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Conserved Mass Models and Particle Systems in One Dimension

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In this paper we study analytically a simple one-dimensional model of mass transport. We introduce a parameter p that interpolates between continuous-time dynamics ($p \to 0$ limit) and discrete parallel update dynamics ($p = 1$). For each p, we study the model with (i) both continuous and discrete masses and (ii) both symmetric and asymmetric transport of masses. In the asymmetric continuous mass model, the two limits $p = 1$ and $p \to 0$ reduce respectively to the q-model of force fluctuations in bead packs [S. N. Coppersmith *et al.*, *Phys. Rev. E* **53**:4673 (1996)] and the recently studied asymmetric random average process [J. Krug and J. Garcia, cond-mat/9909034]. We calculate the steady-state mass distribution function $P(m)$ assuming product measure and show that it has an algebraic tail for small m, $P(m) \sim m^{-\beta}$, where the exponent β depends continuously on p. For the asymmetric case we find $\beta(p) = (1-p)/(2-p)$ for $0 \leqslant p < 1$ and $\beta(1) = -1$, and for the symmetric case, $\beta(p) = (2-p)^2/(8-5p+p^2)$ for all $0 \leqslant p \leqslant 1$. We discuss the conditions under which the product measure ansatz is exact. We also calculate exactly the steady-state mass–mass correlation function and show that while it decouples in the asymmetric model, in the symmetric case it has a nontrivial spatial oscillation with an amplitude decaying exponentially with distance.

KEY WORDS: Interacting particle systems; mass transport; parallel and random sequential dynamics.

I. INTRODUCTION

There is a wide variety of physical systems in nature where the basic microscopic dynamical processes involved are aggregation, fragmentation, adsorption, desorption and transport of mass. These processes are abundant and occur in systems such as colloidal suspensions,[(1)] polymer gels,[(2, 3)]

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river networks,[4] aerosols and clouds[5] and surface growth phenomena involving island formation.[6] These systems can have different types of non-equilibrium stationary states and phase transitions between them as the rates of the underlying microscopic processes are varied. While for systems in thermal equilibrium the stationary state is characterized by the Gibbs measure, there is no such general recipe for non-equilibrium systems. In order to gain more insights on the nature of these steady states and possible phase transitions between them, several simple lattice models involving mass transport have been proposed and studied recently.[7–9] By virtue of their simplicity, these lattice models are often amenable to exact analysis and yet contain rich and nontrivial physics.

These models constitute simple examples of interacting many particle systems out of equilibrium; in particular the dynamics of these systems do not obey detailed balance. The steady states of interacting manly body systems are in general difficult to characterize and only a few exact results are available. These include simple exclusion processes with open and closed boundary conditions,[10] abelian sandpile models of self organized criticality,[11] traffic models[12] and mass aggregation model of Takayasu.[7] Moreover the steady states in some cases are non universal and depend on the detailed nature of the dynamics used for updating. For example the steady state in the asymmetric simple exclusion process depends on whether the update rules are parallel or random sequential.[13] It is therefore desirable to study more of such simple models in a systematic way in order to get insight into the nature of the non-equilibrium steady states. In this paper we study a simple lattice model of mass transport analytically which sheds some light on these general issues pertaining to interacting many body systems.

Besides, the dynamics in seemingly unrelated systems can often be mapped onto simple one dimensional mass transport models evolving with time according to some prescribed rules. These systems include river networks,[7, 14] force fluctuations in granular systems such as bead packs,[15] traffic flows,[16] voting systems,[17, 18] wealth distributions,[19] generalized Hammersley process[20] and inelastic collisions in granular gases.[21]

In this paper we study analytically a model of mass transport in a one dimensional lattice. Each lattice-site contains a nonnegative mass variable and the dynamics consists of transporting a finite amount of mass from each site to its neighbours. The amount to be transported is randomly chosen from a given distribution. We introduce a parameter p that interpolates between continuous time dynamics ($p \to 0$ limit) and discrete parallel update dynamics ($p = 1$). For each p, we study the model with (i) both continuous and discrete masses and (ii) both symmetric and asymmetric transport of masses.

The paper is organized as follows. In Section II, we define the mass model precisely and discuss its mapping to other models of non equilibrium statistical mechanics. We also summarize our main results. In Section III, we discuss the asymmetric continuous mass model and solve for the mass distribution function in the steady state assuming that product measure holds. We then discuss under what conditions the product measure is exact. The two point mass correlation function is also computed exactly. In Section IV, we study the symmetric version of the continuous mass model. We show that product measure becomes exact in a particular limit. We also compute the stationary two point mass correlation function exactly for the symmetric model. In Section V, we study the discrete mass version of the model. Finally we conclude with a summary and outlook in Section VI. Appendix A contains a proof that product measure fails for the asymmetric model for any $p<1$. In Appendix B we prove that product measure is exact for the symmetric model in the $p\to 0$ limit.

II. THE MODEL

Our mass model is defined on a lattice. For simplicity, we define it here on a one dimensional lattice with periodic boundary conditions. The generalization to higher dimensions is straightforward. Each lattice site contains a nonnegative mass variable. We consider two versions of the model: (i) when the mass at each site is a continuous variable and (ii) when the mass at each site is discrete.

First consider the continuous mass model (Fig. 1). We start with a random configuration of masses, $m(i)$ at each site i. The dynamics is

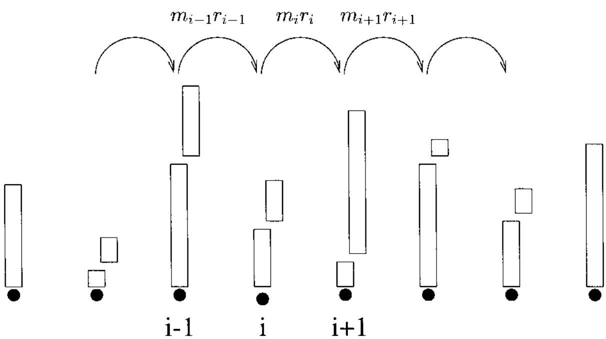

Fig. 1. Asymmetric continuous mass model: A random fraction r_i of each mass m_i is broken off and added to the right neighbour with probability p. With probability $(1-p)$, the broken piece rejoins the original mass.

defined as follows. Each discrete time step consists of two moves: (1) a fraction η_i of each mass m_i is chosen and then (2) with probability p, this fraction is added to a neighbouring site and with probability $(1-p)$ it remains at the original site. In the asymmetric model, the fraction is always added to the right neighbour. In the symmetric case, the fraction goes either to the left or the right neighbour with equal probability. We note that total mass is conserved by the dynamics. Thus the model has two parameters, the probability p and the average mass per site ρ.

The fractions η_i are independent and identically distributed random variables drawn from a probability distribution on $[0, 1]$. In this paper, we will mostly consider a uniform distribution of η_i on $[0, 1]$ though some of our results can be generalized to a class of other distributions.

We note that for $p=1$, all the chosen fractions of masses are definitely transported to their neighbours. This corresponds to fully parallel update dynamics. In this case, the asymmetric version of the model reduces exactly to the q-model introduced by Coppersmith *et al.*[15] to study force fluctuations in random bead packs. In this case, the mass at each site evolves as

$$m_i(t+1)=\eta_{i-1}m_{i-1}(t)+(1-\eta_i)\,m_i(t) \tag{1}$$

In the context of bead packs, the indices i and t index the site i at a depth t in a two dimensional packing. Then for large t, m_i represents the force supported by a bead at (i, t) scaled by the mean weight and $\eta_i m_i$ is the random component of the weight (scaled by the mean weight) transmitted from a bead at depth t to a neighbouring one at depth $(t+1)$ that touches it. The same equation was also studied in ref. 9 in the context of a lattice gas model. The stationary mass distribution $P(m)$ of the q-model was solved exactly[15] and remarkably the mean field theory turned out to be exact in the thermodynamic limit for the case of uniform distribution of the fractions η_i's. This means that the steady state joint distribution of masses at different sites factorises, $P(m_1, m_2, m_3,...)=\prod_i P(m_i)$. In other words, the product measure is exact in this case and $P(m)$ was shown[15, 9] to have a simple distribution,

$$P(m)=\frac{4m}{\rho^2}\,e^{-2m/\rho} \tag{2}$$

In the opposite limit $p\to 0$, the probability that two or more sites will be simultaneously updated in a single move is $O(p^2)$ and hence negligible. With the choice of $p=\Delta t$, this limit thus corresponds to the random sequential continuous time dynamics. This case has been studied recently by Krug and Garcia[16] and *assuming* that product measure holds they derived the steady state mass distribution $P(m)$ for uniform distribution of η_i's,

$$P(m) = \frac{1}{\sqrt{2\pi\rho m}} \exp(-m/2\rho) \tag{3}$$

The difference in the small m behaviour of the mean field $P(m)$ in the parallel and random sequential case was correctly noted by Krug and Garcia.[16] We have studied both the symmetric and asymmetric versions of the model for arbitrary p. Our main results are summarized as follows:

(1) In the asymmetric case, we show that within the mean field approximation $P(m) \sim m^{-\beta}$ for small m, while it decays exponentially for large m. The exponent $\beta(p)$ depends continuously on p with a discontinuity at $p=1$, $\beta(p) = (1-p)/(2-p)$ for $p<1$ and $\beta(1) = -1$. In the symmetric case, the mean field $P(m)$ has a similar behaviour except the exponent $\beta(p) = (2-p)^2/(8-5p+p^2)$ for all p in $[0, 1]$. Note that in the symmetric case, there is no discontinuity at $p=1$.

(2) In the asymmetric case, we prove rigorously that the product measure is exact only for $p=1$. For any $p<1$ (including the random sequential $p=0$ case), we show that the product measure ansatz, $P(m_1, m_2, m_3, \ldots) = \prod_i P(m_i)$ breaks down. But remarkably the mean field $P(m)$ is almost indistinguishable from the $P(m)$ obtained from numerical simulation in one dimension. We note that the breakdown of product measure property does not necessarily mean that the correct single point distribution $P(m)$ is still not given by the mean field $P(m)$; in fact numerical results strongly suggest that the mean field $P(m)$ is exact even though product measure fails. In the symmetric case on the other hand, product measure is exact only for $p \to 0$ but fails for any $p>0$. Besides, as opposed to the asymmetric case, the mean field $P(m)$ is considerably different from the distribution obtained numerically. This is due to strong correlations between masses in the symmetric case as mentioned below.

(3) The two-point mass correlation function $C(r) = \langle m(0)\, m(r) \rangle$ between two sites at distance r can be computed exactly (without recourse to the assumption of product measure) for arbitrary p in both asymmetric arid symmetric models. We find that in the asymmetric case, for all values of p, the connected part of the correlation function vanishes, $C(r) - \rho^2 = 0$ for $r>0$. In the limit $p \to 0$, this fact was noted by Krug and Garcia.[16] This however does not imply the validity of product measure is exact which would require factorization of all higher order correlations as well. For the symmetric case, the correlation function factorises only for $p=0$. However for $p>0$, the function $C(r) - \rho^2$ has a nontrivial spatial dependence. It

oscillates with distance r and the amplitude of the oscillation decays exponentially with r.

We have also studied a discrete mass version of the above model. In this case the mass m_i at any site i can take only discrete non-negative integer values. Instead of a random fraction breaking off a mass as in the continuous case, the mass to be taken out of a site is a random variable that takes only discrete values 0, 1, 2, 3,..., m_i equally likely, i.e., any of these values is chosen with the same probability $1/(m_i+1)$. Then as before, with probability p, the chosen mass is actually transported to a neighbouring site and with probability $(1-p)$ it stays at its original site. We derive the explicit expressions for the mass distribution for the discrete case also.

This model can be mapped onto a model of hard core particles moving with long range jumps in one dimension.[9, 16] First consider the continuous mass model. Each site of the lattice corresponds to a particle (point) on the real line and the mass m_i represents the continuous gap between ith and $(i+1)$th particle. The transport of random fraction of m_i from the ith site to $(i+1)$th site corresponds to the $(i+1)$th particle jumping to the left by a random fraction of the available gap between it and its left neighbour (see Fig. 2). The discrete mass problem similarly corresponds to particles moving on a one dimensional lattice (as opposed to the real line in the continuous case) with hard core repulsion. At each time step a particle moves to a site randomly chosen from the set of empty sites in front of it. This is a generalization of the simple exclusion process where a particle can jump only to a nearest neighbour site provided it is unoccupied. In this generalized case, while the hard core repulsion is respected, long range jumps are allowed.

The discrete mass problem can also be mapped onto an interface growth problem via the usual mapping from a lattice gas model to a growing interface. Starting from a reference height $h=0$, a particle at site i corresponds to $h(i+1)=h(i)-1$ while a hole corresponds to $h(i+1)=h(i)+1$. Under this mapping, our problem corresponds to the following rules: Any stretch of the interface with slope equal to 1 can be split at any randomly chosen point in between into two sections of slope 1 connected by a bond of slope -1.

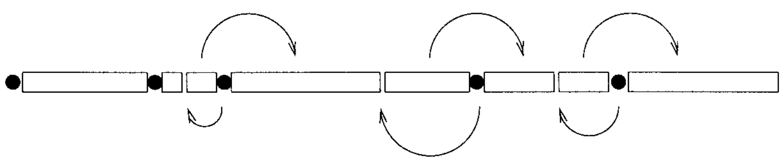

Fig. 2. Mapping of the mass model to a particle model is shown. Each transfer of mass to the right corresponds to the particle (filled circles) jumping to the left.

III. THE ASYMMETRIC MODEL WITH CONTINUOUS MASS

In this section, we study the continuous mass model where in each time step a fraction of the mass from any given site is transported with probability p to its right neighbour. The Langevin equation for the mass update can be written as

$$m_i(t+1) = m_i(t) - \sigma_i \eta_i m_i(t) + \sigma_{i-1} \eta_{i-1} m_{i-1}(t) \tag{4}$$

where the fractions η_i's are random numbers in $[0, 1]$ chosen from a uniform distribution and the random variables σ_i's take values 1 with probability p or 0 with probability $q = 1 - p$. The distribution of both of these variables are independent from site to site. Defining $r_i = \eta_i \sigma_i$, we get

$$m_i(t+1) = m_i(t)(1 - r_i) + m_{i-1}(t)\, r_{i-1} \tag{5}$$

and it is not difficult to see that the effective distribution $f(r_i)$ of the random variable r_i on $[0, 1]$ given by

$$f(r_i) = q\delta(r_i) + p \tag{6}$$

The evolution equation of the single point mass distribution function $P(m_i, t)$ (which is independent of i due to translational invariance) can be written down exactly,

$$P(m_i, t+1) = \int_0^\infty dm_{i-1} \int_0^1 dr_{i-1} f(r_{i-1}) \int_0^\infty dm'_i \int_0^1 dr_i f(r_i)\, P(m_{i-1}, m'_i, t) \times \delta(m'_i(1 - r_i) + m_{i-1} r_{i-1} - m_i) \tag{7}$$

Here $P(m_{i-1}, m_i, t)$ is the joint probability distribution of mass m_{i-1} at site $i-1$ and m_i at site i. The time evolution of the single point probability distribution involves the two point joint probability distribution function. Similarly the n-point probability distribution will involve the $(n+1)$-point joint probability distribution and in general this hierarchy cannot be broken.

A. Mean Field Theory

We first compute the single point mass distribution $P(m)$ from Eq. (7) by assuming that the joint distribution factorises in the steady state, $P(m_{i-1}, m_i) = P(m_{i-1})\, P(m_i)$. This approximation clearly ignores correlations between masses. Within this approximation, Eq. (7) involves only single point distribution function $P(m)$. Taking the stationary limit, $t \to \infty$, and using the explicit form of the distribution $f(r)$ from Eq. (6), we find that the Laplace transform, $Q(s)=\int_0^\infty P(m)\exp(-ms)\,dm$ satisfies the equation,

$$Q(s)=\frac{pV(pV+q)}{1-pqV-q^2} \tag{8}$$

where $V(s)=\int_0^1 Q(sr)\,dr$. We note that $(d/ds)(sV)=Q(s)$. Eliminating Q, we get a first order differential equation for V which can be integrated to give,

$$\frac{1-V}{V^{2-p}}=\rho s/2 \tag{9}$$

where the integration constant has been determined by using the fact that $dQ/ds|_{s=0}=-\rho$ with ρ being the average mass per site. The above equation reduces to a quadratic, cubic and linear equation in V for $p=0$, $p=0.5$ and $p=1$ respectively.

For the fully parallel dynamics $p=1$, we get $V(s)=2/(2+\rho s)$ and hence $Q(s)=4/(2+\rho s)^2$. By inverting the Laplace transform, we recover the result $P(m)=(4m/\rho^2)\,e^{-2m/\rho}$ obtained by Coppersmith *et al.*[15] In the random sequential limit, $p\to 0$, we get

$$V(s)=\frac{-1+\sqrt{1+2\rho s}}{\rho s}, \qquad Q(s)=\frac{1}{\sqrt{1+2\rho s}}$$
$$P(m)=\frac{1}{\sqrt{2\pi\rho m}}\exp(-m/(2\rho)) \tag{10}$$

The same result was obtained by Krug and Garcia[16] by a somewhat indirect method by computing the moments and then guessing the distribution from its moments. When $p=0.5$, the expression for $P(m)$ is complicated and we do not reproduce it here.

For arbitrary p, a closed form expression of $P(m)$ is difficult to obtain. However the asymptotic behaviour of $P(m)$ for large and small m can be easily derived. For large m, we expect $P(m)\sim e^{-\alpha m}$. The decay coefficient α can be derived by noting that the Laplace transform $Q(s)$ must have a pole at $s=-\alpha$. From Eq. (8), we note that the pole of Q occurs when $V=(1-q^2)/pq$. Using this in Eq. (9), we get,

$$\alpha=\frac{2(1-p)^{1-p}}{\rho(2-p)^{2-p}} \tag{11}$$

In the limits, $p=1$ and $p\to 0$, this gives the correct decay coefficient of $P(m)$.

For small m, on the other hand, $P(m)$ has an algebraic tail, $P(m) \sim m^{-\beta}$. From Eq. (9), we note that for large s, $V(s) \approx (2/\rho s)^{1/(2-p)}$. Using $Q(s) = d(sV)/ds$, we get for large s,

$$Q(s) \approx \frac{1-p}{2-p}\left(\frac{2}{\rho s}\right)^{1/(2-p)} \tag{12}$$

This implies that for $p<1$, $P(m) \sim m^{-\beta}$ for small m with $\beta = (1-p)/(2-p)$. Note that for $p=1$, the coefficient of $1/s$ vanishes in Eq. (12) and the leading order term decays as $1/s^2$, implying $\beta = -1$ for $p=1$. Thus there is a discontinuity in the exponent $\beta(p)$ at $p=1$.

How good is the product measure ansatz? In general, we have noted before that the equation of the n-point distribution function contains the (n + 1)-point distribution function. If the product measure ansatz were to be exact, then one has to ensure that every equation of the hierarchy is satisfied by the product measure ansatz. This was in fact proved to be case for $p=1$.[15] It is easy to show that this ansatz is exact only for $p=1$ and fails for all $p<1$. This is proved by showing, that for $p<1$, the second equation of the hierarchy (involving the two-point and three-point distributions) is not satisfied by the $P(m)$ obtained from the first equation of the hierarchy, i.e., Eq. (7) assuming product measure. Algebraic details are given in Appendix A.

For $p<1$, we compared the mean field answer for $P(m)$ with the numerically obtained distribution in one dimension. In the limit $p \to 0$, the mean field $P(m)$ matches extremely well with the numerically computed one. This was also noted by Krug and Garcia.[16] For arbitrary p, we do not have a closed form expression of mean field $P(m)$ to compare with the simulation results. However, various moments of m with the mean field $P(m)$ can be computed exactly for arbitrary p and compared to the numerically obtained moments. The mean field moments are computed by expanding $V(s)$ in powers of s. We list the the moments $\langle m^n \rangle$ for $n = 1,..., 5$ below.

$$\begin{aligned}
\langle m \rangle &= \rho \\
\langle m^2 \rangle &= \frac{3(2-p)}{2}\rho^2 \\
\langle m^3 \rangle &= \frac{3(2-p)(5-3p)}{2}\rho^3 \\
\langle m^4 \rangle &= \frac{5(2-p)(21-26p+8p^2)}{2}\rho^4 \\
\langle m^5 \rangle &= \frac{15(2-p)(504-955p+600p^2-125p^3)}{16}\rho^5
\end{aligned} \tag{13}$$

Table I. Numerically Obtained Moments of the Mass Compared with the Mean Field Values [Eq. (13)] for $p=0.8$ in the Asymmetric Continuous Model[a]

Moments	Numerical	Mean Field
$\langle m^2 \rangle$	1.7998(0)	1.80
$\langle m^3 \rangle$	4.6826(0)	4.68
$\langle m^4 \rangle$	15.9888(3)	15.96
$\langle m^5 \rangle$	67.72(1)	67.50

[a] The excellent agreement with the mean field results in this case is to be contrasted with rather poor agreement with mean field results in case of symmetric continuous model (see Table II).

To check how accurate these mean field moments are, we have computed these moments directly from numerical simulation of the model for different values of p. In Table I, we compare the mean field moments (up to order 5) to the numerical ones for a representative value of $p=0.8$. The closeness of these moments to the numerical values for arbitrary p suggests strongly that the mean field $P(m)$ may be exact for all p even though the product measure fails for $p<1$.

B. Correlation Function

In this subsection we compute the two point mass correlation function exactly for the asymmetric continuous mass models In the continuous time case ($p\to 0$ limit of our model), this was computed exactly by Krug and Garcia[16] for arbitrary probability distributions of the random fraction r. We reproduce their calculation here for completeness. Multiplying $m_i(t+1)$ by $m_j(t+1)$ in Eq. (5) and taking expectation value in the steady state, we find that two point correlations $C_j=\langle m_i m_{i+j}\rangle$ satisfy the following set of linear equations,

$$
\begin{aligned}
C_0(\mu_1-\mu_2)-C_1\mu_1(1-\mu_1)&=0\\
C_0(\mu_1-\mu_2)-2C_1\mu_1(1-\mu_1)+C_2\mu_1(1-\mu_1)&=0\\
C_{j-1}-2C_j+C_{j+1}&=0, \qquad j=2,3,4,\ldots
\end{aligned}
\tag{14}
$$

with the boundary conditions $C_j\to\rho^2$ as $j\to\infty$. Here $\mu_1=\langle r_i\rangle$ and $\mu_2=\langle r_i^{\,2}\rangle$ are the first and second moments of the random fraction r_i

distributed according to Eq. (6). These set of of equations can be solved easily to give,[16]

$$C_0 = \frac{\mu_1(1-\mu_1)}{\mu_1 - \mu_2}\,\rho^2$$
$$C_j = \rho^2, \qquad j = 1, 2, 3,... \tag{15}$$

Thus the two point correlation function $\langle m_i m_j \rangle$ is equal to $\langle m_i \rangle \langle m_j \rangle$ for $i \neq j$. In fact this conclusion holds for any arbitrary distribution of the fractions r_i. This however does not mean that the product measure is exact. That would require that all higher order correlations must also factorize. In fact, for the asymmetric model, it can be shown[22] that the 3-point correlation function does not factorize except for $p = 1$.

IV. THE SYMMETRIC MODEL

In this section, we study the continuous mass model where mass transport has no bias in direction. Once again we have a continuous mass m_i at each site. In each time step, a fraction is chosen at random from a uniform distribution on $[0, 1]$ and this fractional mass is transported to the left or right nearest neighbour with equal probability $p/2$. With probability $q = 1 - p$, the fractional mass stays at the original site. In order to write down the mass evolution equation, it is convenient to define a set of variables s_i at each site i. The variable s_i can be either $+1$ or -1 with equal probability 1/2. If $s_i = 1$, it indicates that the fractional mass from site i is transported to the right neighbour. On the other hand, $s_i = -1$ indicates transport to the left neighbour. Then the mass evolution equation can be written down as in the asymmetric case,

$$m_i(t+1) = (1 - r_i)\, m_i(t) + \frac{1 + s_{i-1}}{2}\, r_{i-1} m_{i-1}(t) + \frac{1 - s_{i+1}}{2}\, r_{i+1} m_{i+1} \tag{16}$$

where the random variables r_i have the same distribution $f(r_i) = q\delta(r_i) + p$ as in the asymmetric case. The evolution of the single point mass distribution function $P(m, t)$ can be written down as in the asymmetric case (Eq. (7)). The only difference is that now the single point equation contains three point distribution (as opposed to the two point function in the asymmetric case) and the additional s_i variables.

A. Mean Field Theory

Assuming product measure, this equation can be solved in the same fashion as in the asymmetric case. It follows that the Laplace transform $Q(s)$ of $P(m)$ in the stationary state satisfies,

$$Q(s) = \frac{pV(q + pV + 1)^2}{4 - q(q + pV + 1)^2} \tag{17}$$

where $V(s) = \int_0^\infty Q(su)\, du$ as in the asymmetric case. Using $Q(s) = d(sV)/ds$, we find the function $V(s)$ is given by the solution of the following nonlinear equation,

$$\left[1 - \frac{p(1-V)}{4}\right]^{p/(4-p)} V^{-(8-5p+p^2)/(4-p)}(1-V) = \frac{\rho s}{2} \tag{18}$$

In the limit $p \to 0$ (random sequential limit), this equation can be solved in closed form and we get,

$$P(m) = \frac{1}{\sqrt{2\pi\rho m}} \exp(-m/(2\rho)) \tag{19}$$

which has the same expression as for the asymmetric $p \to 0$ case. For other values of p, while we are unable to get a closed form expression, the asymptotic behaviour of $P(m)$ for large and small m can be easily derived. We find that for large m, $P(m) \sim \exp(-\alpha m)$ where the coefficient $\alpha(p)$ can be determined in the same way as in the asymmetric case. It is given by a long expression which we do not present here. For small m, $P(m)$ has an algebraic tail, $P(m) \sim m^{-\beta}$ where the exponent $\beta(p)$ can be determined by examining the large s behaviour of $Q(s)$. We find $\beta(p) = (2-p)^2/(8-5p+p^2)$ which decreases continuously from $1/2$ ($p \to 0$) to $1/4$ ($p = 1$).

For the symmetric case, we show in Appendix B that the product measure is exact in the $p \to 0$ limit. For $p > 0$, the product measure fails and unlike the asymmetric case, the mean field $P(m)$ is considerably different from the distribution obtained numerically. This failure of mean field theory for $p > 0$ shows up in the calculation of two point correlation function as done in the next subsection. However, while the mean field theory fails for large m (as evident from expectation value of the moments of the mass distribution shown in Table II), it matches very well with the numerical result for small m (see Fig. 3).

Table II. Comparison of Numerically Obtained Moments of the Mass with the Mean Field Values for $p=0.8$ in the Symmetric Continuous Model[a]

Moments	Numerical	Mean Field
$\langle m^2 \rangle$	2.3237(4)	2.100
$\langle m^3 \rangle$	8.623(5)	6.660
$\langle m^4 \rangle$	44.37(7)	28.260
$\langle m^5 \rangle$	293.2(9)	150.314

[a] This clearly shows that the mean field approximation is not good for the symmetric case as compared to the asymmetric case.

B. Correlation Function

For the symmetric model, the translationally invariant stationary two point mass correlation function, $C_{j-i} = \langle m_i m_j \rangle$ does not factorize for $j \neq i$ as in the asymmetric case. Below we compute the two point correlation exactly and show that the connected part of the correlation function in fact has a nontrivial spatial dependence.

Multiplying Eq. (16) by $m_j(t+1)$ and taking expectation value, we find that in the stationary limit $t \to \infty$, the correlation function C_j satisfies,

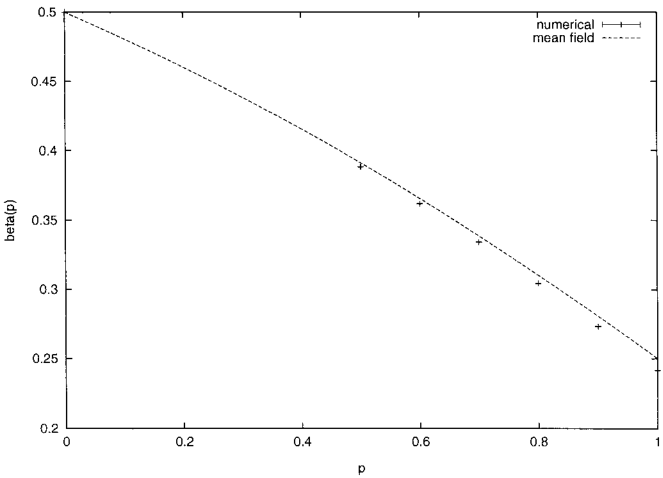

Fig. 3. The analytical mean field answer for $\beta(p)$ is compared with the numerical result for the symmetric continuous mass model. While the numerical single site distribution for masses is quite different from the mean field answer (see Table II), the small m behaviour is predicted well by the mean field.

$$\begin{aligned}
&\mu_1 C_{j-2}(1-\delta_{j2}) + 4(1-\mu_1)\, C_{j-1} + 2(3\mu_1 - 4)\, C_j \\
&\qquad + 4(1-\mu_1)\, C_{j+1} + \mu_1 C_{j+2} = 0, \qquad j = 2, 3,... \\
&4(1-w)\, C_0 + 2(7\mu_1/2 - 4)\, C_1 + 4(1-\mu_1)\, C_2 + \mu_1 C_3 = 0 \\
&4(-1+w)\, C_0 + 4(1-\mu_1)\, C_1 + \mu_1 C_2 = 0
\end{aligned} \tag{20}$$

where $\mu_1 = \langle r_i \rangle$ and $\mu_2 = \langle r_i^2 \rangle$ are respectively the first and second moments of $f(r_i)$ and $w = \mu_2/\mu_1$.

Let $G(z) = \sum_{j=1}^{\infty} C_j z^j$ be the generating function. Multiplying Eq. (20) by z^j and summing over j's, we get

$$G(z) = \frac{z[4(1-w)\, zC_0 + C_1 \mu_1 (1+z)]}{(1-z)[4z + \mu_1 (1-z)^2]} \tag{21}$$

The boundary condition, $C_j \to \rho^2$ as $j \to \infty$ implies that $G(z) \to \rho^2/(1-z)$ as $z \to 1$. This gives us one relation between C_0 and C_1,

$$C_1 = (2\rho^2 - 2(1-w)\, C_0)/\mu_1 \tag{22}$$

We need one more condition to fix both C_0 and C_1. This is obtained by noting that $G(z)$ in Eq. (21) has three poles, $z = 1$ and $z = z_\pm$ where $z_\pm = (\mu_1 - 2 \pm 2\sqrt{1-\mu_1})/\mu_1$. We note that $|z_+| < 1$ which would imply that C_j will blow up exponentially as $|z_+|^j$ for large j. Since this can not happen, the numerator on the right hand side of Eq. (21) must also vanish at $z = z_+$ in order to cancel the pole. This provides an additional condition which together with Eq. (22) gives,

$$C_0 = \frac{\rho^2 (1+z_+)}{(1-w)(1-z_+)} = \frac{\rho^2 \sqrt{1-\mu_1}}{1-w} \tag{23}$$

and C_1 can be determined from Eq. (22). Inverting the generating function, we find that for any $n > 0$,

$$C_n = \rho^2 [1 - z_+^n] \tag{24}$$

Since $z_+ = (\mu_1 - 2 + 2\sqrt{1-\mu_1})/\mu_1$ lies in $[-1, 0]$, clearly the connected part of the correlation function has a nontrivial oscillation with distance with an amplitude that decays exponentially with the distance.

Curiously the function C_n for $n > 0$ depends only on μ_1 but not on μ_2, whereas C_0 involves both μ_1 and μ_2. We also note that the above exact result is valid for any arbitrary distribution $f(r)$ of the fractions r_i and not just for the special distribution given by Eq. (6). For that distribution, we get from Eq. (6), $\mu_1 = p/2$ and $\mu_2 = p/3$ and hence $w = 2/3$. One useful check is that in the limit $p \to 0$, we get $z_+ \to 0$ implying complete decoupling of the two point correlation. This is consistent with the fact that product measure is exact in the symmetric case only in the $p \to 0$ limit. For $p > 0$, the correlation function has a nontrivial spatial dependence and product measure clearly fails.

V. THE DISCRETE MASS MODEL

In this section we study the model when the mass m_i at each site i is a discrete non negative integer. In each time step, a block of size n_i is chosen at each site and is transported to its neighbour with probability p and stays at the original site with probability $q = 1 - p$. The block size n_i is a discrete random variable which can take values $0, 1, 2,..., m_i$, all with equal probability $1/(m_i + 1)$. As in the continuous mass model, the mass transport can be either asymmetric or symmetric. We study here only the asymmetric model but the symmetric version can be studied by using similar procedures.

There is an equivalent lattice gas representation of this model in one dimension as mentioned in Section II. In this mapping, lattice site i of the mass model corresponds to the ith hard core particle and the mass m_i represents the number of holes or empty sides between the ith and $(i+1)$th particle. In the lattice gas dynamics of the asymmetric model, at each time step every particle jumps to any one of the available vacant sites in front of it with equal probability.

The analysis of the stationary mass distribution of the asymmetric discrete model proceeds along the same line as its continuous counterpart. We write down the evolution equation of the single site distribution function $P(m, t)$ in terms of the joint two point distribution $P(m_1, m_2, t)$. Assuming product measure holds, the evolution equation is given by,

$$\begin{aligned} P(m_i, t+1) = {} & p^2 \sum_{m_{i-1}=0}^{\infty} \sum_{m_1=0}^{m_{i-1}} \sum_{m_i'=0}^{\infty} \sum_{m_2=0}^{m_i'} \frac{P(m_{i-1})\, P(m_i')}{(m_{i-1}+1)(m_i'+1)} \\ & \times \delta(m_i' - m_2 + m_1 - m_i) \\ & + pq \sum_{m_i'=0}^{\infty} \sum_{m_2=0}^{m_i'} \frac{P(m_i')}{(m_i'+1)}\, \delta(m_i' - m_2 - m_i) \\ & + pq \sum_{m_{i-1}=0}^{\infty} \sum_{m_1=0}^{m_{i-1}} \sum_{m_i'=0}^{\infty} \frac{P(m_{i-1})\, P(m_i')}{(m_{i-1}+1)}\, \delta(m_i' + m_1 - m_i) \\ & + q^2 \sum_{m_i'=0}^{\infty} P(m_i')\, \delta(m_i' - m_i) \end{aligned} \tag{25}$$

We define the generating function, $Q(x)=\sum_0^\infty P(m)\, x^m$. In the stationary limit, we get from the above equation,

$$Q(x)=\frac{[f(x)-f(1)][p(f(x)-f(1))+q(x-1)]}{(x-1)[(1+q)(x-1)-q(f(x)-f(1))]} \tag{26}$$

where $f(x)=\sum_{m=0}^\infty (P(m)/(m+1))\, x^{m+1}$. Using $Q(x)=df/dx$, one can obtain closed form expressions of $Q(x)$ and hence of $P(m)$ only in the two limits, $p=1$ and $p\to 0$. For the fully parallel dynamics $(p=1)$, we get,

$$Q(x)=\frac{1}{(1-\rho/2(x-1))^2} \tag{27}$$

$$P(m)=\frac{4(m+1)\,\rho^n}{(\rho+2)^{n+2}} \tag{28}$$

For the random sequential case $(p\to 0)$, we find

$$Q(x)=\frac{1}{\sqrt{1-2\rho(x-1)}} \tag{29}$$

$$P(m)=\frac{(2\rho)^m}{(1+2\rho)^{n+1/2}}\,\frac{(2n)!}{(n!)^2\, 2^{2n}} \tag{30}$$

It is easy to check that in the limit of large m and ρ, these distributions reduce to their continuous counterparts (Eq. (2) and Eq. (3), respectively) as expected.

As in the continuous asymmetric model, it turns out that the product measure is exact only in the $p=1$ limit. The proof that the product measure is exact for $p=1$ in the discrete case can be derived by following the same line of arguments as used for the continuous case.[15] Basically, one writes down the exact evolution equation for the n-point joint distribution which involves the $(n+1)$th point joint distribution. One makes the ansatz for product measure and ensures that this ansatz is consistent for all values of n, i.e., all the equations of the hierarchy satisfy the product measure ansatz.

Without giving the details we just outline below few basic steps. For $p=1$, assuming product measure in the equation involving single point and two point distributions (Eq. 25), we obtain $P(m)$ as given by Eq. (28).

Consider first a cluster of n neighbouring sites 1, 2,..., n. The time development for the n-point probability distribution can be written as

$$P(m_1,..., m_n, t+1) = \sum_{m'_0=0}^{\infty} \sum_{r_0=0}^{m'_0} \cdots \sum_{m'_n=0}^{\infty} \sum_{r_n=0}^{m'_n} \frac{P(m'_0, m'_1,..., m'_n, t)}{(m'_0+1)(m'_1+1)\cdots(m'_n+1)} \times \delta(m'_1 - r_1 + r_0 - m_1) \cdots \delta(m'_n - r_n + r_{n-1} - m_n) \tag{31}$$

We have to now show that the product measure ansatz $P(m_1, m_2,...) = \prod_i P(m_i)$ in the steady state with $P(m)$ given by Eq. (28) is consistent with Eq. (31). To show this, we consider the n-variable generating function $Q(x_1,..., x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} P(m_1,..., m_n)\, x_1^{m_1} \cdots x_n^{m_n}$. We assume product measure on the right hand side of Eq. (32), sum over the m_i's and obtain,

$$\begin{aligned} Q(x_1,..., x_n) &= \frac{\alpha - f(x_1)}{1-x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} \cdots \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}} \frac{\alpha - f(x_n)}{1-x_n} \\ &= Q(x_1)\, Q(x_2) \cdots Q(x_n) \end{aligned} \tag{32}$$

where in deriving the last step we have used $Q(x) = df/dx$ and the expression of $Q(x)$ from Eq. (26). One can repeat the same calculation when the

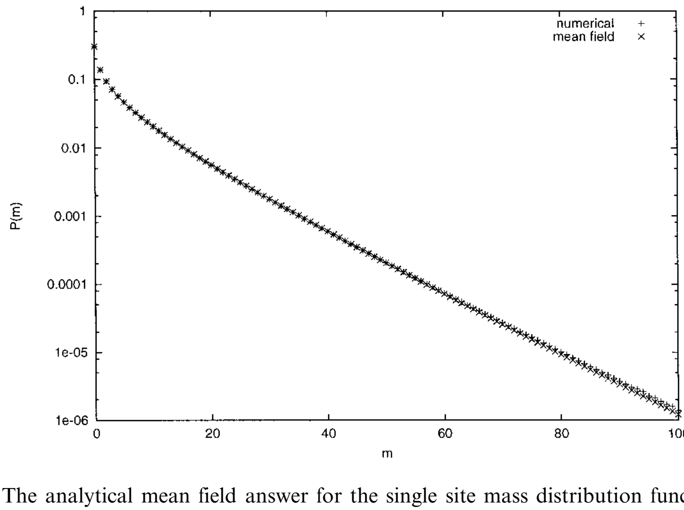

Fig. 4. The analytical mean field answer for the single site mass distribution function $P(m)$ given by Eq. (30) is compared to the numerical result for the asymmetric random sequential *discrete* mass model. The date is for system size $L = 20000$. The closeness of the two curves suggest that mean field $P(m)$ is exact even though the product measure fails.

n sites are not necessarily neighbours. This therefore proves that every equation of the hierarchy of distribution functions satisfies the product measure ansatz for $p=1$. However, this proof fails for $p<1$ as in the continuous case and the same line of argument used for the continuous case (see Appendix A) goes through for the discrete case. Even though the product measure fails, the mean field answer for other values of p match very well with the numerically computed one. For the random sequential case ($p \to 0$), we compare the mean field result for the single site probability distribution with numerical simulation (see Fig. 4).

VI. SUMMARY AND CONCLUSION

In this paper we have studied a simple mass model of chipping and aggregation where a mass at a site can chip off a fraction to its neighbour. A parameter p was introduced which allowed us to interpolate between parallel dynamics anti random sequential dynamics. We studied the model for both continuous and discrete masses as well as for symmetric and asymmetric transport of mass.

We have calculated analytically the mass distribution function $P(m)$ in the steady state for all p assuming product measure, i.e., neglecting correlation between masses. In some cases we proved that this product measure is exact. One of the main results is that the distribution $P(m)$ has an algebraic tail for small m, $P(m) \sim m^{-\beta(p)}$ where the exponent $\beta(p)$ depends on p. Thus the steady state is non universal and depends on the specific nature of the dynamics characterized by the parameter p.

Another interesting point is that for the asymmetric, continuous mass model, we show that even though the two-point mass correlation function decouples for any p, product measure is not valid for $p<1$. This means that the correlations between masses at different sites show up only in 3 or higher order correlation functions but not at the 2 point level. Exact calculation of the 3-point correlation function will be presented elsewhere.[22] Interestingly however the single point mass distribution $P(m)$ obtained using product measure ansatz is extremely close to the numerically obtained distribution.

Interpreting m_i as the distance between two hard core particles labelled i and $(i+1)$, it is easy to see that within product measure ansatz, the steady state probability of a given configuration can be written as, $P(m_1, m_2, ...) \sim \prod_i m_i^{-\beta(p)}$ for small gaps between neighbouring particles. This represents a gas of particles moving on a ring with nearest neighbour interaction $\beta(p) \log(r)$ for small r, where r is the separation between neighbouring particles. Choosing different dynamics via tuning p corresponds

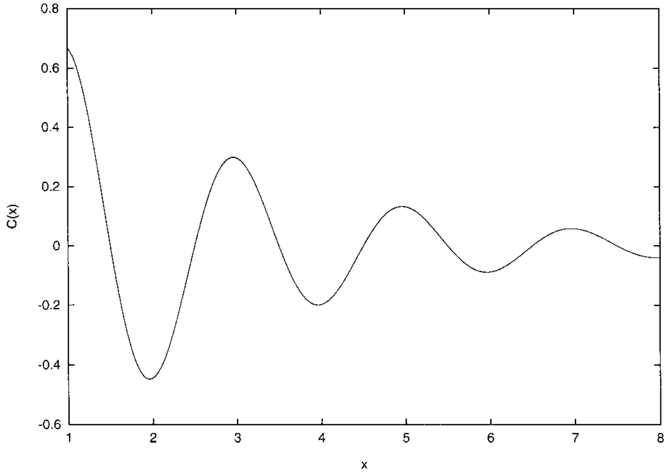

Fig. 5. The correlation function $C(x) = \langle m_0 m_x \rangle - \langle m \rangle^2$ given by Eq. (24) is shown for $\mu_1 = 0.96$. It oscillates with distance with the amplitude decaying exponentially to zero.

to changing the coupling continuously. For the asymmetric model, $\beta(p) = (1-p)/(2-p)$ for $p < 1$ and -1 for $p = 1$. This corresponds to a shift from a potential that prefers "bunching" of particles for $p < 1$ to a repulsive one at $p = 1$. This jump discontinuity is lost for the symmetric model where we have $\beta(p) = (2-p)^2/(8-5p+p^2) > 0$ for all p.

We also calculated exactly the correlation function $C_j = \langle m_0 m_j \rangle$ for the asymmetric and the symmetric models. When the transport is asymmetric the correlation function factorises for $j \neq 0$. Unlike the asymmetric case, theme are nontrivial correlations in the symmetric model. The connected part of the correlation function oscillates with distance and the amplitude of the, oscillation decays exponentially with distance (Fig. 5).

A simple lattice mass model with diffusion, aggregation and chipping of single units of mass was shown to exhibit nonequilibrium phase transition in the steady state.[8, 9, 3] In this paper we have shown that if a random fraction chips off instead of a single unit, the steady state no longer has a phase transition as the rates of microscopic processes are varied.

There are several directions for future work. For the asymmetric continuous mass model with continuous dynamics ($p \to 0$ limit of our model), Krug and Garcia[16] had derived density–density correlations between particles in the lattice gas representation. It would be interesting to extend their calculation to general p. Another interesting direction would be to derive the large scale hydrodynamics for general p and extend the calculation of the tracer diffusion coefficient[16, 23] to general p.

APPENDIX A. NON EXACTNESS OF PRODUCT MEASURE ANSATZ FOR ASYMMETRIC CONTINUOUS MODEL FOR $p<1$

In this appendix, we show that the product measure or the mean field theory is not exact for asymmetric continuous mass model for $p \neq 1$. The steps in the proof are as follows. In general the evolution equation of the n-point joint distribution function will involve the $(n+1)$-point distribution. If the product measure were to be exact, then every equation of this hierarchy has to be consistent with that ansatz. We show below that for the asymmetric case, the second equation of the hierarchy namely the one involving the 2-point and 3-point distributions is not consistent with product measure ansatz.

Firstly, we recall that we can derive an expression for the single point distribution $P(m)$ in the steady state by assuming product measure in the equation involving the single point and two point distributions (namely Eq. (7)). The Laplace transform $Q(s) = pV(pV+q)/(1-pqV-q^2)$ is given by Eq. (8) where $Q(s) = d(sV)/ds$. Next we write down the second equation of the hierarchy, namely k the evolution equation of the joint mass distribution, $P(m_i, m_{i+1})$ of two adjacent sites i and $(i+1)$,

$$\begin{aligned} P(m_i, m_{i+1}, t+1) = & \int dm_{i-1} \int dr_{i-1} f(r_{i-1}) \int dm'_i \int dr_i f(r_i) \\ & \times \int dm'_{i+1} \int dr_{i+1} f(r_{i+1}) \\ & \times P(m_{i-1}, m_i, m_{i+1}, t)\, \delta(m_{i-1} r_{i-1} + m'_i (1-r_i) - m_i) \\ & \times \delta(m'_i r_i + m'_{i+1}(1-r_{i+1}) - m_{i+1}) \end{aligned} \tag{33}$$

All the integrals over dm run from 0 to ∞ while the integrals over dr run from 0 to 1. $P(m_{i-1}, m_i, m_{i+1}, t)$ is the three point joint mass distribution function and $f(r)$ is given by Eq. (6). If the product measure were exact, the joint distributions in the above equation would factorize and the resulting equation must be satisfied by the $P(m)$ obtained from the first equation of the hierarchy, namely Eq. (8).

Assuming factorization $P(m_1, m_2, \ldots) = \prod_i P(m_i)$ in Eq. (33), multiplying both sides by $e^{-m_i s_1 - m_{i+1} s_2}$ and then integrating over m_i and m_{i+1}, we get

$$\begin{aligned} Q(s_1)\, Q(s_2) = & (q + pV(s_1))(qQ(s_2) + pV(s_2)) \\ & \times \left(qQ(s_1) + p \int_0^1 dr_i\, Q(s_1 + (s_2 - s_2)\, r_i) \right) \end{aligned} \tag{34}$$

If product measure were to be true, this equation must be satisfied with the expression of $Q(s)$ obtained from Eq. (8). If we substitute the expression for $Q(s)$ from Eq. (8) in the above equation, we find after a somewhat tedious but straightforward algebra, that Eq. (34) reduces to,

$$\left(\frac{V(s_2)}{V(s_1)}\right)^{2-p} - (2-p)\,\frac{V(s_2)}{V(s_1)} + 1 - p = 0 \tag{35}$$

If product measure is to be true then a necessary condition (but not sufficient) is that the above equation be satisfied for arbitrary values of s_1 and s_2. For $p<1$, this is an algebraic equation for the ratio $V(s_2)/V(s_1)$. Sine the coefficients do not involve s_1 or s_2, the solution for $V(s_2)/V(s_1)$ will be a constant independent of s_1 and s_2. Clearly this can not be true for arbitrary values of s_1 and s_2. Thus product ansatz is not exact for $p<1$ for asymmetric dynamics.

Note however that for $p=1$, Eq. (35) becomes an identity. This however is a necessary but not sufficient condition to prove that product measure is exact for $p=1$. However it was shown[(15)] that for $p=1$, all equations of the hierarchy of distribution functions are actually consistent with product measure ansatz.

APPENDIX B. PROOF OF EXACTNESS OF PRODUCT ANSATZ FOR $p \to 0$ FOR THE SYMMETRIC MODEL

In this appendix we show that the mean field is exact for the $p \to 0$ limit of the symmetric continuous model. Consider a cluster of n consecutive sites 1, 2,..., n. In the steady state, the joint probability distribution function $P(m_1, m_2,..., m_n)$ satisfies the equation,

$$\begin{aligned}
0 = &-(2n+2)\,P(m_1,..., m_n)\\
&+\int_0^\infty dm_0 \int_0^1 dr \int_0^\infty dm_1'\, P(m_0,..., m_n)(\delta(m_1' + m_0 r - m_1)\\
&+\delta(m_1'(1-r) - m_1))\\
&+\int_0^\infty dm_{n+1} \int_0^1 dr \int_0^\infty dm_n'\, P(m_1,..., m_{n+1})(\delta(m_n' + m_{n+1} r - m_n)\\
&+\delta(m_n'(1-r) - m_n))
\end{aligned}$$

$$+\sum_{i=1}^{n-1}\int_0^\infty dm'_i \int_0^1 dr \int_0^\infty dm'_{i+1} P(m_1,..., m_n)$$
$$\times (\delta(m'_i(1-r)-m_i)\, \delta(m'_{i+1}+m'_i r-m_{i+1})$$
$$+\delta(m'_i+m'_{i+1}r-m_i)\, \delta(m'_{i+1}(1-r)-m_{i+1})) \tag{36}$$

The first term is the total rate of going out of the state. The second and third terms describe the mass transfer at the boundary of the n-cluster while the last term accounts for mass transfer within the cluster. Let $P(m_1,..., m_n)=\prod_1^n P(m_i)$. We multiply both sides of the equation by $e^{-m_1 s_1 - \cdots - m_n s_n}$ and sum over $m_1,..., m_n$. The resulting terms in the right hand side can be simplified by using the explicit expression of $V(s)$ from Eq. (10). Then each one of the terms involving the integrals reduces to $2\prod_1^n Q(s_i)$, where $Q(s)=\int_0^\infty P(m)\, e^{-ms}\, dm$ as before. Thus Eq. (36) is indeed satisfied by the product measure ansatz for all n. Joint probability distributions for any n arbitrary sites can be split up into product of distributions for clusters of neighbouring sites, and then the proof can be applied for each of the individual clusters.

We note that for the $p \to 0$ limit of the symmetric model, it was shown in ref. 16 by a different method that the product measure is exact for any finite system of size N.

For symmetric model with $p > 0$, the product measure is not exact as was shown in the text by explicit calculation of two point mass correlation function.

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