with time and the system finds it progressively harder to relax. Similar defect mediated slow relaxation has also recently been observed in other models [7].

In terms of the motion of the domain walls between "$+$" and "$-$" phases, our model can also be viewed as a reaction diffusion process. We need to distinguish between the two types of domain walls $-+ \equiv A$ and $+- \equiv B$. Note that by definition (originating from a spin configuration) the A's and B's always occur alternately. The A's and B's diffuse and, when an A and a B meet, they annihilate only if A is to the left of B, otherwise there is hard core repulsion between them.

To start with, we set up our notations. We define $P_n(t)$ and $R_n(t)$ to be the number of domains of size n per unit length of "$+$" and "$-$" types, respectively. Then, $N(t) = \sum_n P_n = \sum_n R_n$ is the number of domains of either "$+$" or "$-$" spins per unit length. The density of kinks is therefore $2N(t)$. We also define the normalized variables, $p_n = P_n/N$ and $r_n = R_n/N$. p_n (or r_n) denotes the conditional probability that, given a domain of "$+$" (or "$-$") has occurred, it is of length n. Let $L_+(t) = \sum nP_n$ and $L_-(t) = \sum nR_n$ denote the densities of "$+$" and "$-$" spins. Clearly $L_+(t) + L_-(t) = 1$ and the magnetization per unit length is $m(t) = L_+(t) - L_-(t)$. The average size of a "$+$" and a "$-$" domain is denoted, respectively, by $l_+(t) = \sum np_n = L_+(t)/N$ and $l_-(t) = \sum nr_n = L_-(t)/N$.

Following Glauber's calculation for the $\alpha = 1$ case, it is easy to show [8] that, for the $\alpha = 0$ case, the domain density, $N(t) = (1 - \langle S_i S_{i+1}\rangle)/4$, of either phase and the fraction of "$+$" spins, $L_+(t) = (1 + \langle S_i\rangle)/2$, evolve according to the exact equations,

$$\frac{dN}{dt} = -P_1 \tag{2}$$

and

$$\frac{dL_+}{dt} = -R_1\,, \tag{3}$$

where $P_1(t) = \langle(1 - S_{i-1})(1 + S_i)(1 - S_{i+1})\rangle/8$ and $R_1(t) = \langle(1 + S_{i-1})(1 - S_i)(1 + S_{i+1})\rangle/8$ are, respectively, the density of "$+$" and "$-$" domains of unit length, i.e., the density of triplets "$-\ +\ -$" and "$+\ -\ +$." It is easy to see physically the origin of these two exact equations. Equation (2) arises from the fact that the domain density can decrease only via the annihilation of the triplets "$-\ +\ -$." Also, on average, the fraction of "$+$" spins can decrease only due to the blockage by "$+\ -\ +$" triplets giving rise to Eq. (3).

By using $m(t) = \langle S_i\rangle = 2L_+(t) - 1$, we find from Eq. (3) that $dm/dt = -2R_1$. We note that, for the case $\alpha = 1$, $dm/dt = 0$ [5], indicating that the magnetization does not evolve with time. In our case, due to the triplet defects "$+\ -\ +$," the average magnetization decays with time. We also note that, unlike the $\alpha = 1$ case, the evolution equation (3) for the single point correlation function involves two and three point correlations [via $R_1(t)$]. Writing down the analogous equations for $R_n(t)$ gives an infinite hierarchy which makes an exact solution difficult for $\alpha = 0$.

By using $R_1 = r_1 N$ and $L_+ = l_+ N$ in Eq. (3), one can formally solve for $N(t)$ in terms of r_1 and l_+ as

$$\frac{N(t)}{N(t_0)} = \frac{l_+(t_0)}{l_+(t)}\,\exp\!\left(-\int_{t_0}^{t}\frac{r_1(t')}{l_+(t')}\,dt'\right). \tag{4}$$

Furthermore if the density of the "$+$" phase is $L_+(t_0) = \epsilon$, then, by using the relation $N(t) = 1/[l_-(t) + l_+(t)]$ in Eq. (4), we find

$$\frac{l_-(t)}{l_+(t)} = \frac{1}{\epsilon}\,\exp\!\left(\int_{t_0}^{t}\frac{r_1(t')}{l_+(t')}\,dt'\right) - 1\,, \tag{5}$$

clearly showing that the ratio $l_-(t)/l_+(t)$ is growing due to the presence of the triplets "$+\ -\ +$." Note that the asymmetry between the growth of "$-$" and "$+$" domains is evident due to the triplet defects "$+\ -\ +$" present with density $R_1 = r_1 N$.

In order to make further analytic progress, we first use the independent interval approximation (IIA), where correlations between neighboring domains are neglected. The IIA was used successfully for the pure Glauber-Ising model, i.e., the $\alpha = 1$ case [9]. Here we apply it to the $\alpha = 0$ case. The basic idea is to write the rate equations of the domain densities $P_n(t)$ and $R_n(t)$ by counting all the gain and loss terms. Typically the equations for single point densities involve terms containing joint densities of two domains. The IIA assumes that the joint density can be written as a product form. Since the derivation [8] is a straightforward generalization of the $\alpha = 1$ case [9], we just present the final equations,

$$\frac{dP_n}{dt} = P_{n+1} + P_{n-1} - 2P_n + \frac{R_1}{N}(P_n - P_{n-1}), \quad (6)$$

for all $n \geq 1$ with $P_0 = 0$ (absorbing boundary condition) and

$$\frac{dR_n}{dt} = R_{n+1} + R_{n-1} - 2R_n - \frac{P_1}{N}(R_n + R_{n-1}) + \frac{P_1}{N^2}\sum_{i=1}^{n-2} R_i R_{n-i-1}, \qquad n \geq 2, \quad (7)$$

and $dR_1/dt = R_2 - R_1 - P_1 R_1/N$, where $N(t) = \sum P_n = \sum R_n$. It can be easily checked that these two IIA equations satisfy Eqs. (2) and (3) exactly, and consequently also Eqs. (4) and (5).

To calculate $N(t)$ using Eq. (4), we need to evaluate two quantities from the IIA equations: (i) $r_1(t) = R_1/N$ and (ii) $l_+(t) = \sum n p_n$, where $p_n = P_n/N$. In order to calculate these two quantities, it is useful to write the IIA equations in terms of the normalized variables, $p_n = P_n/N$ and $r_n = R_n/N$. From Eqs. (6) and (7), we then get

$$\frac{dp_n}{dt} = p_{n+1} + p_{n-1} - 2p_n + r_1(p_n - p_{n-1}) + p_1 p_n \quad (8)$$

for all $n \geq 1$ with $p_0 = 0$ (absorbing boundary condition) and

$$\frac{dr_n}{dt} = r_{n+1} + r_{n-1} - 2r_n - p_1 r_{n-1} + p_1 \sum_{i=1}^{n-2} r_i r_{n-i-1}, \qquad n \geq 2, \quad (9)$$

and $dr_1/dt = r_2 - r_1$. It is easy to check that the normalization condition $\sum p_n = \sum r_n = 1$ is satisfied by these two equations.

The two IIA equations above are coupled nonlinear equations with infinite number of variables and hence are difficult to solve exactly. Our approach is a combination of a scaling assumption and then rechecking this assumption for self-consistency. Consider first the p_n equation, i.e., Eq. (8). We substitute $p_n(t) = t^{-1/2} f(nt^{-1/2}, t)$ in Eq. (8) and ask if the resulting equation allows for a steady state scaling solution as $t \to \infty$, i.e., if the scaling function becomes explicitly independent of t as $t \to \infty$. It is easy to verify that, if $r_1(t)$ decays faster than $t^{-1/2}$, such a time-independent scaling solution is possible with $f(x) = \frac{x}{2}\exp(-x^2/4)$. In this case, $l_+(t) = \sum n p_n \approx t^{1/2}\int_0^\infty x f(x)\,dx = \sqrt{\pi t}$ at late times.

Next we consider the r_n equation, i.e., Eq. (9). Since $p_1 = -d\log N/dt$, a natural choice would be to write $r_n(t) = N(t) g[nN(t), t]$. Substituting this in Eq. (9), we find that in the $t \to \infty$ limit, the equation allows for a time independent scaling function, $g(x) = c\exp(-cx)$ (c is a constant), provided $N(t)$ decays faster than $t^{-1/2}$. In this case, $r_1 = N(t)g(0) = cN(t)$. Thus if scaling starts holding beyond some time t_0, then $c = r_1(t_0)/N(t_0)$.

Using the results (i) $r_1(t) = r_1(t_0)N(t)/N(t_0)$ and (ii) $l_+(t) = \sqrt{\pi t}$ in the exact equation [Eq. (4)], we find

$$\frac{N(t)}{N(t_0)} = \sqrt{\frac{t_0}{t}}\,\frac{\log(b)}{\log(bt/t_0)}, \quad (10)$$

where $\log(b) = \frac{\sqrt{\pi}}{r_1(t_0)\sqrt{t_0}}$. Substituting this result in the expression for $r_1(t)$, we find

$$r_1(t) = \frac{\sqrt{\pi}}{\sqrt{t}\,\log(bt/t_0)}. \quad (11)$$

We now use the late time result [Eq. (10)] in the exact relation [Eq. (2)] and find

$$p_1 = \frac{1}{2t} + \frac{1}{t\log(bt/t_0)}. \quad (12)$$

From the above expressions, it is clear that both $r_1(t)$ and $N(t)$ decay faster than $t^{-1/2}$ and hence our scaling solutions are completely self-consistent.

It is easy to see that these IIA results become exact in the zero density limit of the "+" phase ($\epsilon \to 0$). In this limit, the average size of a "−" domain is $1/\epsilon$ times larger than the average size of a "+" domain. As time increases, the "+" domains will certainly grow in size. But a typical "+" domain will disappear (via the absorbing boundary condition) long before encountering other "+" domains, i.e., before feeling the presence of the constraint due to

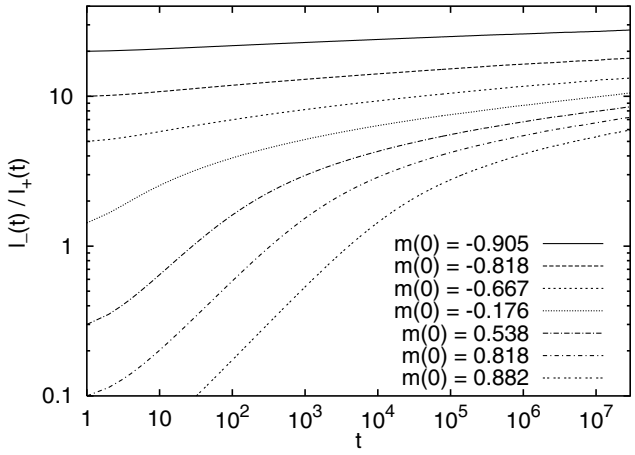

FIG. 1. $l_-(t)/l_+(t)$ versus t for runs with different initial magnetizations.

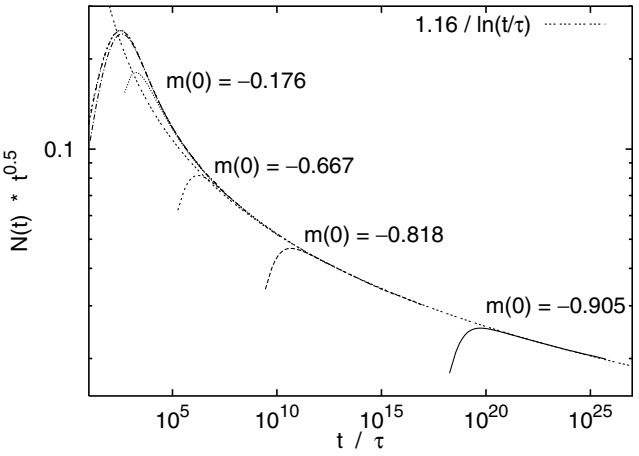

FIG. 2. $\sqrt{t}\,N(t)$ versus $t/\tau[m(0)]$ for the same runs as in Fig. 1. Also shown is the prediction const/ $\log(t/\tau)$.

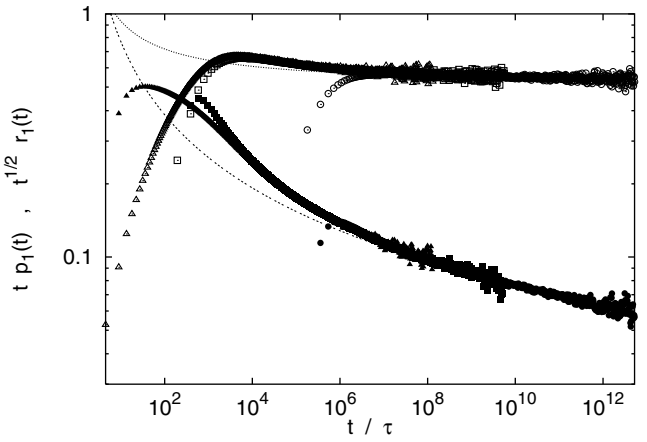

FIG. 3. $tp_1(t)$ (top) and $\sqrt{t}\,r_1(t)$ (bottom) versus $t/\tau[m(0)]$. To avoid overcrowding, results are shown only for $m(0) = -2/3$ (circles), 0 (squares), and $2/3$ (triangles). The dashed curves show the predictions $1/2 + 1/\ln(t/\tau)$ and $\sqrt{\pi}/\ln(t/\tau)$.

triplets "$+\ -\ +$." The probability of such an event is of order $O(\epsilon)$. Thus, effectively, the dynamics of the system will proceed via eating up of the "+" domains. Hence, if there is no correlation between domains in the initial condition, the dynamics is not going to generate correlations between them and hence IIA becomes exact. The picture in this limit is similar to the zero temperature dynamics of the q state Potts model in the limit $q \to 1^+$ [9]. For other volume fractions, it is likely that IIA predicts the correct fixed point picture at late times. This is confirmed by Monte Carlo simulations of the model.

To improve efficiency, these simulations were made for a version of the model with simultaneous updating. This should not change any of the above conclusions. For convenience, we chose initial conditions such that all domains of minority spins had length 1, while the lengths of majority spin domains were distributed exponentially. At each time t, all kink positions were written into an array K, and only this array is used to generate the array K′ for the next time step. For each value of $m(0)$ we simulated between 20 and 200 lattices of 2^{26} sites for 3×10^7 time steps.

Data for $l_-(t)/l_+(t)$, plotted in Fig. 1, show the predicted monotonic increase with t. This increase is logarithmic for $t > t_0$, while it is faster for $t < t_0$. For a detailed comparison with the above theory we need to know how t_0 (and thus also b) depends on $m(0)$. We expect it to be exponential for $m(0) \ll 1$, but this is not sufficient for a detailed analysis. Thus we determine for each $m(0)$ a τ such that the data for $l_-(t)/l_+(t)$, for $\sqrt{t}\,N(t)$, for $tp_1(t)$, and for $\sqrt{t}\,r_1(t)$ collapse when plotted against t/τ. The fact that a single $\tau[m(0)]$ exists which gives a good data collapse in all four plots is highly nontrivial. We show such plots for $N(t)$, $p_1(t)$, and $r_1(t)$ in Figs. 2 and 3. We see good agreement with the theoretical predictions. In particular, data collapse is excellent (showing that the only memory left from the initial conditions is the current value of the magnetization). But the detailed predictions for the scaling function show substantial corrections which seem, however, to disappear for $t \to \infty$.

In summary, we have addressed a rather general question about the effect of kinetic disorders on phase ordering kinetics and have obtained exact results for a particular type of kinetic disorder in a 1D Ising model. Besides, our model can be viewed as a new model of reaction diffusion processes as well as a toy model of granular compaction.

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Note added.—A related reaction diffusion model has recently been studied numerically by Odor and Menyhard [10].

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