An Anisotropic Ballistic Deposition Model with Links to the Ulam Problem and the Tracy-Widom Distribution

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We compute exactly the asymptotic distribution of scaled height in a (1+1)–dimensional anisotropic ballistic deposition model by mapping it to the Ulam problem of finding the longest nondecreasing subsequence in a random sequence of integers. Using the known results for the Ulam problem, we show that the scaled height in our model has the Tracy-Widom distribution appearing in the theory of random matrices near the edges of the spectrum. Our result supports the hypothesis that various growth models in $(1+1)$ dimensions that belong to the Kardar-Parisi-Zhang universality class perhaps all share the same universal Tracy-Widom distribution for the suitably scaled height variables.

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Growth processes are ubiquitous in nature. The past few decades have seen an extensive research on a wide variety of both discrete and contiuous growth models [1, 2, 3]. A large class of these growth models such as the Eden model [4], restricted solid on solid (RSOS) models [5], directed polymers [3], polynuclear growth models (PNG) [6] and ballistic deposition models (BD) [7] are believed to belong to the same universality class as that of the Kardar-Parisi-Zhang (KPZ) equation describing the growth of interface fluctuations [8]. This universality is, however, somewhat restricted in the sense that it refers only to the width or the second moment of the height fluctuations characterized by two independent exponents (the growth exponent β and the dynamical exponent z) and the associated scaling function. Moreover, even this restricted universality is established mostly numerically. Only in very few special discrete (1+1)-D models, the exponents $\beta = 1/3$ and $z = 3/2$ can be computed exactly via the Bethe ansatz technique [9]. A natural and important question is whether this universality can be extended beyond the second moment of height fluctuations. For example, does the full distribution of the height fluctuations (suitably scaled) is universal, i.e. is the same for different growth models belonging to the KPZ class? Moreover, the KPZ-type equations are usually attributed to models with small gradients in the height profile and the question whether the models with large gradients belong to the KPZ universality class is still open.

Recently Prähofer and Spohn [10] found an exact mapping between a specific PNG model and the so-called longest increasing subsequence (LIS) problem, also known as the Ulam problem. The LIS problem was first raised by Ulam in the early 60's [11], then the interest in it reappeared in the mathematical literature in 70's since the work of Vershik and Kerov [12]. The exact mapping of PNG to LIS and the subsequent utilization of the exact results available for the LIS problem allowed Prähofer and Spohn to find (besides the exact KPZ growth exponent $\beta = 1/3$) the exact asymptotic height distribution in the PNG model [10]. This distribution turned to be the well known Tracy-Widom distribution appearing in the theory of edge states of random matrices [13]. Around the same time, Johansson showed rigorously [14] that a specific (1+1)-D directed polymer model, believed to be in the KPZ universality class, also has the same Tracy-Widom distribution for the scaled height (energy) variable. Gravner et. al. found the same Tracy-Widom distribution in another class of $(1 + 1)$ dimensional growth models which they called 'oriented digital boiling' model[15]. It would be interesting to know whether there are other growth models such as the RSOS or the BD ones, which are believed to be in the KPZ universality class as far as the second moment is concerned, also share the same Tracy-Widom distribution for the scaled height.

The purpose of this Letter is to present a BD model which can be mapped exactly to the LIS problem and hence it shares the same Tracy-Widom distribution as the PNG model. This exact result, in combination with the results of [10, 14, 15], then lends support to the hypothesis that perhaps all these different growth models, at least in (1+1) dimensions, share the same universal Tracy-Widom distribution for scaled height. This hypothesis, if true, puts the universality on a much stronger footing going beyond just the second moment. Incidentally, to our knowledge, our model is the first exact solution for the full asymptotic height distribution of BD type systems.

Before describing our model, it is worth summarizing the main results for the LIS problem that we use later. Take a set of n integers $\{1, 2, 3, \ldots, n\}$. Consider all the

$n!$ possible permutations of this sequence. For any given permutation, let us find out all possible increasing subsequences (terms of a subsequence need not necessarily be consecutive elements) and from them find out the longest one. For example, take $n = 10$ and consider a particular permutation $\{8, 2, 7, \underline{1}, \underline{3}, \underline{4}, 10, \underline{6}, \underline{9}, 5\}$. From this sequence, one can form several increasing subsequences such as $\{8, 10\}$, $\{2, 3, 4, 10\}$, $\{1, 3, 4, 10\}$ etc. The longest one of all such subsequences is either $\{1, 3, 4, 6, 9\}$ as shown by the underscores or $\{2, 3, 4, 6, 9\}$. The length l_n of the LIS (in our example $l_n = 5$) is a random variable as it varies from one permutation to another. In the Ulam problem one considers all the $n!$ permutations to be equally likely. Given this uniform measure over the space of permutations, what is the statistics of the random variable l_n? Ulam found numerically for the average length $\langle l_n \rangle$ the asymptotic behavior $\langle l_n \rangle \sim c\sqrt{n}$ for large n. Later this result was established rigorously by Hammersley [16] and the constant $c = 2$ was found by Vershik and Kerov [12]. Recently, in a seminal paper, Baik, Deift and Johansson (BDJ) [17] derived the full distribution of l_n for large n. In particular, they showed that $l_n \to 2\sqrt{n} + n^{1/6}\chi$ for large n, where the random variable χ has an n-independent distribution which happens to be the Tracy-Widom distribution for the largest eigenvalue of a random matrix drawn from the Gaussian Unitary Ensemble [13]. They also showed that when the sequence length n itself is a random variable drawn from a Poisson distribution with mean $\langle n \rangle = \lambda$, the length of the LIS converges for large λ to

$$l_\lambda \to 2\sqrt{\lambda} + \lambda^{1/6}\chi, \tag{1}$$

where χ has the Tracy-Widom distribution. The fixed n and the fixed λ ensembles are like the canonical and the grand canonical ensembles in statistical mechanics. The detailed form of the Tracy-Widom distribution is rather complicated and not very illuminating (see [10] for a picture). The BDJ results led to an avalanche of subsequent mathematical works [18].

In our (1+1)-D BD model columnar growth occurs sequentially on a linear substrate consisting of L columns with free boundary conditions. The time t is discrete and is increased by 1 with every deposition event. One starts at $t = 0$ with an empty substrate. At any stage of the growth, a column (say the k-th column) is chosen at random with probability $p = \frac{1}{L}$ and a "brick" is deposited there which increases the height of this column by one unit, $H_k \to H_k + 1$. Once this "brick" is deposited, it screens all the sites at the same level in all the columns to its right from future deposition, i.e. the heights at all the columns to the right of the k-th column must be strictly greater than or equal to $H_k + 1$ at all subsequent times. For example, in Fig.1, the first brick (denoted by 1) gets deposited at $t = 1$ in the 4-th column and it immediately screens all the sites to its right. Then the second brick

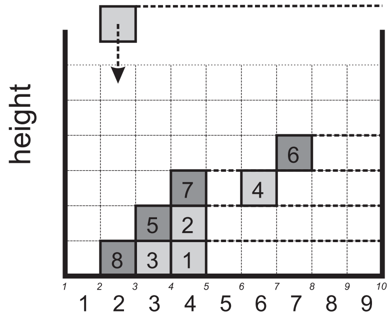

FIG. 1: Growth of a heap with asymmetric long-range interaction. The numbers inside cells show the times at which the blocks are added to the heap.

(denoted by 2) gets deposited at $t = 2$ again in the same 4-th column whose height now becomes 2 and thus the heights of all the columns to the right of the 4-th column must be ≥ 2 at all subsequent times and so on. Formally such growth is implemented by the update rule,

$$H_k(t+1) = \max\{H_k(t), H_{k-1}(t), \ldots, H_1(t)\} + 1. \tag{2}$$

The model is anisotropic and evidently even the average height profile $\langle H_k(t) \rangle$ depends nontrivially on both the column number k and time t. Our goal is to compute the asymptotic height distribution $P_k(H, t)$ for large t.

It is easy to find the height distribution $P_1(H, t)$ of the first column, since the height there does not depend on any other column. At any stage, the height in the first column either increases by one unit with probability $p = \frac{1}{L}$ (if this column is selected for deposit) or stays the same with probability $1 - p$. Thus $P_1(H, t)$ is simply the binomial distribution, $P_1(H, t) = \binom{t}{H} p^h (1 - p)^{t-H}$ with $H \leq t$. The average height of the first column thus increases as $\langle H_1(t) \rangle = pt$ for all t and its variance is given by $\sigma_1^2(t) = tp(1 - p)$. While the first column is thus trivial, the dynamics of heights in other columns is nontrivial due to the right-handed infinite range interactions between the columns. For convenience, we subsequently measure the height of any other column with respect to the first one. Namely, by height $h_k(t)$ we mean the height difference between the $(k+1)$-th column and the first one, $h_k(t) = H_{k+1}(t) - H_1(t)$, so that $h_0(t) = 0$ for all t.

To make progress for columns $k > 0$, we first consider a (2+1)-D construction of the heap as shown in Fig.2, by adding an extra dimension indicating the time t. In Fig.2, the x axis denotes the column number, the y axis stands for the time t and the z axis is the height h. In this figure, every time a new block is added, it "wets" all the sites at the same level to its "east" (along the x axis) and to its "north" (along the time axis). Here "wetting" means "screening" from further deposition at those sites at the same level. This $(2 + 1)$-D system of "terraces" is in one-to-one correspondence with the $(1 +$

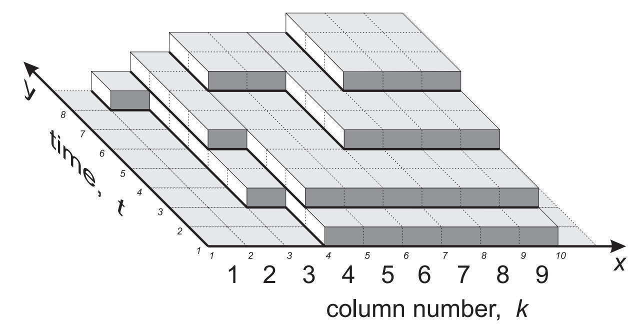

FIG. 2: $(2+1)$ dimensional "terraces" corresponding to the growth of a heap in Fig.1

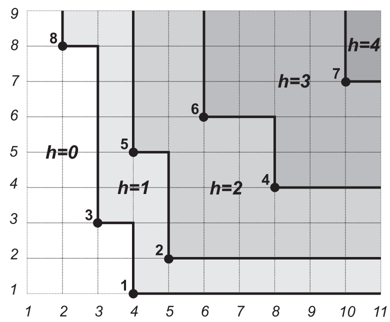

FIG. 3: The directed lines are the level lines separating adjacent terraces with height diffrence 1 in Fig. 2, projected onto the (x, y) plane and shifted by $(x, y) \to (x + h(x, y), y)$ to avoid partial overlap. The black dots denote the deposition events. The numbers next to the dots denote the times of those deposition events.

1)-D heap in Fig.1. This construction is reminiscent of the 3D anisotropic directed percolation (ADP) problem studied by Rajesh and Dhar [19]. Note however, that unlike the ADP problem, in our case each row labelled by t can contain only one deposition event[22].

The next step is to consider the projection onto the 2D (x, y)-plane of the level lines separating the adjacent terraces whose heights differ by 1. In this projection, some of the level lines may overlap partially on the plane. To avoid the overlap for better visual purposes, we make a shift $(x, y) \to (x+h(x, y), y)$ and represent these shifted directed lines on the 2D plane in Fig.3.

The black dots in Fig.3 denote the points where the deposition events took place and the integer next to a dot denotes the time of this event. Note that each row in Fig.3 contains a single black dot, i.e. only one deposition per unit of time can occur. In Fig.3, there are 8 such events whose deposition times form the sequence $\{1, 2, 3, 4, 5, 6, 7, 8\}$ of length $n = 8$. Now let us read the deposition times of the dots sequentially, but now column by column and vertically from top to bottom in each column, starting from the leftmost one. Then this sequence reads $\{8, 3, 5, 1, 2, 6, 4, 7\}$ which is just a permutation of the original sequence $\{1, 2, 3, 4, 5, 6, 7, 8\}$. In the permuted sequence $\{8, 3, 5, 1, 2, 6, 4, 7\}$ there are 3 LIS's: $\{3, 5, 6, 7\}$, $\{1, 2, 6, 7\}$ and $\{1, 2, 4, 7\}$, all of the same length $l_n = 4$. There is a greedy algorithm called the "patience sorting" game devised by Aldous and Diaconis to determine this length of the LIS[18]. This game goes as follows: start forming piles with the numbers in the permuted sequence starting with the first element which is 8 in our example. So, the number 8 forms the base of the first pile. The next element, if less than 8, goes on top of 8. If not, it forms the base of a new pile. One follows a greedy algorithm: for any new element of the sequence, check all the top numbers on the existing piles starting from the first pile and if the new number is less than the top number of an already existing pile, it goes on top of that pile. If the new number is larger than all the top numbers of the existing piles, this new number forms the base of a new pile. Thus in our example, we form 4 distinct piles: $[\{8, 3, 1\}, \{5, 2\}, \{6, 4\}, \{7\}]$. The number of piles (4) is same as the length $l_n = 4$ of the LIS of this permuted sequence. In fact, Aldous and Diaconis proved [18] that the length of the LIS l_n is exactly equal to the number of piles in the corresponding patience sorting game.

Let us note one immediate fact from Fig.3. The numbers belonging to the different level lines in Fig.3 are in one-to-one correspondence with the piles $[\{8, 3, 1\}, \{5, 2\}, \{6, 4\}, \{7\}]$ in Aldous–Diaconis patience sorting game. Hence, each pile can be identified with an unique level line. Now, the height $h(x, t)$ at any given point (x, t) in Fig.3 is equal to the number of level lines inside the rectangle bounded by the corners: $[0, 0], [x, 0], [0, t], [x, t]$. Thus, we have the correspondonce: height $\equiv$ number of level lines $\equiv$ number of piles $\equiv$ length l_n of the LIS. However, to compute l_n, we need to know n which is the number of black dots inside this rectangle.

Once the problem is reduced to finding the number of black dots or deposition events, we no longer need the Fig.3 (as it may confuse due to the visual shift $(x, y) \to (x+h(x, y), y)$) and can go back to Fig.2, where the north-to-east corners play the same role as the black dots in Fig.2. In Fig.2, to determine the height $h_k(t)$ of the k-th column at time t, we need to know the number of deposition events inside the 2D plane rectangle $R_{k,t}$ bounded by the four corners $[0, 0], [k, 0], [0, t], [k, t]$. Let us begin with the last column $k = L$. For $k = L$ the number of deposition events n in the rectangle $R_{L,t}$ is equal to the time t because there is only one deposition event per time. In our example $n = t = 8$. For a general $k < L$ the number of deposition events n inside the rectangle $R_{k,t}$ is a random variable, since some of the rows inside the rectangle may not contain a north-to-east corner or a deposition event. The probability distribution $P_{k,t}(n)$ (for a given $[k, t]$) of this random variable can, however, be easily found as follows. At each step of deposition, a column is chosen at random from any of the L columns. Thus, the probability that a north-to-east corner will fall on the segment of line $[0, k]$ (where $k \leq L$) is equal to k/L. The deposition events are completely independent of each other, indicating the absence of correlations between different rows labelled by t in Fig.2. So, we are asking the question: given t rows, what is the probability that n of them will contain a north-to-east corner? This is simply given by the binomial distribution

$$P_{k,t}(n) = \binom{t}{n} \left(\frac{k}{L}\right)^n \left(1 - \frac{k}{L}\right)^{t-n}, \quad (3)$$

where $n \leq t$. Now we are reduced to the following problem: given a sequence of integers of length n (where n itself is random and is taken from the distribution in Eq.(3)), what is the length of the LIS? Recall that this length is precisely the height $h_k(t)$ of the k-th column at time t in our model. In the thermodynamic limit $L \to \infty$ for $t \gg 1$ and any fixed k such that the quotient $\lambda = \frac{tk}{L}$ remains fixed but is arbitrary, the distribution in Eq.(3) becomes a Poisson distribution $P(n) \to e^{-\lambda}\frac{\lambda^n}{n!}$, with the mean $\lambda = \frac{tk}{L}$. We can then directly use the BDJ result in Eq.(1) to predict our main result for the height in the BD model,

$$h_k(t) \to 2\sqrt{\frac{tk}{L}} + \left(\frac{tk}{L}\right)^{1/6} \chi, \quad (4)$$

for large $\lambda = tk/L$, where the random variable χ has the Tracy-Widom distribution. Using the known exact value $\langle \chi \rangle = -1.7711...$ from the Tracy-Widom distribution[13], we find exactly the asymptotic average height profile in the BD model,

$$\langle h_k(t) \rangle \to 2\sqrt{\frac{tk}{L}} - 1.7721... \left(\frac{tk}{L}\right)^{1/6}. \quad (5)$$

The leading square root dependence of the profile on the column number k has been seen numerically [21]. The Eq.(5) also predicts an exact sub-leading term with $k^{1/6}$ dependence. Similarly, for the variance, $\sigma_k^2(t) = \langle [h_k(t) - \langle h_k(t) \rangle]^2 \rangle$, we find asymptotically: $\sigma_k^2(t) \to c_0 \left(\frac{tk}{L}\right)^{1/3}$, where $c_0 = \langle [\chi - \langle \chi \rangle]^2 \rangle = 0.8132...$ [13]. Eliminating the t dependence for large t between the average and the variance, we get, $\sigma_k^2(t) \approx a \langle h_k(t) \rangle^{2\beta}$ where the constant $a = c_0/2^{2/3} = 0.51228\ldots$ and $\beta = 1/3$, thus recovering the KPZ scaling exponent. In addition to the BD model with infinite range right-handed interaction reported here, we have also analyzed the model (analytically within a mean field theory and numerically) when the right-handed interaction is short ranged[21]. Surprisingly, we found that the asymptotic average height profile is independent of the range of interaction[21].

In summary, we have shown that the asymptotic scaled height in an anisotropic (1+1)D BD model has the Tracy-Widom distribution. Our exact result, in combination with those of Refs.[10, 14, 15] where the same distribution was found in rather different growth models, suggests that the universality in all these growth processes is perhaps much wider than it was thought before, extending to the full asymptotic height distribution and is not restricted only to the second moment of the height.

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