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Extreme value statistics and traveling fronts: various applications

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Abstract

An intriguing connection between extreme value statistics and traveling fronts has been found recently in a number of diverse problems. In this short review we outline a few such problems and consider their various applications.

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1. Introduction

Independence of random events is the most desirable property in probability theory and statistical physics. If we have a collection of independent random variables $X_1,\ldots,X_N$ with finite variance, then the distribution of the sum $\sum X_j/\sqrt{N}$ is Gaussian in the thermodynamic limit $N\to\infty$. Similarly under broad circumstances, the asymptotic distribution of the extreme values, e.g., $X_{\min}=\min(X_1,\ldots,X_N)$, belongs to one of just three possible families [1–3].

However, independence is the exception rather than the rule—random variables are often highly correlated. Little is known on extreme value statistics of correlated random variables yet a vast number of problems can be recast into such scheme. The celebrated example is the traveling salesman problem, that is to find the shortest closed tour visiting every 'city' once. There are $(N-1)!/2$ possible tours and the lengths of the tours are obviously correlated. This and a few other combinatorial optimization

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problems were recently analyzed by using techniques originally developed to study spin glasses [4]. A nice general review of the recent progress in that direction is given by Martin et al. [5];[1] for an outstanding progress in one specific problem, the matching problem, see [6].

In a series of recent publications [7–12] we have shown that there is an intriguing connection between the statistics of extreme values arising in various contexts and traveling fronts. More precisely, the cumulative distributions of extreme variables were shown to admit a traveling front solution. Such a connection was also noted in the context of a particle moving in a random potential [13]. The goal of this short review is to convince the readers that the techniques of traveling fronts is a powerful tool to tackle the extreme value statistics of correlated random variables. By casting the problem in the traveling front framework, one easily determines two leading terms in the asymptotic expansion of the average value of the extreme variable. Furthermore, the variance of the extreme variable is nothing but the width of the traveling wave front and therefore it is usually *finite*. These results are very natural in the traveling wave framework yet very difficult to guess and derive using other methods.

Traveling front solutions have been found in numerous problems for a recent review, see Ref. [14]. To keep the discussion short, we consider the most well-known example—the one-dimensional Kolmogorov, Petrovsky, Piskunov (KPP) equation [15], also known as the Fisher equation [16]. This is a nonlinear partial differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^2 \,, \tag{1}$$

where $\phi(x,t)$ represents, for example, the density of a population at a point x at time t. Clearly this equation has two fixed points or stationary solutions: (i) $\phi(x) = 1$ for all x and (ii) $\phi(x) = 0$ for all x. A simple linear stability analysis shows that the solution (i) is stable while the solution (ii) is unstable. Therefore, if one starts with a sufficiently sharp initial condition, say $\phi(x, t = 0) = 1$ for $x < 0$ and $\phi(x, t = 0) = 0$ for $x \geqslant 0$ it is easy to see (for example by numerical simulation) that as time proceeds, the front separating the stable solution $\phi = 1$ and the unstable solution $\phi = 0$ advances in the forward direction with a unique velocity $v_{\rm f}$. Besides, the front retains its shape in the sense that the width of the front remains finite even at large times. The front velocity $v_{\rm f}$ is determined by analyzing the tail region $x \to \infty$. In this region, ϕ is small and one can ignore the nonlinear term ϕ^2 in Eq. (1). The resulting linear equation allows a spectrum of decaying solutions $\phi(x,t) \propto {\rm e}^{-\lambda[x - v(\lambda)t]}$ provided $v(\lambda)$ satisfies the dispersion relation

$$v(\lambda) = \lambda + \frac{1}{\lambda} \,. \tag{2}$$

Thus, a whole family of solutions parametrized by λ is in principle allowed. However, the front actually advances with a unique velocity $v_{\rm f}$. Thus, there must be a selection principle to choose the right velocity from the whole spectrum $v(\lambda)$. Note that the dispersion spectrum (2) has a unique minimum at $\lambda = \lambda^* = 1$, where $v(\lambda^*) = 2$. It

[1] This issue also contains several other interesting reviews, e.g., Martens [5] discusses a physicist's approach to number partitioning.

was shown [15,17] that for sufficiently steep initial conditions, the extremum of the dispersion curve is selected by the front, i.e., $v_f = v(\lambda^*) = 2$. Note that while the spectrum is determined solely by the linearized equation, for a given initial condition the nonlinear term plays a crucial role in selecting the final velocity from the full spectrum allowed by the linear equation. Subsequently, it was shown [17–19] that the front position $x_f(t)$, apart from the leading $v(\lambda^*)t$ term, has a slow logarithmic correction

$$x_f(t) = v(\lambda^*)t - \frac{3}{2\lambda^*}\ln t + \cdots . \tag{3}$$

Although this velocity selection principle was originally proved only for the KPP equation, this strategy of selecting the extremum of the dispersion spectrum of the linearized equation was subsequently shown to apply to various traveling front solutions provided certain conditions are satisfied [14]. We will show how this selection principle can be successfully used to derive exact asymptotic results for the statistics of extreme variables in a number of problems. In all the problems discussed below, we will find a traveling front solution of the same generic form (with a leading 'linear' term followed by a subleading 'logarithmic' correction) as in Eq. (3). While the velocity dispersion spectrum $v(\lambda)$ will be widely different from problem to problem, the principle of selecting the extremum of the spectrum, namely $v(\lambda^*)$, will still be valid. Finally, while λ^* and $v(\lambda^*)$ are thus nonuniversal, the prefactor $\frac{3}{2}$ in the logarithmic correction term turns out to be universal and is just the first excited state energy of a quantum harmonic oscillator [19,7,12]. For a short derivation of this correction term, see e.g. Appendix A of Ref. [12].

2. Directed polymer on a Cayley tree

As a first example, consider the problem of directed polymer on a Cayley tree studied by Derrida and Spohn [20] and recently resurfaced in a number of apparently unrelated problems [13,21–23]. The primary emphasis of this work was on the spin glass like transition occurring at *finite* temperature and on *fluctuation* properties. In contrast, we consider exactly *zero* temperature and focus on the basic *macroscopic* quantity, namely the ground state energy [8,10].

We consider a tree rooted at O (see Fig. 1) where a random energy ε is associated with every bond of the tree. The variables ε's are independent and each drawn from the same distribution $\rho(\varepsilon)$. A directed polymer of size n goes down from the root O to any of the 2^n nodes at the level n. There are $N = 2^n$ possible paths for the polymer of size n and the energy of any of these paths is

$$E_{\text{path}} = \sum_{i \in \text{path}} \varepsilon_i . \tag{4}$$

The set of $N = 2^n$ variables $E_1, E_2, \ldots, E_N$ are clearly correlated in a hierarchical (i.e., ultrametric) way and the two-point correlation between the energies of any two paths is proportional to the number of bonds they share. The ground state energy $E_{\min}(n) = \min[E_1, E_2, \ldots, E_{2^n}]$ is then a random variable and we are interested in its statistics.

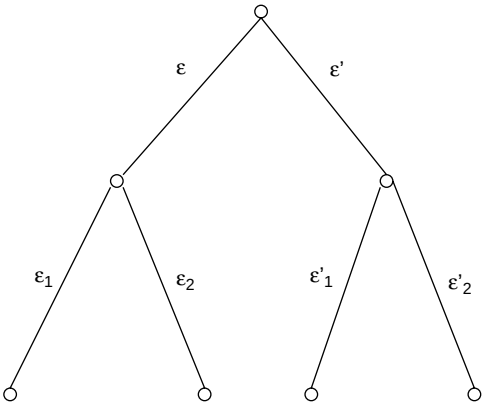

Fig. 1. The directed polymer on a Cayley tree. The ε's denote the bond energies.

The suitable quantity that has the traveling front solution is the cumulative distribution $P_n(x) = \text{Prob}[E_{\min} \geqslant x]$. It satisfies [8] a nonlinear recursion relation,

$$P_{n+1}(x) = \left[\int \mathrm{d}\varepsilon \rho(\varepsilon) P_n(x-\varepsilon) \right]^2 \tag{5}$$

and the initial condition $P_0(x) = \theta(-x)$, where $\theta(x)$ is the Heaviside step function. The front velocity can be determined by following the 'linearizing' strategy outlined above for the KPP equation. Unlike the KPP equation, however, we now need to linearize recursion equation (5) in the 'backward' tail region $x \to -\infty$ [8]. By inserting $1 - P_n(x) \propto \mathrm{e}^{\lambda[x - v(\lambda)n]}$ into Eq. (5), one finds [8] a dispersion spectrum

$$v(\lambda) = -\frac{1}{\lambda} \ln \left[2 \int \mathrm{d}\varepsilon \rho(\varepsilon) \mathrm{e}^{-\lambda\varepsilon} \right] . \tag{6}$$

For generic distributions $\rho(\varepsilon)$, this spectrum has a unique maximum at $\lambda = \lambda^*$ and by the general velocity selection principle the maximum velocity $v(\lambda^*)$ will be selected by the front. We now illustrate it for two distributions.

(i) The bimodal distribution, $\rho(\varepsilon) = p\delta(\varepsilon - 1) + (1 - p)\delta(\varepsilon)$, where $0 \leqslant p \leqslant 1$. In this case one can also think of the energy of a bond as a 'length' variable and a bond is present with probability p and absent with probability $(1 - p)$. By analyzing Eqs. (5) and (6) it was found [8] that the polymer undergoes a 'depinning' transition at the critical value $p_c = \frac{1}{2}$. For $p > \frac{1}{2}$, the ground state energy is 'extensive' and increases linearly with n. On the other hand, for $p < \frac{1}{2}$ the polymer is 'localized' and the ground state energy remains finite even in the $n \to \infty$ limit. More precisely, the average ground state energy $\langle E_{\min}(n) \rangle = \int_0^\infty P_n(x)\,\mathrm{d}x$ has the asymptotic behaviors [8]

$$\langle E_{\min}(n) \rangle \simeq \begin{cases} v_{\min}(p)n, & p > \frac{1}{2} , \\ (\ln 2)^{-1} \ln\ln n, & p = \frac{1}{2} , \\ \text{finite}, & p < \frac{1}{2} , \end{cases} \tag{7}$$

where $v_{\min}(p) = v(\lambda^*)$ with $v(\lambda^*)$ being the maximum of the dispersion spectrum in Eq. (6). Taking into account the correction term as in the KPP equation, we found that for $p > \frac{1}{2}$, the average ground state energy is given by [8]

$$\langle E_{\min}(n)\rangle \simeq v_{\min}(p)n + \frac{3}{2\lambda^*}\ln n + \cdots \ . \tag{8}$$

In a similar way, one can compute the average maximum energy. $\langle E_{\max}(n)\rangle \simeq v_{\max}(p)\,n$. Interestingly, the velocities satisfy a duality relation $v_{\min}(p) + v_{\max}(1-p) = 1$.

(ii) For the unbounded one-sided distribution $\rho(\varepsilon) = \mathrm{e}^{-\varepsilon}\theta(\varepsilon)$, dispersion spectrum (6) becomes

$$v(\lambda) = -\frac{1}{\lambda}\ln\left[\frac{2}{\lambda+1}\right] \ . \tag{9}$$

The average ground state energy is given by Eq. (8) with $\lambda^* = 3.31107\ldots$, and $v(\lambda^*) = 0.23196\ldots$.

We have also studied the asymptotic behaviors of the full distribution of the ground state energy for various $\rho(\varepsilon)$ and found [10] that the hierarchical correlations between the random variables $[E_1, E_2, \ldots, E_{2^n}]$ violate the well-known Gumbel-type behaviors [3] exhibited by the distribution of the extreme of a set of *independent* random variables.

3. Random binary search tree

We now outline an application of the traveling front techniques to the analysis of search algorithms in computer science [12]. A computer is constantly fed with enormous amount of data. It is therefore essential to organize or sort the data in an efficient way, so that the computer spends the minimum time to search for a data if required later. There are various 'sorting' and 'search' algorithms devised for this purpose [24]. One particular algorithm that has been widely studied by computer scientists is the random binary search tree (RBST) algorithm [25]. To understand this algorithm, consider a simple example. Suppose the incoming data string consists of the 12 months of the year appearing in the following random order: July, September, December, May, April, February, January, October, November, March, June and August. The RBST algorithm stores this data on a binary tree in the following way. A chronological order (January, February, etc.) is first chosen. Now the first element of the input string (July) is put at the root of a tree (see Fig. 2). The next element of the string is September. One compares with the root element (July) and sees that September is bigger than July (in chronological order). So one assigns September to a daughter node of the root in the right branch. On the other hand, if the new element were less than the root, it would have gone to the daughter node of the left branch. Then the next element is December. We compare at the root (July) and decide that it has to go to the right, then we compare with the existing right daughter node (September) and decide that December has to go to the node which is the right daughter of September. The process continues till all the elements are assigned their nodes on the tree. For the particular data string in the above example, we get the unique tree shown in Fig. 2. If the incoming data

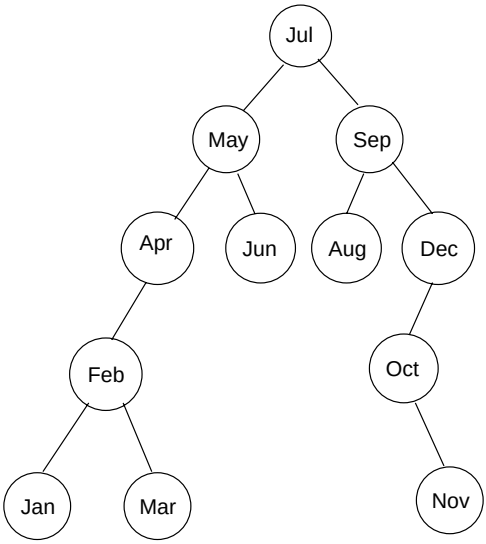

Fig. 2. The binary search tree corresponding to the data string in the order: July, September, May, April, February, January, October, November, March, June, and August. The tree has size $N = 12$ and height $H_N = 5$.

(consisting of N elements) is completely random, then all $N!$ possible binary trees have equal probability to occur and are called random binary search trees.

A quantity that is widely used to measure the efficiency of such an algorithm is the maximum search time required to find an element. This is the worst case scenario. The maximum search time is quantified by the height H_N of a tree, i.e., the distance from the node to the farthest element on the tree. In the example of Fig. 2, $H_N = 5$. Thus, determining the statistics of H_N is an extreme value problem. Apart from a slight modification, this problem can be mapped [12] onto the directed polymer problem described in the previous example. In the modified problem, the bond energies are not completely uncorrelated as in the usual polymer problem, but are correlated in a special way: the energies of two bonds emanating from the same parent (see Fig. 1) satisfy the constraint, $e^{-\varepsilon} + e^{-\varepsilon'} = 1$ and this constraint holds at every level of the tree. For example, in Fig. 1, we also have $e^{-\varepsilon_1} + e^{-\varepsilon_2} = 1$, $e^{-\varepsilon_1'} + e^{-\varepsilon_2'} = 1$, etc. The statistics of the height variable H_N in RBST problem was shown to be identical to the ground state energy $E_{\min}(n)$ of this modified polymer problem with the identification $N = 2^n$. The traveling front analysis then gives the average height [12]

$$\langle H_N \rangle \simeq \alpha_0 \ln N + \alpha_1 \ln \ln N \,. \tag{10}$$

Here $\alpha_0 = 1/v(\lambda^*)$ and $\alpha_1 = -3/2\lambda^* v(\lambda^*)$ where $\lambda^* = 3.31107\ldots$, is the maximum of the dispersion curve (9) with $v(\lambda^*) = 0.23196\ldots$. The traveling front retains its shape asymptotically, i.e., its width remains finite. This shows that the variance of the height H_N remains finite even in the large N limit. While some of these results mentioned here were also derived by the computer scientists using rigorous mathematical bounds

[26–28], the existence of a traveling front was not realized before. Besides, the traveling front techniques allow us to derive more detailed results such as the asymptotic behaviors of the full probability distribution of H_N (not just its moments) and also obtain completely new results for trees generated with arbitrary distributions, not necessarily uniform [12]. The statistics of other observables such as the 'balanced' height of a tree (which corresponds to the maximum energy in the modified directed polymer problem) has also been derived exactly using the traveling front techniques [12].

4. Aggregation dynamics of growing random trees

In the RBST example discussed above, the trees have fixed size N. Alternately one can consider random trees generated dynamically via an aggregation mechanism where the size of the trees grows indefinitely with time t. Apart from the computer science problems discussed above, such growing trees also arise in physical situations such as collision processes in gases [29] where the largest Lyapunov exponent is related to the maximum height problem.

We studied a simple tree generation model [11], where initially we have an infinite number of trivial (single-leaf) trees. Then, two trees are picked at random and attached to a common root. This merging process is repeated indefinitely (to simplify formulas the rate is set equal to 2). Let $c(t)$ be the number density of trees at time t. Initially, $c(0)=1$, and then it evolves according to $\mathrm{d}c/\mathrm{d}t=-c^2$ whose solution is $c(t)=(1+t)^{-1}$. Hence the average number of leaves per tree $\langle N\rangle$ grows linearly with time, $\langle N\rangle=c^{-1}=1+t$. We are interested in the minimal and maximal heights of such a growing tree. Rather than considering the two extremal height distributions separately, we studied a more general model that interpolates between the two cases. In this model, each tree carries an extremal height k. The result of a merger between trees with extremal heights k_1 and k_2 is a new tree with extremal height k given by

$$k=\begin{cases}\min(k_1,k_2)+1 & \text{with prob. } p\,,\\ \max(k_1,k_2)+1 & \text{with prob. } 1-p\,.\end{cases} \tag{11}$$

Here, p is a mixing parameter whose limits $p=1$ and 0 correspond to the minimal and the maximal heights problems, respectively.

The number density of trees with extremal height k, $c_k(t)$, evolves according to the master equation

$$\frac{\mathrm{d}c_k}{\mathrm{d}t}=c_{k-1}^2-2cc_k+2pc_{k-1}\sum_{j=k}^{\infty}c_j+2(1-p)c_{k-1}\sum_{j=0}^{k-2}c_j\,. \tag{12}$$

Introducing the cumulative fractions $A_k=c^{-1}\sum_{j=k}^{\infty}c_j$ and a new time variable $T=\int_0^t \mathrm{d}\tau\, c(\tau)=\ln(1+t)$, we recast Eqs. (12) into

$$\frac{\mathrm{d}A_k}{\mathrm{d}T}=-A_k+2(1-p)A_{k-1}+(2p-1)A_{k-1}^2\,, \tag{13}$$

which should be solved subject to the step function initial conditions, $A_k(0)=1$ for $k\leqslant 0$ and $A_k(0)=0$ otherwise.

In the long time limit, $A_k(T)$ approaches a traveling wave form, $A_k(T) \to f(k-vT)$. The velocity v can be determined [11] as in the above problems. Knowing this velocity one can compute the expected extremal tree height from the relation, $\langle k\rangle = c^{-1}\sum_k kc_k$. Let us note one interesting fact. For $p=0$ (the maximal height case), we can express $\langle k\rangle$ as a function of $\langle N\rangle$ (after eliminating t using $\langle N\rangle = 1+t$) for large $\langle N\rangle$ and find

$$\langle k\rangle = \alpha_0 \ln\langle N\rangle + \alpha_1 \ln\ln\langle N\rangle\ , \tag{14}$$

where α_0 and α_1 identical to those in Eq. (10) for the fixed N trees. This shows that the dynamically growing trees have the same asymptotic properties as those of the fixed size trees for large trees if one replaces N in Eq. (10) by $\langle N\rangle$. This dynamic approach is thus similar to the grand canonical approach in statistical mechanics with the time t playing the role of the chemical potential that can be chosen to fix the average size.

5. The dynamics of efficiency in a simple model

We now move from trees to economy and consider a model that mimics the dynamics of efficiencies of competing agents [9]. We represent the efficiency of each agent by a single nonnegative number. The efficiency of every agent can, independent of other agents, increase or decrease stochastically by a certain amount which we set equal to unity. In addition, the agents interact with each other which is the fundamental driving mechanism for economy. We assume that the interaction equates the efficiencies of underachievers to the efficiencies of better performing agents and set the rate of this process equal to one; we denote the rates of the increase and decrease of the efficiency by p and q, respectively.

Let $h_i(t)$ is the efficiency of the agent i at time t and $P(h,t)$ is the fraction of agents with efficiency h at time t. The evolution equation for $P(h,t)$ is obtained by counting all possible gain and loss terms. For $h \geqslant 1$, this equation reads [9]

$$\frac{\mathrm{d}P(h,t)}{\mathrm{d}t} = -P(h,t)\sum_{h'=h+1}^{\infty} P(h',t) - (p+q)P(h,t) + qP(h+1,t) + pP(h-1,t) + P(h,t)\sum_{h'=0}^{h-1} P(h',t)\ . \tag{15}$$

The cumulative distribution $F(h,t) = \sum_{h'\geqslant h} P(h',t)$ satisfies

$$\frac{\mathrm{d}F(h,t)}{\mathrm{d}t} = -F^2(h,t) + (1-p-q)F(h,t) + qF(h+1,t) + pF(h-1,t)\ . \tag{16}$$

The function $F(h,t)$ approaches a traveling wave form, $F(h,t) = f(h-vt)$. The velocity can be determined [9] by repeating the steps detailed in the previous examples. The analysis is more cumbersome due to appearance of the critical line $p_c(q)$ in the (p,q) plane,

$$p_c(q) = \begin{cases} 1+q-2\sqrt{q} & \text{for } q \geqslant 1\ , \\ 0 & \text{for } q \leqslant 1 \end{cases} \tag{17}$$

separating different behaviors. For $p > p_c(q)$, the system is in the developing phase with the average efficiency $\langle h \rangle$ increasing according to

$$\langle h \rangle = v_{\min} t - \frac{3}{2\lambda^*} \ln t + O(1) \,. \tag{18}$$

Here again λ^* is the decay rate, $f(x) \propto \mathrm{e}^{-\lambda^* x}$ as $x \to \infty$. For $p \leqslant p_c(q)$ with $q > 1$, the system is localized and $\langle h \rangle$ approaches a time-independent constant in the long time limit. For $p = 0$ and $q < 1$, the system is in the developing phase for unbounded initial efficiency distributions, with the growth rate dependent on initial conditions. For economically relevant compact initial conditions, the regime $p = 0$ and $q < 1$ belongs to the stagnant phase.

To summarize, we have exemplified that the asymptotic statistics of extreme values can often be analytically determined using the powerful traveling front techniques. There might be a deeper hidden connection between the extreme value statistics and traveling fronts, and establishing such a connection is a challenging task. Finally, we mention that the recently computed [30] correction to the celebrated Bekenstein–Hawking (BH) formula for the black hole entropy $S_{\mathrm{BH}} = A_{\mathrm{H}}/4l_{\mathrm{Pl}}^2$ (here A_{H} is the classical horizon area and l_{Pl} is the Planck length) is logarithmic. Specifically, for the four-dimensional non-rotating black hole, the entropy expansion [30]

$$S = S_{\mathrm{BH}} - \frac{3}{2} \ln S_{\mathrm{BH}} + \cdots \,, \tag{19}$$

strikingly resembles the time dependence of the front position, Eq. (3). The logarithmic correction term also appears for other black holes [31]. It is therefore tempting to speculate a traveling front structure in the black hole entropy. This speculation gets further strengthened by the fact that the BH entropy is the 'maximal' entropy that a black hole can have [30]. In other words this may be considered as an extreme value problem. The verification of the existence, if any, of a traveling front structure in the black hole problem remains an outstanding open problem.

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