Area distribution and the average shape of a Lévy bridge

Grégory Schehr

Laboratoire de Physique Théorique, CNRS-UMR 8627, Université Paris-Sud, 91405 Orsay Cedex, France

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques, CNRS-UMR 8626, Université Paris-Sud, Bât. 100, 91405 Orsay Cedex, France

Abstract. We consider a one dimensional Lévy bridge x_B of length n and index $0 < \alpha < 2$, *i.e.* a Lévy random walk constrained to start and end at the origin after n time steps, $x_B(0) = x_B(n) = 0$. We compute the distribution $P_B(A, n)$ of the area $A = \sum_{m=1}^{n} x_B(m)$ under such a Lévy bridge and show that, for large n, it has the scaling form $P_B(A, n) \sim n^{-1-1/\alpha} F_\alpha(A/n^{1+1/\alpha})$, with the asymptotic behavior $F_\alpha(Y) \sim Y^{-2(1+\alpha)}$ for large Y. For $\alpha = 1$, we obtain an explicit expression of $F_1(Y)$ in terms of elementary functions. We also compute the average profile $\langle \tilde{x}_B(m) \rangle$ at time m of a Lévy bridge with fixed area A. For large n and large m and A, one finds the scaling form $\langle \tilde{x}_B(m) \rangle = n^{1/\alpha} H_\alpha \left(m/n, A/n^{1+1/\alpha} \right)$, where at variance with Brownian bridge, $H_\alpha(X, Y)$ is a non trivial function of the rescaled time m/n and rescaled area $Y = A/n^{1+1/\alpha}$. Our analytical results are verified by numerical simulations.

1. Introduction

Random walks, and the associated continuous-time Brownian motion (BM), are ubiquitous in nature. As such, they are not only the cornerstones of statistical physics [1, 2, 3] but have also found many applications in a variety of areas such as biology [4], computer science [5, 6] and finance [7]. Continuous time Brownian motion is simply defined by the equation of motion

$$x(0) = 0 \ , \ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \eta(t) \ , \tag{1}$$

where $\eta(t)$ is a Gaussian white noise of zero mean $\langle \eta(t) \rangle = 0$ and short range correlations $\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$ where D is the diffusion constant (in the following we set $D = 1$). An interesting variant of Brownian motion in a given time interval $[0, T]$ is the so called Brownian bridge $x_B(t)$ which is a Brownian motion conditioned to start and end at zero, *i.e.* $x_B(T) = x_B(0) = 0$. Here we focus on two interesting observables associated with this bridge, namely

- the distribution of the area A under the bridge (see Fig. 1 a))

$$A = \int_0^T x_B(t) \, \mathrm{d}t \ , \tag{2}$$

  which is obviously a random variable, being the sum of (strongly correlated) random variables. For the Brownian bridge, the distribution of A can easily be computed using the fact that $x_B(t)$ is a Gaussian random variable. This can be seen from the well known identity in law [2]

$$x_B(t) := x(t) - \frac{t}{T}x(T) \ , \tag{3}$$

  where $x(t)$ is a standard Brownian motion (1). For the Brownian bridge, A is thus also a centered Gaussian random variable. A direct computation of the second moment $\langle A^2 \rangle$ yields straightforwardly

$$P_B(A, T) = \sqrt{\frac{3}{\pi T^3}} \exp\left(-\frac{3A^2}{T^3}\right) \ . \tag{4}$$

- the average shape of the bridge, $\langle \tilde{x}_B(t) \rangle$ for a fixed area A (see Fig. 1 b)). For the Brownian bridge, it takes a simple form [8]

$$\langle \tilde{x}_B(t) \rangle = \frac{A}{T}f\left(\frac{t}{T}\right) \ , \ f(x) = 6x(1 - x) \ . \tag{5}$$

The distribution of the area under a Brownian bridge (4) is a standard result and its extension to various constrained Brownian motions has recently attracted much attention [9, 10, 11, 12, 13, 14, 15]. For instance, the distribution of the area under a Brownian excursion (*i.e.* a Brownian motion conditioned to start and end at 0 and constrained to stay positive in-between), the so called Airy-distribution, describes the statistics of the maximal relative height of one-dimensional elastic interfaces [9, 11, 15]. Another example is the area A under a Brownian motion till its first-passage time t_f [10], which has an interesting application to the description of the avalanches in the directed Abelian sandpile model proposed in Ref. [16], such that t_f relates to the avalanche duration and A to the size of the avalanche cluster. Related quantities were recently studied in the statistics of avalanches near the depinning transition of elastic manifolds in random media [17]. On the other hand, the average shape of random walk bridges with a fixed area A (5) has been studied some time ago in the context of wetting [8] to prove the validity of the Wulff construction in $1 + 1$ dimensions and more recently in the context of mass transport models [18]. In these models, where the transport rules depend on the environment of the departure site, the steady state has a pair-factorized form [19], which generalizes the factorized steady states found in simpler system like the zero range process [20, 21, 22]. As the mass density crosses some critical value, the system exhibits a condensation transition which is governed by interactions, which in turn give rise to a spatially extended condensate. It was shown in Ref. [18] that the shape of this condensate can be described by the average shape of a random walk bridge with fixed area, and the results of Ref. [8] were recovered.
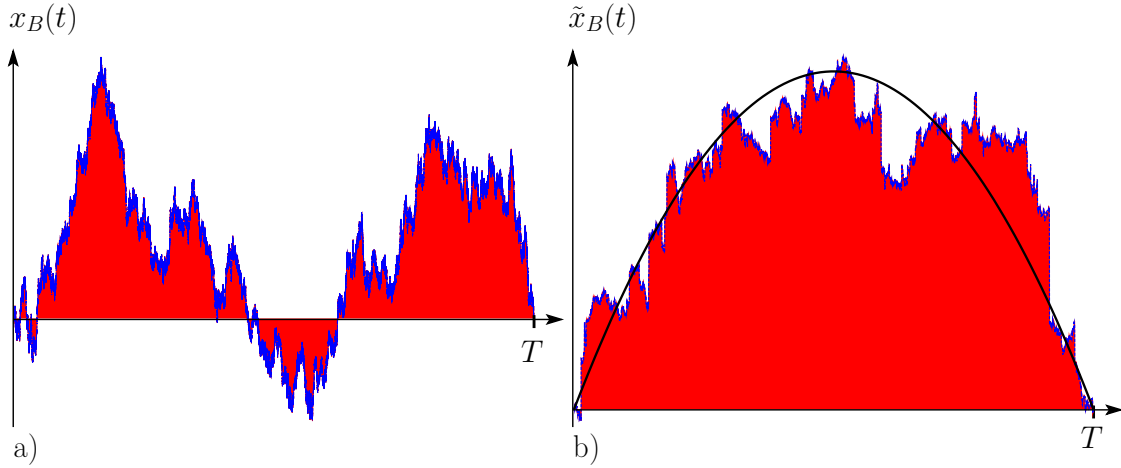
Figure 1. a): The blue line is the trajectory of a Brownian bridge $x_B(t)$ on $[0, T]$, $x_B(0) = x_B(T) = 0$ and in red is the area A under this Brownian bridge. **b)**: The typical trajectory of a Brownian bridge $\tilde{x}_B(t)$ (in blue) with a fixed area A (in red) on the interval $[0, T]$. The solid black line is the average profile $\langle \tilde{x}_B(t) \rangle$ as given by Eq. (5).

While these quantities are well understood for a Brownian bridge, much less is known for the case of a Lévy bridge. The aim of the present paper is to compute the distribution of the area and the average shape for a fixed area A in that case. To this purpose, it is convenient to consider a random walk $x(m)$, in discrete time (see Fig. 2 a)), starting at $x(0) = x_0$ at time 0 and evolving according to

$$x(m) = x(m-1) + \eta(m) \,, \tag{6}$$

where $\eta(m)$ are independent and identically distributed (i.i.d.) random variables distributed according to a common distribution $\phi(\eta)$. Here we focus on the case where $\phi(\eta) = \mathcal{S}_\alpha(\eta)$ where $\mathcal{S}_\alpha(\eta)$ is a symmetric α-stable (Lévy) distribution. Its characteristic function is given by $\int_{-\infty}^{\infty} \mathcal{S}_\alpha(\eta) e^{ik\eta} d\eta = e^{-|k|^\alpha}$. In particular, for large η, the distribution of η has a power law tail $\phi(\eta) \sim \eta^{-(1+\alpha)}$, with $0 < \alpha < 2$. The Lévy bridge $x_B(m)$, on the interval $[0, n]$, is a Lévy random walk conditioned to start and end at 0, *i.e.* $x_B(n) = x_B(0) = 0$. In the following we will compute the distribution $P_B(A, n)$ of the area A under the bridge of length n, *i.e.* $A = \sum_{m=0}^{n} x_B(m)$ and the average shape of a bridge $\langle \tilde{x}_B(m) \rangle$, with $0 \leq m \leq n$, for fixed A. Here we consider the natural scaling limit where $x_B \sim n^{1/\alpha}$, while $A \sim n^{1+1/\alpha}$, whereas the aforementioned previous works [8, 18] focused on a different scaling limit which, for Brownian motion, corresponds to $x_B \sim \sqrt{n}$ and $A \sim n^2$. Note that, even in this natural scaling limit, the identity in law valid for the Brownian bridge (3) does not hold for a Lévy bridge [23, 24] and one thus expects the distribution of the area A to be non trivial. Our results can be summarized as follows:

- for a free Lévy random walk starting at $x_0 = 0$, one finds that the distribution of the area $P(A, n)$ takes the form

$$P(A, n) = \frac{1}{\gamma_n} \mathcal{S}_\alpha \left(\frac{A}{\gamma_n} \right) \,, \; \gamma_n = \left(\sum_{m=1}^{n} m^\alpha \right)^{1/\alpha} , \tag{7}$$

where $\mathcal{S}_\alpha(x)$ is a symmetric α-stable distribution (21). For a Lévy bridge, one finds, in the scaling limit $A \to \infty$, $n \to \infty$, keeping $A/n^{1+1/\alpha}$ fixed, that the distribution of the area $P_B(A, n)$ takes the scaling form

$$P_B(A,n) \sim \frac{1}{n^{1+1/\alpha}} F_\alpha\left(\frac{A}{n^{1+1/\alpha}}\right) , \tag{8}$$

where $F_\alpha(y)$ is a monotonically decreasing function, with asymptotic behaviors

$$F_\alpha(Y) \sim \begin{cases} F_\alpha(0) , & Y \to 0 , \\ \dfrac{a_\alpha}{Y^{2(1+\alpha)}} , & Y \to \infty , \end{cases} \tag{9}$$

where $F_\alpha(0)$, see Eq. (36), and a_α, see Eq. (37), are computable constants. For $\alpha = 1$, we obtain an explicit expression for $F_1(Y)$ in terms of elementary functions (C.6).

- on the other hand, in the aforementioned scaling limit, one obtains the average profile $\langle \tilde{x}_B(m) \rangle$ for a Lévy bridge as well as the average profile $\langle \tilde{x}(m) \rangle$ for a free Lévy walk with a fixed area A. For $\langle \tilde{x}(m) \rangle$ one obtains a farely simple expression

$$\langle \tilde{x}(m) \rangle = \frac{A}{n} \frac{\alpha+1}{\alpha} \left[1 - \left(1 - \frac{m}{n}\right)^\alpha \right] . \tag{10}$$

For a Lévy bridge, the expression is more involved. For generic α one finds the scaling form

$$\langle \tilde{x}_B(m) \rangle = n^{1/\alpha} H_\alpha\left(\frac{m}{n}, \frac{A}{n^{1+1/\alpha}}\right) , \tag{11}$$

which, in general, has a non-trivial dependence on A. One recovers a linear dependence in A (as for the Brownian bridge in Eq. (5)) only in the limits $A \to 0$ (68) and $A \to \infty$ (69).

The paper is organized as follows. In section 2, we compute the joint distribution of the position and the area under a Lévy random walk. In section 3, we use these results to compute the distribution of the area A under a Lévy bridge of size n while in section 4, we use them to compute the average profile of a Lévy walk with a fixed area. Finally, in section 5 we present a numerical method, based on a Monte-Carlo algorithm, to compute numerically $P_B(A,n)$ and $\langle \tilde{x}_B(m) \rangle$ before we conclude in section 6. Some technical (and useful) details have been left in Appendices A,B and C.

2. Free Lévy walk : joint distribution of the position and the area

We start with the computation of the joint distribution $P(x, A, m|x_0, x_0, 0)$ of the position and the area after m steps given that $x(0) = x_0$ (see Fig. 2 a)). If we denote by $A(m)$ the area under the random walk after m time steps, this random variable evolves according to the equation

$$A(0) = x_0 , \tag{12}$$
$$A(m) = A(m-1) + x(m) . \tag{13}$$

Therefore $P(x, A, m|x_0, x_0, 0)$ satisfies the following recursion relation:

$$P(x, A, 0|x_0, x_0, 0) = \delta(x - x_0)\delta(A - x_0) ,$$
$$P(x, A, m|x_0, x_0, 0) = \int_{-\infty}^{\infty} P(x - \eta, A - x, m-1|x_0, x_0, 0)\phi(\eta)\mathrm{d}\eta . \tag{14}$$

Introducing $\hat{\phi}(k) = \int_{-\infty}^{\infty} \phi(\eta)e^{ik\eta}\mathrm{d}\eta$ the Fourier transform of $\phi(\eta)$, and thus $\hat{\phi}(k) = e^{-|k|^\alpha}$ for a Lévy random walk, and $\hat{P}(k_1, k_2, m|x_0, x_0, 0)$ the double Fourier transform of $P(x, A, m|x_0, x_0, 0)$ with respect to both x and A, *i.e.* $\hat{P}(k_1, k_2, m|x_0, x_0, 0) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}A P(x, A, m|x_0, x_0, 0)e^{ik_1 x + ik_2 A}$ the recursion relation (14) reads

$$\hat{P}(k_1, k_2, 0|x_0, x_0, 0) = e^{i(k_1+k_2)x_0} , \tag{15}$$
$$\hat{P}(k_1, k_2, m|x_0, x_0, 0) = \hat{\phi}(k_1 + k_2)\hat{P}(k_1 + k_2, k_2, m-1|x_0, x_0, 0) , \tag{16}$$
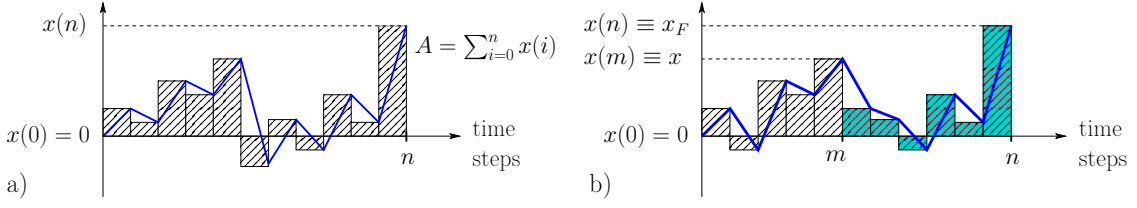
Figure 2. a): The area A under a random walk in discrete time. **b)**: Illustration of the method, see Eq. (46), to compute the probability $\tilde{P}(x,m|A,n)$ that the position of the random walker, starting in $x(0) = 0$ is x after m time steps, given that the area, after n time steps, is fixed to A. The light area corresponds to A_1, the area under the walk over the interval $[0,m]$ while the shaded area corresponds to the area $A - A_1$, the area under the walk over the interval $[m,n]$.

which can be solved, yielding

$$\hat{P}(k_1, k_2, n|x_0, x_0, 0) = \prod_{m=1}^{n} \hat{\phi}(k_1 + mk_2) e^{i(k_1 + (n+1)k_2)x_0} . \tag{17}$$

Hence for a Lévy walk of index α one has simply

$$P(x, A, n|x_0, x_0, 0) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-\sum_{m=1}^{n} |k_1 + mk_2|^\alpha} e^{-ik_1(x-x_0)} e^{-ik_2(A-(n+1)x_0)} . \tag{18}$$

Note that this expression (18) can also be obtained directly by noticing that $x(n) = x_0 + \sum_{m=1}^{n} \eta(i)$ and thus $A(n) - (n+1)x_0 = \sum_{m=1}^{n} x(m) = \sum_{m=1}^{n} \sum_{l=1}^{m} \eta(l) = \sum_{m=1}^{n} m\eta(n+1-m)$ such that

$$P(x, A, n|x_0, x_0, 0) = \prod_{m=1}^{n} \int_{-\infty}^{\infty} d\eta(m) \prod_{m=1}^{n} \phi[\eta(m)] \delta\left(x - x_0 - \sum_{m=1}^{n} \eta(m)\right)$$

$$\times \delta\left(A - (n+1)x_0 - \sum_{m=1}^{n} m\eta(n+1-m)\right) . \tag{19}$$

After a double Fourier transform with respect to x and A, this Eq. (19) yields immediately the expression $\hat{P}(k_1, k_2, n|x_0, x_0, 0)$ in Eq. (17). Of course the marginal distribution of the position $P(x, n|x_0, 0)$ and of the area $P(A, n|x_0, 0)$ are also stable laws. Indeed one has

$$P(x, n|x_0, 0) = \int_{-\infty}^{\infty} P(x, A, n|x_0, x_0, 0) dA = \frac{1}{n^{1/\alpha}} \mathcal{S}_\alpha\left(\frac{x - x_0}{n^{1/\alpha}}\right) , \tag{20}$$

$$P(A, n|x_0, 0) = \int_{-\infty}^{\infty} P(x, A, n|x_0, x_0, 0) dx = \frac{1}{\gamma_n} \mathcal{S}_\alpha\left(\frac{A - (n+1)x_0}{\gamma_n}\right) ,$$

$$\gamma_n = \left(\sum_{m=1}^{n} m^\alpha\right)^{1/\alpha} \sim \frac{n^{1+1/\alpha}}{(\alpha+1)^{1/\alpha}} , \ n \gg 1 ,$$

where

$$\mathcal{S}_\alpha(x) = \int_{-\infty}^{\infty} e^{-|k|^\alpha - ikx} \frac{dk}{2\pi} . \tag{21}$$

For example, $\mathcal{S}_1(x)$ is the Cauchy distribution while $\mathcal{S}_2(x)$ is a Gaussian distribution :

$$\mathcal{S}_1(x) = \frac{1}{\pi} \frac{1}{1 + x^2} , \ \mathcal{S}_2(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} . \tag{22}$$

Note also the explicit expression

$$\mathcal{S}_\alpha(0) = \int_{-\infty}^{\infty} e^{-|k|^\alpha} \frac{dk}{2\pi} = \frac{\Gamma(1 + \alpha^{-1})}{\pi} , \tag{23}$$

which will be useful in the following.

We now want to study $P(x, A, n|x_0, x_0, 0)$ in the limit of large n. The marginal distributions in Eq. (20) suggest the scaling $x \sim n^{1/\alpha}$ and $A \sim n^{1/\alpha+1}$. From the expression in Eq. (18) one

checks explicitly that in the limit $n \to \infty$, keeping $X = x/n^{1/\alpha}$ and $Y = A/n^{1/\alpha+1}$ fixed, the joint distribution takes the scaling form

$$P(x, A, n|x_0, x_0, 0) = \frac{1}{n^{2/\alpha+1}} G\left(\frac{x}{n^{1/\alpha}}, \frac{A}{n^{1/\alpha+1}} \middle| \frac{x_0}{n^{1/\alpha}}\right) , \tag{24}$$

where the function $G(X, Y|X_0)$ is given by

$$G(X, Y|X_0) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-\int_0^1 |k_1+k_2 z|^\alpha dz - ik_1(X-X_0) - ik_2(Y-X_0)} . \tag{25}$$

After the change of variable $k_2 = k$ and $k_1 = kr$, we obtain

$$G(X, Y|X_0) = \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| e^{-|k|^\alpha \gamma(r) - ikr(X-X_0) - ik(Y-X_0)} , \quad \gamma(r) = \int_0^1 |r + z|^\alpha dz , \tag{26}$$

where the function $\gamma(r)$ is explicitly given by

$$\gamma(r) = \begin{cases} \dfrac{1}{\alpha+1}\left((-r)^{\alpha+1} - (-1-r)^{\alpha+1}\right) , & r < -1 , \\[2mm] \dfrac{1}{\alpha+1}\left((r+1)^{\alpha+1} + (-r)^{\alpha+1}\right) , & -1 \le r \le 0 , \\[2mm] \dfrac{1}{\alpha+1}\left((r+1)^{\alpha+1} - r^{\alpha+1}\right) , & r > 0 . \end{cases} \tag{27}$$

For $\alpha = 2$, one has $\gamma(r) = r^2 + r + 1/3$ and one finds

$$G(X, Y|X_0) = \frac{\sqrt{3}}{2\pi} \exp\left[-3(Y - X_0)(Y - X) - (X - X_0)^2\right] , \tag{28}$$

which yields back the propagator of the so-called random acceleration process [25].

3. Lévy bridge

In the absence of any constraint for the walker, the area under a Lévy random walk is the sum of Lévy random variables and is thus again a Lévy random variable. However, if one considers constrained Lévy walks, this is not true anymore and the area may become different from a simple Lévy random variable. In the first subsection, we compute the distribution of the position for a Lévy bridge while the second subsection is devoted to the distribution of the area under this bridge.

3.1. Distribution of the position

Here we study the Lévy bridge $\{x_B(m)\}_{0 \le m \le n}$ which starts at 0 at time 0, $x_B(0) = 0$, and is constrained to come back to 0 after n time steps, *i.e.* $x_B(n) = 0$. In that case, one can compute the distribution of the position $P_B(x, m)$ after m time steps for such a bridge as $P_B(x, m) = P(x, m|0, 0)P(x, n - m|0, 0)/P(0, n|0, 0)$, such that in the scaling limit one has

$$P_B(x, m) = \frac{1}{n^{1/\alpha}} G_B\left(\frac{x}{n^{1/\alpha}}, \frac{m}{n}\right) ,$$

$$G_B(X, \tau) = \frac{\pi}{\Gamma(1 + \alpha^{-1})} \frac{1}{(\tau(1 - \tau))^{1/\alpha}} \mathcal{S}_\alpha\left(\frac{X}{\tau^{1/\alpha}}\right) \mathcal{S}_\alpha\left(\frac{X}{(1 - \tau)^{1/\alpha}}\right) , \tag{29}$$

where we have used $\mathcal{S}_\alpha(0) = \Gamma(1 + \alpha^{-1})/\pi$, see Eq. (23). For $\alpha = 2$ it is easy to see from Eq. (29) that $x_B(m)$ is a Gaussian variable. However, for $\alpha < 2$, the Lévy bridge is not any more a Lévy random variable. For instance, for $\alpha = 1$ one obtains a non trivial distribution

$$G_B(X, \tau) = \frac{1}{\pi} \frac{\tau(1 - \tau)}{(\tau^2 + X^2)((1 - \tau)^2 + X^2)} . \tag{30}$$

It is also easy to see that, for any $\alpha < 2$ one has the asymptotic behavior

$$G_B(X, \tau) \sim c'_\alpha \tau(1 - \tau) X^{-2(\alpha+1)} , \quad X \gg 1 , \tag{31}$$

where c'_α is independent of τ, which implies that $\langle (x_B(m))^2 \rangle$ is well defined for $\alpha > 1/2$ (of course $\langle x_B(m) \rangle = 0$ by symmetry for all α). A straightforward calculation shows that

$$\frac{\langle x_B^2(m) \rangle}{n^{2/\alpha}} = \tilde{a}_\alpha \frac{m}{n}\left(1 - \frac{m}{n}\right) , \quad \tilde{a}_\alpha = \frac{\alpha \Gamma(2 - \alpha^{-1})}{\Gamma(1 + \alpha^{-1})} . \tag{32}$$

It is interesting to notice that $\langle x_B^2(m)\rangle/n^{2/\alpha}$ depends on α only through the amplitude $\tilde{a}_\alpha$ but the parabolic shape in $(m/n)(1-m/n)$ holds for all values of $2 \geq \alpha > 1/2$. Besides $\tilde{a}_\alpha$ is diverging for $\alpha \to 1/2^+$ while one has $\tilde{a}(1) = 1$ and $\tilde{a}(2) = 2$. One finds, curiously, that it reaches a minimum for a non-trivial value $\alpha^* = 0.74122\ldots$ for which $\tilde{a}(\alpha^*) = 0.85264\ldots$. In view of these properties (29, 30) one expects that, for $\alpha < 2$, the area under such a Lévy bridge has a non trivial distribution, which we now focus on.

3.2. Distribution of the area

In this subsection, we consider a Lévy bridge, *i.e.* a Lévy walk which starts at the origin $x_0 = 0$ and is conditioned to come back to the origin after n steps and we ask : what is the distribution $P_B(A, n)$ of the area A under this *Lévy* bridge ? One can obtain $P_B(A, n)$ from Eqs (18, 20) as

$$P_B(A, n) = \frac{P(0, A, n|0, 0, 0)}{P(x = 0, n|0, 0)} . \tag{33}$$

Therefore, using $P(x = 0, n|0, 0) = n^{-1/\alpha}\mathcal{S}_\alpha(0) = n^{-1/\alpha}\Gamma(1 + \alpha^{-1})/\pi$, see Eq. (23), together with the scaling form (24, 26) one obtains, in the limit $n \to \infty$, keeping $A/n^{1+1/\alpha}$ fixed:

$$P_B(A, n) = \frac{1}{n^{1+1/\alpha}}F_\alpha\left(\frac{A}{n^{1+1/\alpha}}\right) ,$$

$$F_\alpha(Y) = \frac{1}{2\Gamma(1 + \alpha^{-1})}\int_{-\infty}^\infty dr \int_{-\infty}^\infty \frac{dk}{2\pi}|k|e^{-|k|^\alpha \gamma(r)-iYk} . \tag{34}$$

Using the explicit expression of $\gamma(r)$ above (27), one computes the Fourier transform $\hat{F}_\alpha(k)$ of $F_\alpha(Y)$ as

$$\hat{F}_\alpha(k) = \int_{-\infty}^\infty F_\alpha(Y)e^{ikY}dY , \tag{35}$$

$$= \frac{|k|}{\Gamma(1 + \frac{1}{\alpha})}\left[\int_0^\infty e^{-\frac{|k|^\alpha}{\alpha+1}\left[(r+1)^{\alpha+1}-r^{\alpha+1}\right]}dr + \int_0^{1/2} e^{-\frac{|k|^\alpha}{\alpha+1}\left[(1/2+r)^{\alpha+1}+(1/2-r)^{\alpha+1}\right]}dr\right] .$$

For generic α, it seems quite difficult to perform explicitly the integrals over r and k in the expression for the distribution $F_\alpha(Y)$ in Eq. (34). One can however extract from this expression the asymptotic behaviors both for $Y \to 0$ and $Y \to \infty$.

Asymptotic behavior for small argument. For small argument, it is straightforward to see on the expression (34) above that the leading behavior of $F_\alpha(Y)$ when $Y \to 0$ is given by

$$F_\alpha(Y) \sim F_\alpha(0) , \; Y \to 0 , \text{with}$$

$$F_\alpha(0) = \frac{(\alpha + 1)^{\frac{2}{\alpha}}}{2\pi}\frac{\Gamma(1 + \frac{2}{\alpha})}{\Gamma(1 + \frac{1}{\alpha})}\left(\int_0^\infty \frac{dr}{((r + 1)^{\alpha+1} - r^{\alpha+1})^{\frac{2}{\alpha}}} + \int_0^{1/2} \frac{dr}{((1/2 + r)^{\alpha+1} + (1/2 - r)^{\alpha+1})^{\frac{2}{\alpha}}}\right) .$$

A study of this function $F_\alpha(0)$ shows that it is a decreasing function of α on the interval $]0, 2]$, which is diverging when $\alpha \to 0$. For $\alpha = 1$ and $\alpha = 2$, $F_\alpha(0)$ assumes simple values

$$F_1(0) = 1 + \frac{4}{\pi} = 2.27324... , \; F_2(0) = \sqrt{\frac{3}{\pi}} = 0.977205... \tag{36}$$

Asymptotic behavior for large argument. The analysis of the large argument behavior of $F_\alpha(Y)$ is more involved. A careful analysis, left in Appendix B, shows that for large Y one has:

$$F_\alpha(Y) \sim \frac{a_\alpha}{Y^{2(1+\alpha)}} , \; Y \gg 1 , \tag{37}$$

$$a_\alpha = \frac{2^{-2(2+\alpha)}\sqrt{\pi}\Gamma(2 + 2\alpha)\tan(\alpha\pi/2)}{\Gamma(2 + \alpha^{-1})\Gamma(1 - \alpha)\Gamma(\frac{5}{2} + \alpha)} . \tag{38}$$

When $\alpha \to 2$, one has from (37), $a_\alpha \sim \frac{\sqrt{\pi}}{21}(\alpha - 2)^2$. From Eq. (37) one obtains that the second moment of the distribution $\langle Y^2\rangle$ is defined only for $\alpha > 1/2$ where it takes the value (see Eq. (B.27))

$$\langle Y^2\rangle = \frac{\tilde{a}_\alpha}{12} = \frac{\alpha\Gamma(2 - \alpha^{-1})}{12\Gamma(1 + \alpha^{-1})} , \; \alpha > 1/2 , \tag{39}$$

where the amplitude $\tilde{a}_\alpha$ appears in the expression for $\langle x_B^2(m)\rangle$ computed above (32). This power law tail of the area distribution (37) with an exponent $2(1+\alpha)$ is quite interesting. Indeed, the area itself is the sum of non-identical and strongly correlated variables $x_B(m)$ all having a similar power law tail also with exponent $2(1+\alpha)$ (31). For $\alpha > 1/2$, their variance is finite and the non-Gaussianity of A can a priori be due both to the correlations between the $x_B(m)$'s and to the fact that A is the sum of non-identical random variables. To test which of these features is responsible for the non-Gaussianity of A, we study the sum of n random variables X_m which are *independent* and such that X_m has the same distribution as $x_B(m)$. Defining $S_n = \sum_{m=1}^{n} X_m$ and $\Sigma_n^2 = \sum_{m=1}^{n} \langle X_m^2 \rangle$, it is known that S_n/Σ_n converges to a centered Gaussian variable of unit variance if the following condition (known as the Lindeberg's condition) is satisfied [2]

$$\lim_{n\to\infty} \frac{1}{\Sigma_n^2} \int_{|x|>\epsilon\Sigma_n} x^2 \mathrm{Proba}(X_m = x)\mathrm{d}x = 0 \ , \ \forall \epsilon > 0 \ . \tag{40}$$

Intuitively, this Lindeberg condition (40) ensures that the probability that any term X_m will be of the same order of magnitude as the sum S_n must tend to zero (see Appendix A for an example of non-identical independent random variables which do not satisfy the Lindeberg condition). In the present case of the Lévy bridge, one can check (see Appendix A) that if X_m is distributed like $x_B(m)$ then the above Lindeberg condition (40) is satisfied (see Appendix A). Therefore the deviations from Gaussianity (37) are purely due to the *strong correlations* between the positions of the walker $x_B(m)$'s.

The special case $\alpha = 1$. In the Cauchy case, $\alpha = 1$, the integral over k can be done in Eq. (26) to obtain

$$G(X, Y|0) = \int_{-\infty}^{\infty} \frac{\gamma^2(r) - (rX + Y)^2}{(\gamma^2(r) + (rX + Y)^2)^2} \frac{\mathrm{d}r}{2\pi^2} \ , \tag{41}$$

where the function $\gamma(r)$ (27) takes here a rather simple form

$$\gamma(r) = \begin{cases} -r - \dfrac{1}{2} \ , \ r < -1 \\[2mm] r^2 + r + \dfrac{1}{2} \ , \ -1 \le r \le 0 \\[2mm] r + \dfrac{1}{2} \ , r > 0 \ . \end{cases} \tag{42}$$

The distribution of the area under a Cauchy bridge is thus given by

$$P_B(A, n) = \frac{1}{n^2} F_1\left(\frac{A}{n^2}\right) \ ,$$

$$F_1(Y) = \frac{1}{\pi} \frac{2}{1 + 4Y^2} + \frac{1}{\pi} \int_0^{1/2} \frac{(u^2 + 1/4)^2 - Y^2}{((u^2 + 1/4)^2 + Y^2)^2} \mathrm{d}u \ . \tag{43}$$

Under this form (43), one can easily obtain the asymptotic behaviors as

$$F_1(Y) \sim \begin{cases} 1 + \dfrac{4}{\pi} \ , \ Y \to 0 \\[2mm] \dfrac{1}{20\pi Y^4} \ , \ Y \to \infty \end{cases} \tag{44}$$

in agreement with the asymptotic behaviors obtained above (36, 37). In fact the integral over u in the expression above (43) can be done explicitly yielding the expression

$$F_1(Y) = \frac{1}{\pi} \frac{2}{1 + 4Y^2} \tag{45}$$

$$+ \frac{1}{\pi} \left(\frac{2(1 - 8Y^2)}{(1 + 4Y^2)(1 + 16Y^2)} + \frac{4}{(1 + 16Y^2)^{\frac{3}{2}}} \mathrm{Re}\left[(1 - 4iY)^{\frac{3}{2}} \arctan\left((1 + 4iY)^{-\frac{1}{2}} \right) \right] \right) \ ,$$

where $\mathrm{Re}(z)$ denotes the real part of the complex number z. In Appendix C we show how this expression (45) can be written explicitly in terms of elementary functions (C.4, C.5, C.6).

4. Average profile for fixed area A.

4.1. The case of a free Lévy walk

We first consider the case of a free Lévy walk of constrained area. We compute the probability $\tilde{P}(x, m|A, n)$ that the position of the random walker, starting at $x(0) = x_0 = 0$ at time 0, is x after m time steps given that the area, after n time steps, is fixed to A. From this probability, one obtains the average profile as $\langle \tilde{x}(m) \rangle = \int_{-\infty}^{\infty} x \tilde{P}(x, m|A, n) \mathrm{d}x$. To compute this probability $\tilde{P}(x, m|A, n)$, we divide the interval $[0, n]$ into two intervals $[0, m]$ and $[m, n]$. Over $[0, m]$ the process starts at $x_0 = 0$ with area $A_0 = 0$ and reaches to x with area A_1 (see the light area on Fig. 2 b)). Over the interval $[m, n]$, the process starts in x and reaches to x_F with area $A - A_1$ (see the shaded area on Fig. 2 b)). Therefore this probability $\tilde{P}(x, m|A, n)$ can be simply expressed in terms of the propagator $P(x, A, n|x_0, x_0, 0)$ computed above (18) as (see Fig. 2 b)):

$$\tilde{P}(x, m|A, n) = \frac{1}{P(A, n)} \int_{-\infty}^{\infty} \mathrm{d}x_F \int_{-\infty}^{\infty} \mathrm{d}A_1 P(x, A_1, m|0, 0, 0) P(x_F, A - A_1, n - m|x, x, 0) , \quad (46)$$

where we have used the Markov property of the Lévy random walk. In the above expression (46), x_F is the end point of the walk (see Fig. 2 b)), which is free here. Hence $\tilde{P}(x, m|A, n)$ is obtained by integration over this end point x_F. Notice that it is normalized according to $\int_{-\infty}^{\infty} \tilde{P}(x, m|A, n) \mathrm{d}x = 1$ (and therefore we have divided by $P(A, n)$ in the expression above (46) because the measure is restricted to random walks of fixed area A after n time steps). Using the explicit expressions computed above (18) one obtains after integration over x_F and A_1:

$$\tilde{P}(x, m|A, n) = \frac{1}{P(A, n|0, 0)} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_2}{2\pi} e^{-\sum_{\nu=0}^{m} |k_1 + \nu k_2|^\alpha - |k_2|^\alpha \sum_{\nu=0}^{n-m} |\nu|^\alpha} e^{-ik_1 x - ik_2(A - (n-m)x)} . \tag{47}$$

In the large n limit, keeping $X = x/n^{1/\alpha}$, $Y = A/n^{1+1/\alpha}$ and $\tau = m/n$ fixed one has

$$\tilde{P}(x, m|A, n) = \frac{1}{n^{1/\alpha}} \tilde{G}(X, \tau|Y) ,$$

$$\tilde{G}(X, \tau|Y) = \frac{(1+\alpha)^{-1/\alpha}}{\mathcal{S}_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} |k| e^{-|k|^\alpha (\tilde{\gamma}(r, \tau) + \tilde{\gamma}(0, 1 - \tau))} e^{-ikrX - ik(Y - (1-\tau)X)} , \tag{48}$$

where we have used the expression of $P(A, n|0, 0)$ given in Eq. (20) and we have introduced

$$\tilde{\gamma}(r, \tau) = \int_0^\tau |r + z|^\alpha dz , \tag{49}$$

which is a generalization of the function $\gamma(r) \equiv \tilde{\gamma}(r, 1)$ in (26). It reads

$$\tilde{\gamma}(r, \tau) = \begin{cases} \dfrac{1}{\alpha+1} \left((-r)^{\alpha+1} - (-\tau - r)^{\alpha+1} \right) , & r \leq -\tau , \\[2mm] \dfrac{1}{\alpha+1} \left((r+\tau)^{\alpha+1} + (-r)^{\alpha+1} \right) , & -\tau \leq r \leq 0 , \\[2mm] \dfrac{1}{\alpha+1} \left((r+\tau)^{\alpha+1} - r^{\alpha+1} \right) , & r \geq 0 . \end{cases} \tag{50}$$

We can now compute $\langle \tilde{x}(m) \rangle$ which, in the large n limit, takes the scaling form

$$\frac{\langle \tilde{x}(m) \rangle}{n^{1/\alpha}} = h_\alpha \left(\frac{m}{n}, \frac{A}{n^{1+1/\alpha}} \right) \tag{51}$$

$$h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{\mathcal{S}_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \mathrm{d}X X \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} |k| e^{-|k|^\alpha g(r, \tau)} e^{-ikrX - ik(Y - (1-\tau)X)} .$$

with $g(r, \tau) = \tilde{\gamma}(r, \tau) + \tilde{\gamma}(0, 1 - \tau)$. This function $h_\alpha(\tau, Y)$ can be written as

$$h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{\mathcal{S}_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{|k|}{(-ik)} e^{-|k|^\alpha g(r, \tau)} \frac{\partial}{\partial r} \left(e^{-ikrX - ik(Y - (1-\tau)X)} \right) \tag{52}$$

which suggests to perform an integration by part in the integral over r, yielding (one can check that the boundary terms vanish)

$$h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{\mathcal{S}_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \mathrm{d}X \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{|k|^{1+\alpha}}{(-ik)} \frac{\partial g(r, \tau)}{\partial r} e^{-|k|^\alpha g(r, \tau)} e^{-ikrX - ik(Y - (1-\tau)X)} \tag{53}$$

On this expression (53), the integral over X can be done yielding simply a delta function of r, namely $2\pi|k|^{-1}\delta(r-(1-\tau))$. This allows us to perform then the integral over r to obtain

$$h_\alpha(\tau,Y) = \frac{(1+\alpha)^{-1/\alpha}}{\mathcal{S}_\alpha((\alpha+1)^{1/\alpha}Y)}i\left(\frac{\partial g(r,\tau)}{\partial r}\right)_{r=1-\tau}\int_{-\infty}^\infty |k|^\alpha k^{-1}e^{-\frac{|k|^\alpha}{1+\alpha}}e^{-ikY}\frac{\mathrm{d}k}{2\pi} \ , \ (54)$$
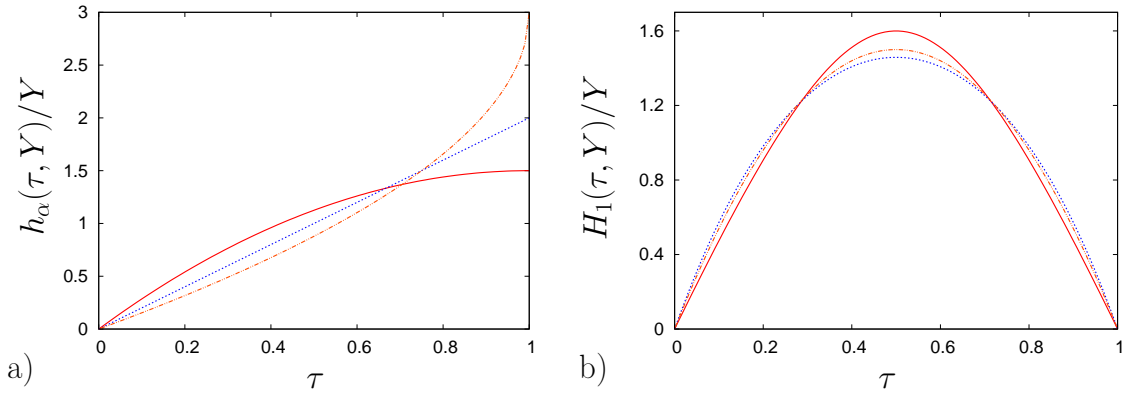
where we have used the relation $\tilde\gamma(1-\tau,\tau)+\tilde\gamma(0,1-\tau)=1/(1+\alpha)$. It is then easy to check that

$$i\int_{-\infty}^\infty |k|^\alpha k^{-1}e^{-\frac{|k|^\alpha}{1+\alpha}}e^{-ikY}\frac{\mathrm{d}k}{2\pi} = \frac{\alpha+1}{\alpha}Y(1+\alpha)^{1/\alpha}\mathcal{S}_\alpha\left[(1+\alpha)^{1/\alpha}Y\right] \ , \qquad (55)$$

so that finally one obtains the simple result

$$h_\alpha(\tau,Y) = Y\frac{\alpha+1}{\alpha}(1-(1-\tau)^\alpha) \ , \qquad (56)$$

where we have used $\left(\frac{\partial g(r,\tau)}{\partial r}\right)_{r=1-\tau} = 1-(1-\tau)^\alpha$. In Fig. (3) a), we show a plot of $h_\alpha(\tau,Y)/Y$ as a function of τ.

Figure 3. Average shape of a free Lévy walk in **a)** and of a Lévy bridge in **b)**. **a)**: Plot of $h_\alpha(\tau,Y)/Y$ given in Eq. (56) as a function of τ for $\alpha=2$ (solid line), $\alpha=1$ (dotted line) and $\alpha=1/2$ (dashed line). **b)**: Plot of $H_1(\tau,Y)/Y$ as a function of τ in the limit $A\to 0$ (solid line), as given by Eq. (68), and in the limit $A\to\infty$ (dotted line), as given in Eq. (69). For comparison, we have also plotted the parabola (dashed line) corresponding to the result for a Brownian bridge, *i.e.* $\alpha=2$, as given in Eq. (5)

4.2. The case of a Lévy bridge

Here we consider a Lévy bridge, i.e. a Lévy random walker starting in 0 at initial time and constrained to come back to the origin after n time steps. We compute the probability $\tilde{P}_B(x,m|A,n)$ that the position of the random walker is x after m time steps given that the area, after n time steps, is fixed to A. From this probability, one obtains the average profile as $\langle \tilde{x}_B(m)\rangle = \int_{-\infty}^\infty x\tilde{P}_B(x,m|A,n)\mathrm{d}x$. This probability $\tilde{P}_B(x,m|A,n)$ can be expressed, as in Eq. (46) in terms of the propagator $P(x,A,n|x_0,x_0,0)$ computed above in Eq. (18) as:

$$\tilde{P}_B(x,m|A,n) = \frac{1}{P(0,A,n|0,0,0)}\int_{-\infty}^\infty P(x,A_1,m|0,0,0)P(x,A-A_1,n-m|0,0,0)\mathrm{d}A_1 \ , \qquad (57)$$

where we have used the Markov property of the Lévy random walk. It is normalized according to $\int_{-\infty}^\infty \tilde{P}_B(x,m|A,n)\mathrm{d}x = 1$ (and therefore we have divided by $P(0,A,n|0,0,0)$ because the measure is restricted to bridges of fixed area A). Using the explicit expressions obtained above (18) one has

$$\tilde{P}_B(x,m|A,n) = \frac{1}{P(0,A,n|0,0,0)}\int_{-\infty}^\infty \frac{\mathrm{d}k_1}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{d}k_1'}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{d}k_2}{2\pi}e^{-i(k_1+k_1')x-ikA}$$

$$\times e^{-\sum_{\nu=1}^n |k_1+\nu k_2|^\alpha - \sum_{\nu=1}^{n-m}|k_1'+\nu k_2|^\alpha} \ . \qquad (58)$$

In the large n limit, keeping $X = x/n^{1/\alpha}$, $Y = A/n^{1+1/\alpha}$ and $\tau = m/n$ fixed one has

$$\tilde{P}_B(x, m|A, n) = \frac{1}{n^{1/\alpha}} \tilde{G}_B(X, \tau|Y) \,,$$

$$\tilde{G}_B(X, \tau|A) = \frac{\pi}{\Gamma(1 + \alpha^{-1}) F_\alpha(Y)} \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}r'}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} k^2 e^{-ik(r+r')X - ikA} \quad (59)$$

$$\times\, e^{-|k|^\alpha [\tilde{\gamma}(r,\tau) + \tilde{\gamma}(r', 1-\tau)]} \quad (60)$$

We can now compute $\langle \tilde{x}_B(m) \rangle$, which in the large n limit takes the scaling form

$$\frac{\langle \tilde{x}_B(m) \rangle}{n^{1/\alpha}} = H_\alpha \left(\frac{m}{n}, \frac{A}{n^{1+1/\alpha}} \right) \,, \quad (61)$$

where

$$H_\alpha(\tau, Y) = i \frac{1}{2\Gamma(1+\alpha) F_\alpha(Y)} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}r |k|^{\alpha-1} k \partial_r \tilde{\gamma}(r, \tau) e^{-|k|^\alpha \tilde{\Gamma}(r,\tau) - ikY} \quad (62)$$

$$= -\frac{1}{2\Gamma(1+\alpha) F_\alpha(Y)} \frac{\partial}{\partial Y} \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}r |k|^{\alpha-1} \partial_r \tilde{\gamma}(r, \tau) e^{-|k|^\alpha \tilde{\Gamma}(r,\tau) - ikY} \right) \,,$$

where we have introduced the notation $\tilde{\Gamma}(r, \tau) = \tilde{\gamma}(r, \tau) + \tilde{\gamma}(-r, 1 - \tau)$ which we compute straightforwardly from Eq. (50) as:

$$\tilde{\Gamma}(r, \tau) = \tilde{\gamma}(r, \tau) + \tilde{\gamma}(-r, 1 - \tau) = \begin{cases} \dfrac{1}{\alpha+1} \left((-r + 1 - \tau)^{\alpha+1} - (-r - \tau)^{\alpha+1} \right) \,, \; r \leq -\tau \\[2mm] \dfrac{1}{\alpha+1} \left((r + \tau)^{\alpha+1} + (-r + 1 - \tau)^{\alpha+1} \right) \,, \; -\tau \leq r \leq 1 - \tau \\[2mm] \dfrac{1}{\alpha+1} \left((r + \tau)^{\alpha+1} - (r - 1 + \tau)^{\alpha+1} \right) \,, \; r \geq 1 - \tau \,. \end{cases} \quad (63)$$

Note that this function $\tilde{\Gamma}(r, \tau)$ satisfies the identity

$$\tilde{\Gamma}(1 - \tau - r, \tau) = \int_0^1 |z - r|^\alpha \mathrm{d}z = \gamma(-r) \,, \quad (64)$$

independently of τ.

For generic α, the expression above (62) is quite difficult to handle. For $\alpha = 2$ (Brownian motion) and $\alpha = 1$, further analytical progress is however possible. For $\alpha = 2$, one has $\partial_r \tilde{\gamma}(r, \tau) = \tau(2r + \tau)$ and $\tilde{\Gamma}(r, \tau) = r^2 + r(2\tau - 1) - \tau(1 - \tau) + 1/3$ and therefore one checks

$$\int_{-\infty}^{\infty} \partial_r \tilde{\gamma}(r, \tau) e^{-k^2 \tilde{\Gamma}(r,\tau)} \mathrm{d}r = \frac{\sqrt{\pi}}{|k|} e^{-\frac{k^2}{12}} \tau(1 - \tau) \,. \quad (65)$$

Using this identity (65) and integrating over r in the expression above (62), and using $F_2(Y) = \sqrt{3/\pi} \exp\left(-3Y^3\right)$ one obtains for $\alpha = 2$:

$$H_2(\tau, Y) = 6Y\tau(1 - \tau) \,. \quad (66)$$

Another interesting case where analytical progress is possible is $\alpha = 1$. Given the expression of $\tilde{\gamma}(r, \tau)$ in Eq. (50) and $\tilde{\Gamma}(r, \tau)$ in Eq. (63), one observes that the integral over r in Eq. (62) gives rise to 4 different terms, corresponding to $r \in]-\infty, -\tau]$, $r \in [-\tau, 0]$, $r \in [0, 1 - \tau]$ and finally $r \in [1 - \tau, +\infty[$. For $\alpha = 1$ it turns out that the first and fourth terms, corresponding to $r \in]-\infty, -\tau]$ and $r \in [1 - \tau, +\infty[$ do cancel each other (which is the case only for $\alpha = 1$) resulting in the following expression:

$$H_1(\tau, Y) = \frac{Y}{\pi \Gamma(1+\alpha) F_1(Y)} \left(\int_{-\tau}^{0} (2r + \tau) \frac{\tilde{\Gamma}(r, \tau)}{(\tilde{\Gamma}(r, \tau)^2 + Y^2)^2} \mathrm{d}r + \tau \int_0^{1-\tau} \frac{\tilde{\Gamma}(r, \tau)}{(\tilde{\Gamma}(r, \tau)^2 + Y^2)^2} \mathrm{d}r \right) \quad (67)$$

with $\tilde{\Gamma}(r, \tau) = \frac{1}{2}((r + \tau)^2 + (r + \tau - 1)^2)$. In the asymptotic limit $Y \to 0$ one obtains

$$H_1(\tau, Y) \sim \frac{Y}{4 + \pi} \left(3\pi - 2 + \frac{2}{1 + 2\tau(\tau - 1)} + 12(2\tau - 1) \arctan(1 - 2\tau) \right) \,. \quad (68)$$

In the opposite limit $Y \to \infty$ one obtains

$$H_1(\tau, Y) \sim Y \frac{10}{3} \tau(1 - \tau)(2 - \tau(1 - \tau)) \,. \quad (69)$$

Note that although the two functions of τ entering these asymptotic expansions in Eq. (68) and Eq. (69) have very different analytical expressions, they are actually quite close to each other on the interval $[0, 1]$ (see Fig. (3) b)).

5. Numerical results

We now come to numerical simulations of Lévy bridges. As mentioned above, one can not use the relation above (3), which is only valid for $\alpha = 2$ [23] to simulate a Lévy bridge. Instead, we consider the joint probability distribution function (pdf) of the increments $\eta(m)$ for a Lévy bridge of size n. Indeed, these increments are independent random variables, distributed according to $\phi(\eta)$ with the global constraint that $x(n) = \sum_{m=1}^{n} \eta(n) = 0$. Therefore the joint pdf of the increments $P_B(\eta(1), \eta(2), \cdots, \eta(n))$ is simply given by

$$P_B(\eta(1), \eta(2), \cdots, \eta(n)) \propto \prod_{m=1}^{n} \phi[\eta(m)] \delta\left(\sum_{m=1}^{n} \eta(m)\right) \tag{70}$$

$$\propto \exp\left[\sum_{m=1}^{n} \ln[\phi[\eta(m)]]\right] \delta\left(\sum_{m=1}^{n} \eta(m)\right) .$$

This joint distribution can thus be considered as a Boltzmann weight with an effective energy $E = -\sum_{m=1}^{n} \ln[\phi[\eta(m)]]$ and effective inverse temperature $\beta = 1$. This thus leads us to use a Monte-Carlo algorithm, with a global constraint, to generate "configurations" of the increments distributed according to the distribution above (70). We implement it in the following way. We start with a random initial configuration of the $\eta(m)$'s which satisfies the global constraint $\sum_{m=1}^{n} \eta(m) = 0$ (it can also be $\eta(m) = 0$, for all m). At each time step we choose randomly two sites i and j among $1, 2, \cdots, n$ and the simple following moves are proposed

$$\eta(i) \to \eta'(i) = \eta(i) + \Delta\eta ,$$
$$\eta(j) \to \eta'(j) = \eta(j) - \Delta\eta , \tag{71}$$

such that the global constraint of zero sum is automatically satisfied. This move is then accepted, in Metropolis algorithm that we use here, with a probability P_{ij} given by

$$P_{ij} = \min\left(1, \frac{\phi[\eta'(i)] \phi[\eta'(j)]}{\phi[\eta(i)] \phi[\eta(j)]}\right) \tag{72}$$

$$= \min(1, \exp(-\Delta E)) , \ \Delta E = \log\left(\frac{\phi[\eta(i)] \phi[\eta(j)]}{\phi[\eta'(i)] \phi[\eta'(j)]}\right) \tag{73}$$

This Monte Carlo algorithm is thus very similar to the Kawasaki dynamics for ferromagnetic spin systems relaxing towards equilibrium with a conserved global magnetization [26]. Once the increments $\eta(k)$'s are generated according to this joint probability (70), we can generate the random walk bridge $x_B(m) = \sum_{k=1}^{m} \eta(k)$ and compute the distribution of the area $A = \sum_{m=1}^{n} x_B(m)$ under the Lévy bridge. In Fig. 4 a), we show a plot of this distribution $P_B(A, n)$ for $\alpha = 1$ and $n = 100$. To compute it we have first run 10^7 Monte Carlo steps to equilibrate the system and the distribution was then computed as an average over 10^7 samples generated in the time interval $[10^7, 2.10^7]$. In Fig. 4, we also show a plot of the exact explicit expression for $F_1(Y)$ given in Eq. (C.6), showing a very good agreement with our numerics. We have also computed numerically this distribution for other values of $\alpha \in]0, 2[$, showing a good agreement with the power law tail obtained in Eq. (37). Note however that for small α, it is actually quite difficult to equilibrate the system such that a precise estimate of the exponent characterizing the power law tail of $P_B(A, n)$ is quite difficult for $\alpha < 1$.

We can use a similar Monte Carlo approach to generate a random walk bridge with a fixed area $A = \sum_{m=1}^{n} (n + 1 - m) \eta(m)$. In that case, the joint pdf of the increments $\tilde{P}_B(\eta(1), \eta(2), \cdots, \eta(n))$ is simply given by

$$P_B(\eta(1), \eta(2), \cdots, \eta(n)) \propto \prod_{m=1}^{n} \phi[\eta(m)] \delta\left(\sum_{m=1}^{n} \eta(m)\right) \delta(\sum_{m=1}^{n} (n + 1 - m)\eta(m) - A) . \tag{74}$$

We start with an initial configuration of the $\eta(m)$'s which satisfies the both global constraints. In practice, we start with $\eta(m) = 6A(N + 1 - 2m)/((N - 1)N(N + 1))$. Then, to satisfy both constraints (74), at each time step we choose randomly three sites i, j and k among $1, 2, \cdots, n$ and the simple following moves are proposed

$$\eta(i) \to \eta'(i) = \eta(i) + \Delta\eta ,$$
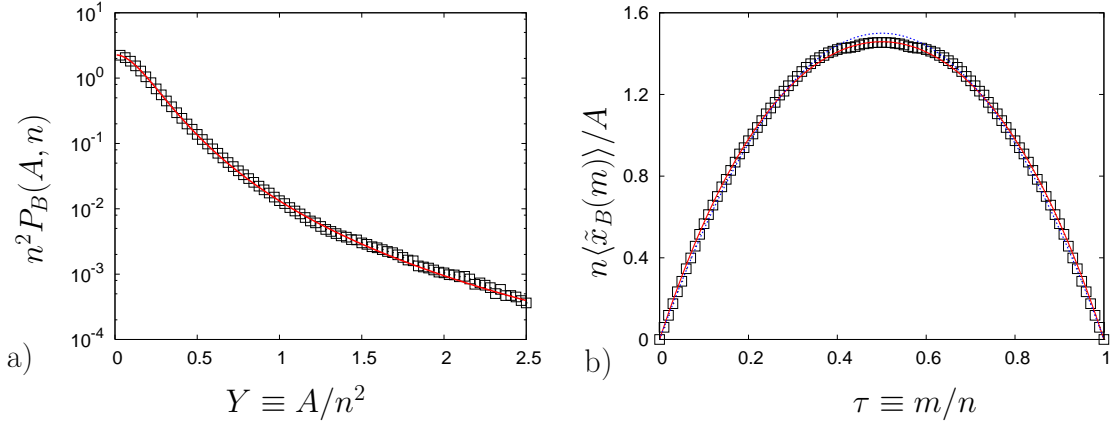
Figure 4. a): The squares represent the numerical data for $n^2 P_B(A, n)$ as a function of $Y \equiv A/n^2$ for a Lévy bridge of index $\alpha = 1$ of length $n = 100$. The solid line is our exact expression for $F_1(Y)$ given in Eq. (C.6). **b)**: The squares represent our numerical data for $n\langle \tilde{x}_B(m) \rangle / A$ as a function of $\tau = m/n$ for a Lévy bridge of index $\alpha = 1$ of length $n = 100$ and $A/n^2 = 20$. The solid line corresponds to our asymptotic result for large A given in Eq. (69) while the dotted line represents the result for the Brownian bridge in Eq. (5).

$$\eta(j) \to \eta'(j) = \eta(j) + \frac{i-k}{k-j}\Delta\eta \, ,$$

$$\eta(k) \to \eta'(k) = \eta(k) + \frac{j-i}{k-j}\Delta\eta \, . \tag{75}$$

Note that to converge to the correct probability measure (74) one has to choose η either positive or negative with equal probability. This move (75) is then accepted with a probability P_{ijk} given by

$$P_{ijk} = \min\left(1, \frac{\phi\left[\eta'(i)\right]\phi\left[\eta'(j)\right]\phi\left[\eta'(k)\right]}{\phi\left[\eta(i)\right]\phi\left[\eta(j)\right]\phi\left[\eta(k)\right]}\right) \tag{76}$$

$$= \min(1, \exp(-\Delta E)) \, , \quad \Delta E = \log\left(\frac{\phi\left[\eta(i)\right]\phi\left[\eta(j)\right]\phi\left[\eta(k)\right]}{\phi\left[\eta'(i)\right]\phi\left[\eta'(j)\right]\phi\left[\eta'(k)\right]}\right) \tag{77}$$

Once the increments $\eta(k)$'s are generated according to this joint probability (74), we can generate the random walk bridge $\tilde{x}_B(m) = \sum_{k=1}^{m} \eta(k)$ with fixed area A and compute the profile $\langle \tilde{x}_B(m) \rangle$. In Fig. 4 b), we show a plot of this average profile for $\alpha = 1$, $n = 100$ and $A/n^2 \sim 20$. To compute it we have first run 10^7 Monte Carlo steps to equilibrate the system and the average was then computed over 10^7 samples generated in the time interval $[10^7, 2.10^7]$. In Fig. 4 b), we also plot, with a solid line, our asymptotic result in Eq. (69), showing a relatively good agreement with our numerics (note that here $A/n^2 = 20$). On the same plot, Fig. 4 b), we also show in dotted line, the result for the Brownian bridge (5), which is independent of A. It is quite remarkable that these two profiles are very similar which show that the global constraints that we impose here have strong consequences on the statistics of the Lévy random walk.

6. Conclusion

To conclude, we have studied two main properties of a Lévy bridge $x_B(m)$ of length n : (i) the distribution $P_B(A, n)$ of the area under a Lévy bridge and (ii) the average profile $\langle \tilde{x}_B(m) \rangle$ of a Lévy bridge with fixed area A.

- For $P_B(A, n)$ we have found the scaling form, valid for large n, $P_B(A, n) \sim n^{-1-1/\alpha}F_\alpha(Y)$ with an interesting power law behavior $F_\alpha(Y) \sim Y^{-2(1+\alpha)}$. For $\alpha = 1$, we have obtained an explicit expression for $F_1(Y)$ in terms of elementary functions (C.6). We have also shown, using the Lindeberg condition that the non-Gaussianity of $P_B(A, n)$, for $\alpha > 1/2$, is due only to the correlations between the positions of the walkers $x_B(m)$'s.

- For the average profile, $\langle \tilde{x}_B(m) \rangle$, we have found the scaling form $\langle \tilde{x}_B(m) \rangle \sim n^{1/\alpha} H_\alpha(m/n, A/n^{1+1/\alpha})$ where, at variance with Brownian motion, $H_\alpha(X, Y)$ is a non trivial function of the rescaled area Y. For $\alpha = 1$, we have obtained simple analytical expressions for $H_1(X, Y)$ in both limits $Y \to 0$ and $Y \to \infty$. In particular, we have shown that the average profile of the Lévy random walk with a fixed area is not very far from the profile of a Brownian bridge with fixed area.

- We have finally compared our analytical results with Monte Carlo simulations of these Lévy random walks with global constraints.

In view of recent developments in the study of area distributions for variants of Brownian motions [13, 9, 11, 15], it would be very interesting to extend the results presented here to other constrained Lévy walk, including in particular Lévy random walks conditioned to stay positive (Lévy excursion), which is a challenging open problem.

Appendix A. On the use of the Lindeberg condition

Let us consider *independent* and *non-identical* random variables $X_1, \cdots, X_n$ which have the same distribution as the Lévy bridge $x_B(m)$ (29), *i.e.*

$$\text{Proba}(X_m = x) = \frac{\pi}{\Gamma(1 + \alpha^{-1/2})} \frac{n^{1/\alpha}}{m^{1/\alpha}(n-m)^{1/\alpha}} \mathcal{S}_\alpha \left(\frac{x}{m^{1/\alpha}} \right) \mathcal{S}_\alpha \left(\frac{x}{(n-m)^{1/\alpha}} \right) \quad (A.1)$$

For $\alpha > 1/2$, $\sigma_m^2 = \langle X_m^2 \rangle$ is well defined and one has (32)

$$\sigma_m^2 = \langle X_m^2 \rangle = \langle x_B^2(m) \rangle = \tilde{a}_\alpha n^{2/\alpha - 2} m(n - m) . \quad (A.2)$$

Given that the variables X_i are not identical, one can not apply directly the Central Limit Theorem. However, one can show that these random variables X_m do satisfy the Lindeberg condition which guarantees that their sum $A_n = \sum_{m=0}^n X_m$ is distributed according a Gaussian distribution in the large n limit. Let us first introduce Σ_n^2

$$\Sigma_n^2 = \sum_{m=1}^n \sigma_m^2 = \frac{\tilde{a}_\alpha}{2} n^{2/\alpha - 2}(n+1)n(n-3) \sim \frac{\tilde{a}_\alpha}{2} n^{2/\alpha + 1} , \ n \gg 1 , \quad (A.3)$$

which implies $\Sigma_n \sim n^{1/2 + 1/\alpha}$ for large n. To apply the Lindeberg condition, we need to estimate for any $\epsilon > 0$

$$\langle X_m^2 \rangle_\epsilon = \int_{|x| > \epsilon \Sigma_n} x^2 \text{Proba}(X_m = x) \mathrm{d}x \quad (A.4)$$

$$= 2 \frac{\pi}{\Gamma(1 + \alpha^{-1/2})} \frac{n^{1/\alpha}}{m^{1/\alpha}(n-m)^{1/\alpha}} \int_{\epsilon \Sigma_n}^\infty x^2 \mathcal{S}_\alpha \left(\frac{x}{m^{1/\alpha}} \right) \mathcal{S}_\alpha \left(\frac{x}{(n-m)^{1/\alpha}} \right) \mathrm{d}x$$

$$\sim c'_\alpha n^{2/\alpha} n^{-(3/2 + \alpha)} m(n - m) , \quad (A.5)$$

where c'_α is independent of m and n. Therefore one has

$$\frac{\sum_{m=0}^n \langle X_m^2 \rangle_\epsilon}{\Sigma_n^2} \sim n^{-(\alpha - 1/2)} . \quad (A.6)$$

For random variables for which the above ratio (A.6) goes to zero in the limit $n \to \infty$ (which is the case here for $\alpha < 1/2$), a theorem due to Lindeberg (thus called the 'Lindeberg condition') [2], says that their sum $A_n/\Sigma_n = \Sigma_n^{-1} \sum_{m=0}^n X_m$ is distributed, in the limit $n \to \infty$, according to a Gaussian distribution of unit variance. The fact that, for a Lévy bridge, the area is not a Gaussian distribution (37) is thus, for $\alpha > 1/2$, a consequence of the correlations between the random variables $x_B(m)$.

To conclude this paragraph, we discuss a simple case where the Lindeberg condition does not hold. Consider the case where $X_1, \cdots, X_n$ are independent random variables distributed according to [27]

$$\text{Proba}(X_m = x) = m e^{-mx} , \quad (A.7)$$

such that one has

$$\langle X_m \rangle = \frac{1}{m} , \ \langle (X_m - \langle X_m \rangle)^2 \rangle = \frac{1}{m^2} . \quad (A.8)$$

Then in that case one has immediately

$$\Sigma_n^2 = \sum_{m=1}^n \langle (X_m - \langle X_m \rangle)^2 \rangle = \sum_{m=1}^n \frac{1}{m^2} \to \frac{\pi^2}{6} , \ n \to \infty . \quad (A.9)$$

One computes straightforwardly, for $\epsilon > 0$

$$\int_{\epsilon \Sigma_n}^\infty (X_m - \langle X_m \rangle)^2 \text{Proba}(X_m = x) \mathrm{d}x = \frac{e^{-\epsilon \Sigma_n m}}{m^2} \left(1 + m^2 (\epsilon \Sigma_n)^2 \right) , \quad (A.10)$$

such that here one has

$$\frac{1}{\Sigma_n^2} \sum_{m=1}^\infty \int_{\epsilon \Sigma_n}^\infty (X_m - \langle X_m \rangle)^2 \text{Proba}(X_m = x) \mathrm{d}x \to \frac{6}{\pi^2} \sum_{m=1}^\infty \frac{e^{-\epsilon' m}}{m^2} (1 + (\epsilon' m)^2) > 0 , \ (A.11)$$

with $\epsilon' = \epsilon \pi^2/6$. Therefore the Lindeberg condition (40) does not hold here. In fact, it can be shown that the distribution of the variable $(S_n - \sum_{k=1}^n k^{-1})/\Sigma_n$ converges to a Gumbel distribution [27].

Appendix B. Asymptotic behavior of $F_\alpha(Y)$ for large Y

To analyse the large argument behavior of $F_\alpha(Y)$, we analyse the small k behavior of its Fourier transform $\hat{F}_\alpha(k)$ given in the text in Eq. (35):

$$\hat{F}_\alpha(k) \quad = \int_{-\infty}^{\infty} \mathrm{d}Y F_\alpha(Y) e^{ikY} = \hat{F}_{\alpha,1}(k) + \hat{F}_{\alpha,2}(k) \ , \tag{B.1}$$

$$\hat{F}_{\alpha,1}(k) = \frac{|k|}{\Gamma(1+\alpha^{-1})} \int_0^{\infty} e^{-\frac{|k|^\alpha}{\alpha+1}[(r+1)^{\alpha+1}-r^{\alpha+1}]} \mathrm{d}r \ , \tag{B.2}$$

$$\hat{F}_{\alpha,2}(k) = \frac{|k|}{\Gamma(1+\alpha^{-1})} \int_0^{1/2} e^{-\frac{|k|^\alpha}{\alpha+1}[(1/2+r)^{\alpha+1}+(1/2-r)^{\alpha+1}]} \mathrm{d}r \ . \tag{B.3}$$

The analysis of the small k behavior of $\hat{F}_{\alpha,2}(k)$ is simply obtained by expanding the exponential under the integral. It yields straightforwardly:

$$\hat{F}_{\alpha,2}(k) = \frac{1}{\Gamma(1+\alpha^{-1})} \left[\frac{|k|}{2} - \frac{|k|^{1+\alpha}}{(\alpha+1)(\alpha+2)} + \frac{|k|^{1+2\alpha}}{2(\alpha+1)^2} \left(\frac{1}{3+2\alpha} + \frac{\sqrt{2\pi}\Gamma(2+\alpha)}{\Gamma(\frac{5}{2}+\alpha)2^{3+2\alpha}} \right) \right] + \mathcal{O}(|k|^{1+3\alpha}) \tag{B.4}$$

The asymptotic expansion of $\hat{F}_{\alpha,1}(k)$ is a bit a more subtle. To get the two first terms of the expansion, one performs the change of variable $z = |k|r$ and then expand $(r+1)^{\alpha+1}$ using the binomial formula,

$$\frac{1}{\alpha+1}((r+1)^{\alpha+1} - r^{\alpha+1}) = r^\alpha + \frac{\alpha}{2}r^{\alpha-1} + \dots \ . \tag{B.5}$$

This yields

$$\hat{F}_{\alpha,1}(k) \sim \frac{1}{\Gamma(1+\alpha^{-1})} \int_0^{\infty} e^{-r^\alpha - \frac{\alpha}{2}|k|r^{\alpha-1}} \mathrm{d}r + \mathcal{O}(k^{1+\eta}) \ , \tag{B.6}$$

where $\eta > 0$ is yet unknown (see below). From this expression (B.6), one immediately obtains the two first terms of the expansion of $\hat{F}_{\alpha,1}(k)$ as

$$\hat{F}_{\alpha,1}(k) = 1 - \frac{|k|}{2\Gamma(1+\alpha^{-1})} + \mathcal{O}(k^{1+\eta}) \ . \tag{B.7}$$

Combining Eq. (B.4) and Eq. (B.7) one sees that the first non-trivial term, proportional to $|k|$ cancel in $\hat{F}_\alpha(k)$. Therefore, one needs to develop $\hat{F}_{\alpha,1}(k)$ beyond the first terms (B.7). To this purpose, we need to study separately the cases $0 < \alpha \le 1/2$, $1/2 < \alpha \le 1$ and $1 < \alpha \le 2$.

Appendix B.1. The case $0 < \alpha \le 1/2$

Let us first analyse the term $\hat{F}_{\alpha,1}(k)$ which we decompose as

$$\hat{F}_{\alpha,1}(k) = B_1(k) + B_2(k) + B_3(k) \ , \tag{B.8}$$

$$B_1(k) = \frac{|k|}{\Gamma(1+\alpha^{-1})} \int_0^1 \mathrm{d}r e^{-\frac{|k|^\alpha}{\alpha+1}[(r+1)^{\alpha+1}-r^{\alpha+1}]} \ ,$$

$$B_2(k) = \frac{|k|}{\Gamma(1+\alpha^{-1})} \left[\int_1^{\infty} \mathrm{d}r e^{-\frac{|k|^\alpha}{\alpha+1}[(r+1)^{\alpha+1}-r^{\alpha+1}]} - e^{-|k|^\alpha[r^\alpha + \frac{\alpha}{2}r^{\alpha-1}]} \right] \ , \tag{B.9}$$

$$B_3(k) = \frac{k}{\Gamma(1+\alpha^{-1})} \int_1^{\infty} \mathrm{d}r e^{-|k|^\alpha(r^\alpha + \frac{\alpha}{2}|k|r^{\alpha-1})} \ . \tag{B.10}$$

It is easy to expand $B_1(k)$ for small k as

$$B_1(k) = \frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})} \frac{2 - 2^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})} \int_0^1 \left[\frac{\mathrm{d}r}{\alpha+1} \left((r+1)^{\alpha+1} - r^{\alpha+1} \right) \right]^2 + \mathcal{O}(k^{1+3\alpha}) \tag{B.11}$$

To expand $B_2(k)$, one checks the asymptotic behaviors, for large r

$$\frac{1}{\alpha+1}[(r+1)^{\alpha+1} - r^{\alpha+1}] - (r^\alpha + \frac{\alpha}{2}r^{\alpha-1}) = \mathcal{O}(r^{\alpha-2}) \ , \tag{B.12}$$

$$\left(\frac{1}{\alpha+1}[(r+1)^{\alpha+1} - r^{\alpha+1}] \right)^2 - (r^\alpha + \frac{\alpha}{2}r^{\alpha-1})^2 = \mathcal{O}(r^{2\alpha-2}) \ , \tag{B.13}$$

so that, for $\alpha < 1/2$ one can safely expand the exponentials in the integrand of $B_2(k)$ (B.9) up to second order to obtain

$$B_2(k) = -\frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})}\int_1^\infty dr\left[\frac{1}{\alpha+1}[(r+1)^{\alpha+1}-r^{\alpha+1}]-(r^\alpha+\frac{\alpha}{2}r^{\alpha-1})\right] \tag{B.14}$$

$$+\frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})}\int_1^\infty dr\left(\left(\frac{1}{\alpha+1}[(r+1)^{\alpha+1}-r^{\alpha+1}]\right)^2-(r^\alpha+\frac{\alpha}{2}r^{\alpha-1})^2\right) \tag{B.15}$$

$$+\mathcal{O}(|k|^{\min(2,1+3\alpha)}) . \tag{B.16}$$

Summing up the contributions from $B_1(k)$ and $B_2(k)$ and performing the integrals yields

$$B_1(k)+B_2(k) = \frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})}\left(\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{\alpha+1}-\frac{1}{2}\right) \tag{B.17}$$

$$+\frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})}\left[\frac{1}{2}+\frac{1}{1+2\alpha}-\frac{1}{(1+\alpha^2)(3+2\alpha)}+\frac{\alpha^2}{8\alpha-4}+2\frac{\Gamma(-3-2\alpha)\Gamma(1+\alpha)}{\Gamma(-\alpha)}\right] \tag{B.18}$$

$$+\mathcal{O}(|k|^{\min(2,1+3\alpha)}) ,$$

which we have carefully checked using Mathematica.

Let us now expand $B_3(k)$ (B.10) for small k. It is easily seen from Eq. (B.7) that the first terms of this expansion are indeed given by

$$B_3(k) = 1 - \frac{|k|}{2\Gamma(1+\alpha^{-1})}+\mathcal{O}(|k|^{1+\mu}) , \tag{B.19}$$

with $\mu > 0$. To go beyond the lowest orders, we first perform a change of variable $x = |k|r$ and then compute $B_3''(k)$ and finally expand it for small k. This yields

$$B_3''(k) = \frac{1}{\Gamma(1+\alpha^{-1})}\left(\alpha(\frac{1}{2}+(1+\frac{\alpha}{2}))k^{\alpha-1}e^{-(1+\frac{\alpha}{2})k^\alpha}+I_3(k)\right) , \tag{B.20}$$

$$I_3(k) = \left(\frac{\alpha}{2}\right)^2\int_k^\infty e^{-x^\alpha-\frac{\alpha}{2}kx^{\alpha-1}}x^{2(\alpha-1)}\,dx , \tag{B.21}$$

where $2(\alpha-1) < -1$ for $\alpha < 1/2$. One then obtains the small k behavior of $I_3(k)$ by simply expanding the term $e^{-\frac{\alpha}{2}kx^{\alpha-1}}$ in the integrand. This yields, to lowest order

$$I_3(k) = \frac{1}{1-2\alpha}\left(\frac{\alpha}{2}\right)^2 k^{2\alpha-1}+\mathcal{O}(k^{3\alpha-1}) . \tag{B.22}$$

From Eq. (B.20) and Eq. (B.22), one obtains straightforwardly

$$B_3(k) = 1-\frac{|k|}{\Gamma(1+\alpha^{-1})}+\frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})}\left(\frac{1}{2}+\frac{1}{\alpha+1}\right) \tag{B.23}$$

$$+\frac{|k|^{1+2\alpha}}{\Gamma(1+\alpha^{-1})}\left(-\frac{1+\alpha/2}{2(1+2\alpha)}\left(\frac{1}{2}+(1+\alpha/2)\right)+\frac{1}{2\alpha(1+2\alpha)}\frac{1}{1-2\alpha}\left(\frac{\alpha}{2}\right)^2\right) \tag{B.24}$$

Finally, combining Eq. (B.17) and Eq. (B.23) together with the small k expansion of $\hat{F}_{\alpha,2}(k)$ above (B.4), one sees that the term proportional to $|k|^{1+\alpha}$ actually cancels, yielding

$$\hat{F}_\alpha(k) = 1+c_\alpha|k|^{1+2\alpha}+\mathcal{O}(k^{1+3\alpha}) , \tag{B.25}$$

$$c_\alpha = \frac{1}{\Gamma(1+\alpha^{-1})}\frac{2^{-2(2+\alpha)}\pi^{3/2}\tan(\alpha\pi/2)}{\cos(\alpha\pi)(1+\alpha)\Gamma(-\alpha)\Gamma(5/2+\alpha)} . \tag{B.26}$$

This singular behavior of $\hat{F}_\alpha(k)$ for small k (B.25) yields the power law behavior of $F_\alpha(Y)$ for large Y

$$F_\alpha(Y) \propto \frac{a_\alpha}{Y^{2(1+\alpha)}} , \quad Y \gg 1 , \tag{B.27}$$

$$a_\alpha = -\frac{1}{\pi}\Gamma(2+2\alpha)\cos(\alpha\pi)c_\alpha = \frac{2^{-2(2+\alpha)}\sqrt{\pi}\Gamma(2+2\alpha)\tan(\alpha\pi/2)}{\Gamma(2+\alpha^{-1})\Gamma(1-\alpha)\Gamma(\frac{5}{2}+\alpha)} . \tag{B.28}$$

Appendix B.2. The case $1/2 < \alpha \leq 1$

This case can be studied along the same line as above except that in that case, $1 + 2\alpha > 2$ and therefore one has to handle with care the analysis of terms which are proportional to k^2, while the coefficient proportional to $k^{1+2\alpha}$ has the same form (B.25). We will not repeat the analysis and simply give the result. One finds that $\hat{F}_\alpha(k)$ behaves for small k as

$$\hat{F}_\alpha(k) = 1 - \frac{b_\alpha}{2}k^2 + c_\alpha k^{1+2\alpha} + \mathcal{O}(k^{\min(3, 1+3\alpha)}) \ , \ b_\alpha = \frac{\alpha\Gamma(2 - \alpha^{-1})}{12\Gamma(1 + \alpha^{-1})} \ . \qquad (B.29)$$

The expression for b_α given above (B.29) yields the expression for $\langle Y^2 \rangle$ given in the text in Eq. (39).

Appendix B.3. The case $1 < \alpha \leq 2$

In this case one can again perform a similar analysis but in this case one has $1 + 2\alpha > 3$. And therefore one has to handle carefully the term proportional to $|k|^3$. A quite lengthy calculation shows that this term actually vanishes for $\alpha > 1$, while the coefficients of the terms proportional to k^2 and $|k|^{1+2\alpha}$ are still given by the expressions above (B.29). This yields again as above (B.29)

$$\hat{F}_\alpha(k) = 1 - \frac{b_\alpha}{2}k^2 + c_\alpha k^{1+2\alpha} + \mathcal{O}(k^5) \ . \qquad (B.30)$$

Appendix C. Explicit expression of $F_\alpha(Y)$ for $\alpha = 1$

In this appendix, we give an explicit expression of $F_1(Y)$ for $\alpha = 1$. The starting point of our analysis is the expression (45) given in the text:

$$F_1(Y) = \frac{1}{\pi}\frac{2}{1 + 4Y^2} \qquad (C.1)$$
$$+ \frac{1}{\pi}\left(\frac{2(1 - 8Y^2)}{(1 + 4Y^2)(1 + 16Y^2)} + \frac{4}{(1 + 16Y^2)^{\frac{3}{2}}}\mathrm{Re}\left[(1 - 4iY)^{\frac{3}{2}}\arctan\left((1 + 4iY)^{-\frac{1}{2}}\right)\right]\right)$$

For z a complex number, the following elementary relations are useful :

$$\arctan z = \frac{1}{2i}\left(\log(1 + iz) - \log(1 - iz)\right) \ , \qquad (C.2)$$

$$\log(x + iy) = \log\left(\sqrt{x^2 + y^2}\right) + 2i\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right) \ . \qquad (C.3)$$

On the other hand has

$$\frac{1}{\sqrt{1 + 4iY}} = a + ib \ , \ a = \frac{1}{(1 + 16Y^2)^{1/4}}\cos\left(\frac{\theta}{2}\right) \ , \qquad (C.4)$$
$$b = \frac{-1}{(1 + 16Y^2)^{1/4}}\sin\left(\frac{\theta}{2}\right) \ ,$$
$$\theta = \arctan(4Y) \ .$$

Defining λ and μ as

$$\lambda = \arctan\left[\frac{a^2 + b^2 - 1 + \sqrt{a^2 + (1 - b)^2}\sqrt{a^2 + (1 + b)^2}}{2a}\right] \ , \qquad (C.5)$$

$$\mu = -\frac{1}{4}\log\left[\frac{(1 - b)^2 + a^2}{(1 + b)^2 + a^2}\right] \ ,$$

in terms of a, b defined above (C.4), one obtains finally (after straightforward algebra)

$$F_1(Y) = \frac{1}{\pi}\frac{2}{1 + 4Y^2} \qquad (C.6)$$
$$+ \frac{1}{\pi}\left(\frac{2(1 - 8Y^2)}{(1 + 4Y^2)(1 + 16Y^2)} + \frac{4}{(1 + 16Y^2)^{3/4}}\left(\lambda\cos\left(\frac{3\theta}{2}\right) + \mu\sin\left(\frac{3\theta}{2}\right)\right)\right) \ .$$

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