Phase Transition in a Random Minima Model: Mean Field Theory and Exact Solution on the Bethe Lattice

Peter Sollich[1], **Satya N. Majumdar**[2], **Alan J. Bray**[3]

[1] Department of Mathematics, King's College London, London WC2R 2LS, UK

[2] Laboratoire de Physique Théorique et Modèles Statistiques, Université de Paris-Sud, 91405 Orsay Cedex, France

[3] School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, UK

E-mail: peter.sollich@kcl.ac.uk

Abstract. We consider the number and distribution of minima in random landscapes defined on non-Euclidean lattices. Using an ensemble where random landscapes are reweighted by a fugacity factor z for each minimum they contain, we construct first a 'two-box' mean field theory. This exhibits an ordering phase transition at $z_c = 2$ above which one box contains an extensive number of minima. The onset of order is governed by an unusual order parameter exponent $\beta = 1$, motivating us to study the same model on the Bethe lattice. Here we find from an exact solution that for any connectivity $\mu + 1 > 2$ there is an ordering transition with a conventional mean field order parameter exponent $\beta = 1/2$, but with the region where this behaviour is observable shrinking in size as $1/\mu$ in the mean field limit of large μ. We show that the behaviour in the transition region can also be understood directly within a mean field approach, by making the assignment of minima 'soft'. Finally we demonstrate, in the simplest mean field case, how the analysis can be generalized to include both maxima and minima. In this case an additional first order phase transition appears, to a landscape in which essentially all sites are either minima or maxima.

1. Introduction

The statistics of the number of stationary points (maxima, minima and saddles) in a random landscape plays an important role in understanding both the static and the dynamical properties of many systems such as structural glasses [1], spin glasses [2], clusters and biomolecules [3], continuum percolation [4], rugged landscapes in evolutionary biology [5], quantum cosmology [6], string theory [7, 8], particles in a random potential [9, 10], and also in several associated problems in random matrices [8, 11, 12, 13]. In particular, the statistics of the total number of minima is important to understand in the context of glassy materials where the system typically gets trapped for a long time in a local minimum of the energy (or free energy) landscape [1, 2].

In theoretical studies one typically models a random energy landcsape as a smooth random manifold (typically Gaussian) sitting on an underlying continuous space. For such smooth Gaussian random surfaces in a continuum, there is a finite density of local minima (expected number of local minima per unit volume) which can be computed from the celebrated Kac-Rice formula [14] and its multi-dimensional generalizations [15, 16, 17]. Recently these formulae have been extended to compute the expected number of saddle points at a given fixed energy and also with a fixed index number of saddles [18, 19, 20]. Similarly the variance and higher moments can also be computed in principle. However, this machinery is not easily extendable to cases (i) where the surface is non-Gaussian and (ii) where the underlying space on which the energy landscape resides is discrete, *e.g.* a regular Euclidean lattice. The latter case is particularly relevant in the practical context of numerical simulations of the random energy landscape where one is obliged to discretize the underlying space. Hence it is interesting to study the statistics of the number of local minima in lattice models of energy landscapes, in particular where the energy distribution at each site is non-Gaussian in general.

With these two motivations in mind, a simple lattice model of an energy landscape has recently been introduced (hereafter referred to as the 'random minima' model) [21]. In this random minima model, a random energy E_i sits at site i of a lattice of N sites with periodic boundary conditions. The energies E_i are drawn, independently from site to site, from a common continuous distribution $p(E)$, not necessarily Gaussian. Any such choice of the set $\{E_i\}$ defines a realization of a random energy landscape. For a given realization, a site i is a local minimum if $E_i < E_j$ for all sites j which are nearest neighbours of site i. Let M denote the total number of local minima. Evidently M will vary from one realization of the landscape to another and one is interested in the probability distribution $P(M, N)$ of M for a given size N of the system. The energies at different sites are uncorrelated in the random minima model. Hence it can be viewed as an effective 'coarse grained' lattice model of a continuous random energy manifold in the limiting situation where the correlation length between the energies at different points in space is smaller than the lattice spacing.

The advantage of this simple model is that many questions regarding the statistics of the number of minima are analytically tractable [21]. Besides, the same distribution $P(M, N)$ has appeared recently in seemingly unrelated problems such as random permutations [22, 23], ballistic deposition models [24] and also simple models of glasses [25]. The distribution $P(M, N)$ turns out to be strictly universal, in the sense of being (even for finite N) independent of the on-site energy distribution $p(E)$ as long as the latter is continuous [21, 24]. This is not difficult to see: transform from the E_i to new variables $x_i = q(E_i)$, with $q(E) = \int_{-\infty}^{E} dE' p(E')$ the cumulative distribution function. For continuous $p(E)$, this transformation is monotonic so that minima in the E_i-landscape are identical with minima in the landscape defined by the x_i. But the x_i are uniformly distributed in the interval $[0, 1]$, so it suffices to consider this particular distribution – denoted $Q(x)$ below – to obtain $P(M, N)$ for *any* continuous $p(E)$.

The average number of minima can be trivially computed, $\langle M\rangle = N/[(\mu+1)+1]$ where $\mu+1$ is the co-ordination number of the lattice. For example, for a d-dimensional hypercubic lattice, $\mu = 2d-1$ whereas for a Bethe lattice μ is just the branching ratio. Similarly, the variance of M can also be computed exactly for various lattices such as a 1-d chain [21, 24], the 2-d square lattice [24] and the Bethe lattice [21]. The distribution $P(M,N)$ has a Gaussian peak near its mean (of width $\sim \sqrt{N}$), but a non-Gaussian tail far from the mean. The non-Gaussian tail is described by a large deviation function that can be computed exactly in 1-d [21]. Also, on any given lattice, M can at most take a value $M_{\rm max}$. This follows from the fact that if a site is a local minimum, none of its neighbours can be a local minimum (nearest neighbour minima exclusion principle). For example, on a bipartite lattice, consisting of two 'boxes' each containing N sites and where each site has nearest neighbour connection to all sites in the other box, one cannot have minima in both boxes and so $M_{\rm max} = N$. In Ref. [21], the probability of the maximal packing configuration $P(M_{\rm max},N)$ was studied and was shown to decay for large N as $P(M_{\rm max},N) \sim \gamma^{-N}$, where the constant γ was exactly computed for a number of lattices.

The purpose of this paper is to go beyond the 'counting problem' in the random minima model and study its thermodynamics and the associated phase transition by introducing a fugacity z for each local minimum. For this purpose, the relevant object of interest is the generating function (or grand partition function)

$$G(z,N) = \sum_M P(M,N)\, z^M \tag{1}$$

and the associated equation of state. The latter tells us how the density of minima $\rho = \langle M\rangle/N$ depends on z, where the average is over the original ensemble of random landscapes reweighted by a factor z^M for each configuration.

At this point it is important to note that due to the nearest neighbour minima exclusion principle, the thermodynamics of the random minima model is similar in spirit, though not in its details, to the well studied 'hard sphere lattice gas' model [26, 27, 28, 29, 30, 31]. In the latter model, when a molecule or a hard sphere occupies a lattice site, a similar exclusion principle holds in that all neighbouring sites have to be empty. If $W(M,N)$ denotes the number of ways of putting M hard particles (with this constraint of nearest neighbour exclusion) on a lattice of N sites, the corresponding grand partition function is defined as

$$Z(z,N) = \sum_M W(M,N)\, z^M. \tag{2}$$

Note the important difference between the two models. In the hard sphere model with M particles, one attaches a uniform weight 1 to each allowed configuration of the M particles. On the other hand, in the random minima model, for each configuration of M local minima, the associated weight comes from an entropic factor obtained by integrating over all possible E_i's associated with the given configuration of the M local minima.

The hard sphere model is well known to undergo a thermodynamic phase transition as one increases the fugacity z through a critical value z_c in two or higher dimensions [28, 31]. For $z < z_c$, the system is in a low density 'disordered' or 'fluid' phase and for $z > z_c$ it is in a high density 'ordered' or 'crystalline' phase. Based on the qualitative analogy between the two models one therefore expects a similar phase transition from a disordered to an ordered state in the random minima model, also upon increasing the fugacity z. Indeed, recent numerical studies by Derrida for a 2-d random minima model indicate the presence of such a phase transition [32]. It is important to understand whether this phase transition in the random minima model is similar/different from that of the hard sphere lattice gas model.

With this in mind, we study the thermodynamics of the random minima model on the Bethe lattice. The hard sphere model was solved exactly on the Bethe lattice many years back [31] and has been revisited recently [33]. In this paper we present an exact solution of the random minima model on the Bethe lattice which turns out to be technically somewhat harder than the hard sphere solution on the same lattice. In addition, we study analytically a rather simple mean field theory of the random minima model which also exhibits a phase transition at a critical value $z_{\rm c} = 2$. We show that in the low density phase (for $z < 2$) the average number of minima is of order unity in the thermodynamic limit, while in the high density phase (for $z > 2$) the average number of minima is extensive with a finite density.

We survey the hard sphere lattice gas briefly in Sec. 2. There is no non-trivial mean field theory for hard particles, so we start directly with the Bethe lattice case (Sec. 2.1). A mean field theory *can* be constructed if particles are made soft, i.e. if occupation of neighbouring sites is permitted subject to some penalty. As we show in Sec. 2.2, when the penalty parameter is made large this approach nicely retrieves the results for the Bethe lattice in the limit of large connectivity.

We turn to our main subject, the random minima problem, in Sec. 3. Here there is a non-trivial two-box mean field theory and we discuss this first, in Sec. 3.1, and also extend it to study the joint statistics of the number of minima and maxima (Sec. 3.2). Next we analyse the random minima problem on the Bethe lattice (Sec. 3.3) and finally we consider a mean field theory with soft assignments of minima in Sec. 3.4. Again we will see that these two approaches give the same results in the respective limits of large connectivity and almost-hard assignments. Considering these limit cases also helps to clarify why the direct mean field approach of Sec. 3.1 gives an unusual apparent order parameter exponent near the phase transition. We summarize and list some open questions in Sec. 4.

2. Hard sphere lattice gas

In this section we revisit briefly the hard sphere lattice gas model, on a Bethe lattice [31, 33] and in a two-box mean field theory with soft particles. This will serve to introduce the techniques we will deploy later for the problem of minima in random landscapes. Note that the simplest mean field theory, a fully connected lattice, makes no sense as the presence of a single particle would exclude particles from all other sites. Also the simplest improvement over this, a fully connected bipartite lattice, is trivial: one of the two boxes, i.e. partitions of the graphs, is always empty, and in the other the particles are then non-interacting. In the random minima problem, on the other hand, already this approach produces a phase transition as discussed in Sec. 3.1 below.

2.1. Bethe lattice

Consider first a Cayley tree‡ with branching ratio μ, of depth l, *i.e.* with l layers below the single root node. Call $Z_{0,1}^{(l)}$ the grand partition function constrained to run over all configurations that do not (or do, respectively) have a particle at the root. The full partition function is them $Z^{(l)} = Z_0^{(l)} + Z_1^{(l)}$. We do not write explicitly the dependence on z, while the superscript (l) indicates indirectly the number of sites $N = (\mu^{l+1}-1)/(\mu-1)$ in the tree.

The quantities $Z_{0,1}^{(l)}$ obey the following recursions over the tree depth:

$$Z_0^{(l+1)} = (Z_0^{(l)} + Z_1^{(l)})^\mu \tag{3}$$

$$Z_1^{(l+1)} = z(Z_0^{(l)})^\mu \tag{4}$$

with $Z_0^{(0)} = 1$, $Z_1^{(0)} = z$. For example, if no particle is present at the root node of a tree of depth $l+1$, then the μ sites in the next level of the tree are each allowed to be either occupied or not; the partition sum is then the product of μ unconstrained partition functions $Z_0^{(l)} + Z_1^{(l)}$ for each of the subtrees of depth l. This gives the first equation above. For the second equation, one notes that if a particle is present at the root then each of the μ sites below must be empty. The partition sum is then a product of the appropriate constrained partition sums $Z_0^{(l)}$ for the subtrees, with an extra factor z to account for the particle at the root.

The two recursions can be combined into one for the ratio $S^{(l)} = Z_1^{(l)}/Z_0^{(l)}$, giving

$$S^{(l+1)} = \frac{z}{(1+S^{(l)})^\mu} \tag{5}$$

with $S^{(0)} = z$. From $S^{(l)}$ one can determine the density at the centre of a Bethe lattice of depth $l+1$, obtained by connecting $\mu+1$ Cayley trees of depth l to a central node (see Fig. 1). Taking the appropriate ratio of the partition sum with the central site occupied to the total partition sum yields

$$\rho^{(l+1)} = \frac{z(Z_0^{(l)})^{\mu+1}}{(Z_0^{(l)} + Z_1^{(l)})^{\mu+1} + z(Z_0^{(l)})^{\mu+1}} \tag{6}$$

$$= \frac{z}{(1+S^{(l)})^{\mu+1} + z} \; . \tag{7}$$

‡ We note that in some of the literature what we call a Cayley tree is termed "rooted Cayley tree", while the term "Cayley tree" is then used for what we call a Bethe lattice, i.e. a tree where every node except those on the boundary has $\mu+1$ neighbours.

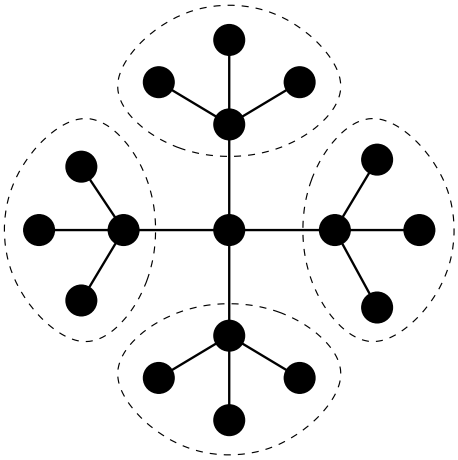

Figure 1. Sketch of a Bethe lattice with $\mu = 3$: every interior node has $\mu + 1 = 4$ neighbours. This lattice can be obtained by connecting $\mu + 1 = 4$ Cayley trees – indicated by the dashed lines – of branching ratio $\mu = 3$ (and in this case depth $l = 1$) to the central node.

For low z the recursion for $S^{(l)}$ has a single fixed point. The bifurcation to an ordered state occurs when $\partial S^{(l+1)}/\partial S^{(l)} = -1$ at the fixed point§. This requirement together with the fixed point condition itself gives

$$S_\mathrm{c} = \frac{1}{\mu - 1}, \qquad z_\mathrm{c} = \mu^\mu/(\mu-1)^{\mu+1}, \qquad \rho_\mathrm{c} = \frac{1}{\mu+1} \ . \tag{8}$$

The divergence of z_c at $\mu = 1$ makes sense: for $\mu = 1$ we have a chain, which as a one-dimensional system with only short-range interactions cannot exhibit a phase transition.

For $z > z_\mathrm{c}$ the recursion for $S^{(l)}$ converges to a period-two sequence, $S^{(2k)} \to S$, $S^{(2k+1)} \to \bar{S}$, where

$$S = \frac{z}{(1+\bar{S})^\mu}, \qquad \bar{S} = \frac{z}{(1+S)^\mu} \ . \tag{9}$$

As anticipated, this means the system is ordered, with alternating layers of the lattice preferentially occupied/empty; the densities are:

$$\rho = \frac{z}{(1+\bar{S})^{\mu+1} + z} = \frac{1}{z^{1/\mu} S^{-(\mu+1)/\mu} + 1} \tag{10}$$

$$\bar{\rho} = \frac{z}{(1+S)^{\mu+1} + z} = \frac{1}{z^{1/\mu} \bar{S}^{-(\mu+1)/\mu} + 1} \ . \tag{11}$$

(Given the initial condition $S^{(0)} = z > S_\mathrm{c}$, the even layers should be the occupied ones, *i.e.* $S > \bar{S}$.) Mathematically, further bifurcations could occur for larger z, but physically this is implausible. In general, the equations for S and $\bar{S}$ need to be solved numerically. For large μ simplifications occur, however. It is to this mean field limit that we now turn.

§ If we write the recursion as $S^{(l+1)} = f(S^{(l)})$, then the bifurcation is to a cycle of two solutions, $S = f(\bar{S})$ and $\bar{S} = f(S)$. Near the bifurcation S and $\bar{S}$ are close, so one can expand $S = f(S) + (\bar{S} - S)f'(S) + O((\bar{S} - S)^2)$ which gives $(S - \bar{S})[1 + f'(S)] = O((\bar{S} - S)^2)$ and hence $f'(S) = -1$ at the bifurcation itself.

2.1.1. Large μ, above the transition Here we take fixed $z > z_{\rm c}$. It is then not hard to see that for $\mu \to \infty$ one gets $S = z$, $\bar{S} = z(1+z)^{-\mu}$. (This is self-consistent, since $\bar{S}$ is exponentially small and so $(1+\bar{S})^{\mu} \to 1$.) The resulting densities are, to leading order,

$$\rho = 1/(z^{-1}+1) = z/(z+1), \qquad \bar{\rho} = z(1+z)^{-(\mu+1)} \ . \tag{12}$$

This is plausible: $\bar{\rho}$ is very small so that the odd layers are basically empty. Then ρ is just determined by the activity z, which attributes weights z and 1, respectively, to the configuration with or without a particle on a site of the even sublattice.

2.1.2. Large μ, around the transition For $\mu \to \infty$, the critical activity from (8) becomes $z_{\rm c} = e/\mu$. We therefore set $z = \tilde{z}/\mu$ to explore the region around the ordering transition. Since the critical density $\rho_{\rm c} = 1/(\mu+1)$ and partition sum ratio $S_{\rm c} = 1/(\mu-1)$ are also $\mathcal{O}(1/\mu)$, we put likewise $\rho = \tilde{\rho}/\mu$ and $S = \tilde{S}/\mu$. The fixed point in the disordered phase then obeys, from the large-μ limit of the fixed point of (5),

$$\tilde{S} = \tilde{z}e^{-\tilde{S}} \tag{13}$$

From (7) the density becomes $\tilde{\rho} = \tilde{z}e^{-\tilde{S}} = \tilde{S}$, so the 'equation of state' is simply

$$\tilde{z} = \tilde{\rho}e^{\tilde{\rho}} \ . \tag{14}$$

In the ordered phase, on the other hand, one has from (9)

$$\tilde{S} = \tilde{z}e^{-\tilde{\bar{S}}}, \qquad \tilde{\bar{S}} = \tilde{z}e^{-\tilde{S}} \tag{15}$$

with again $\tilde{\rho} = \tilde{z}e^{-\tilde{\bar{S}}} = \tilde{S}$ and $\tilde{\bar{\rho}} = \tilde{\bar{S}}$. So the activity and the densities in the even/odd layers are related by

$$\tilde{z} = \tilde{\rho}e^{\tilde{\bar{\rho}}} = \tilde{\bar{\rho}}e^{\tilde{\rho}} \ . \tag{16}$$

The two densities obey

$$\tilde{\rho}e^{-\tilde{\rho}} = \tilde{\bar{\rho}}e^{-\tilde{\bar{\rho}}} \tag{17}$$

and hence the critical point is at $\tilde{\rho} = \tilde{\bar{\rho}} = \tilde{\rho}_{\rm c} = 1$, $\tilde{z}_{\rm c} = e$ as expected. For higher $\tilde{z}$, the densities deviate from each other with a standard square root singularity, to leading order, $\tilde{\rho} - 1 = 1 - \tilde{\bar{\rho}} \sim (\tilde{z} - \tilde{z}_{\rm c})^{1/2}$. This corresponds to an order parameter critical exponent $\beta = 1/2$ as expected for a mean field model. For general $\tilde{z} > \tilde{z}_{\rm c}$, the last two equations – which together determine the equation of state of the ordered phase – need to be solved numerically. (One could choose, say, $\tilde{\rho} > 1$, find the corresponding $\tilde{\bar{\rho}}$ from (17), then determine $\tilde{z}$.) The asymptotic behaviour for $\tilde{z} \gg 1$ is $\tilde{\rho} = \tilde{z}$, $\tilde{\bar{\rho}} = \tilde{z}e^{-\tilde{z}}$ which matches with the $z \ll 1$ limit of (12) as it should.

2.2. Mean field theory with soft particles

The large connecitivity limit $\mu \to \infty$ discussed above must correspond to a mean field theory that one ought to be able to construct directly, without having to first solve for lattices of finite connectivity. As explained above, a fully connected lattice makes no sense as the presence of a single particle would exclude particles from all other sites. One is therefore led to considering a fully connected bipartite lattice. This can be thought of as two boxes ('left' and 'right') with N sites each; every site is connected to all others in the *other* box. If we now directly enforce the hard repulsion of particles on neighbouring (connected) sites, the model is trivial: as soon as one box contains any particles, the other one must be completely empty. The density in the non-empty box is then just $\rho = z/(z+1)$ as determined by the fugacity, and the system is always ordered.

To retrieve the ordering phase transition, one needs to introduce a soft repulsion. Here we give a configuration with M and $\bar{M}$ particles in the two boxes weight $z^{M+\bar{M}} \exp(-\alpha M\bar{M}/N)$. Sending $\alpha \to \infty$ then recovers the hard repulsion.

The partition function for this soft repulsion model is

$$Z(z, N) = \sum_{M,\bar{M}} \binom{N}{M} \binom{N}{\bar{M}} z^{M+\bar{M}} e^{-\alpha M\bar{M}/N} \tag{18}$$

and can be evaluated by introducing the densities $\rho = M/N$, $\bar{\rho} = \bar{M}/N$ and evaluating using steepest descents for $N \to \infty$:

$$N^{-1} \ln Z = \max_{\rho,\bar{\rho}} \{\mathcal{H}(\rho) + \mathcal{H}(\bar{\rho}) + (\rho + \bar{\rho}) \ln z - \alpha\rho\bar{\rho}\} \tag{19}$$

with the entropy

$$\mathcal{H}(\rho) = -\rho \ln \rho - (1-\rho) \ln(1-\rho) \ . \tag{20}$$

The resulting saddle point conditions are

$$\ln[(1-\rho)/\rho] + \ln z - \alpha\bar{\rho} = 0 \tag{21}$$

$$\ln[(1-\bar{\rho})/\bar{\rho}] + \ln z - \alpha\rho = 0 \ . \tag{22}$$

We will be interested in the large α limit where the repulsion is 'nearly hard'; when the symmetry between boxes is broken, we assume without loss of generality that it is the left box that has the higher density, *i.e.* $\rho > \bar{\rho}$.

2.2.1. Large α, above the transition Taking α large at fixed z, we see that to satisfy the second saddle point equation to $\mathcal{O}(\alpha)$ one needs $\bar{\rho} = z \exp(-\alpha\rho)$ to leading order. The α-dependent term in the first saddle point equation then becomes negligible, so that $(1-\rho)/\rho = 1/z$ or $\rho = z/(z+1)$. This is just the result (12) on the Bethe lattice for large μ, as expected, and is consistent with the simple expression obtained from the balance of the weights of unoccupied and occupied configurations (see above). Note that the density of the almost empty box is $\bar{\rho} = z \exp[-\alpha z/(z+1)]$ to leading order; this *does not* match with the Bethe lattice result if one naively identifies α with μ. So only the leading order densities ($z/(z+1)$ and 0) match while the subleading (exponentially small, in the nearly empty box) corrections are not related.

2.2.2. Large α, around the transition Here we set $\rho = \tilde{\rho}/\alpha$ and $\bar{\rho} = \tilde{\bar{\rho}}/\alpha$, by analogy with the large μ treatment on the Bethe lattice. This gives for $\alpha \to \infty$ the saddle point equations

$$\ln(\tilde{z}/\tilde{\rho}) - \tilde{\bar{\rho}} = 0, \qquad \ln(\tilde{z}/\tilde{\bar{\rho}}) - \tilde{\rho} = 0 \ . \tag{23}$$

These can be rewritten as

$$\tilde{z} = \tilde{\rho} e^{\tilde{\bar{\rho}}} = \tilde{\bar{\rho}} e^{\tilde{\rho}} \tag{24}$$

which is *exactly* the same as the large-μ result on the Bethe lattice. So near their respective transitions the two models behave identically, demonstrating that the fully connected two-box model with soft repulsion captures the same physics as the Bethe lattice for high connectivity.

3. Random minima

In this section we turn to our main subject, the arrangement of the local minima of a random function on a Bethe lattice. As explained in the introduction, we can without loss of generality take the function value at each site i to be a random variable x_i sampled from a uniform distribution $Q(x)$ over $[0, 1]$. We will define binary indicator variables m_i, setting $m_i = 1$ if site i is a minimum, *i.e.* if none of its neighbours has a larger x; otherwise we set $m_i = 0$. (With this convention, leaves of a tree are counted as minima if their x is smaller than that of the parent node directly above.) The m_i are analogous to hard particle occupation numbers since no two neighbouring nodes can have $m = 1$; but the random values x_i introduce other, non-trivial correlations. As in the hard sphere model we will multiply the weight of any configuration of the m_i, produced by a random draw of the x_i, by a fugacity factor z^M, where now $M = \sum_i m_i$ is the total number of minima. We wish to calculate the density of minima as a function of z, and understand whether an ordering transition does again take place for sufficiently large z.

3.1. Mean field theory

We begin with the simplest calculation, which is the two-box mean field theory. Each box contains N sites as before, and the particles in each box are regarded as neighbors of all the particles in the other box. Clearly, all sites that are minima must belong to the same box. Call the random variables in the left box x_i and those in the right box $\bar{x}_i$, with $i = 1, \ldots, N$.

To work out the generating function (1) we need to find $P(M, N)$, the probability of having M minima in our system. Assume first that the minima are in the left box. There are M minima if precisely M among the x_i are smaller than all of the $\bar{x}_j$, *i.e.* smaller than $\bar{x}_-$, the smallest of the $\bar{x}_j$. Given that the $\bar{x}_j$ are uniformly distributed over $[0, 1]$, the probability distribution of $\bar{x}_-$ is

$$P(\bar{x}_-) = N(1 - \bar{x}_-)^{N-1} \,, \tag{25}$$

so the probability that M of the x_i's are smaller than $\bar{x}_-$ is

$$\binom{N}{M} \left\langle \bar{x}_-^M (1 - \bar{x}_-)^{N-M} \right\rangle \tag{26}$$

where the average is taken using the distribution $P(\bar{x}_-)$. Accounting for the configurations where the roles of the two boxes are swapped, the probability of having

M minima is twice as large:

$$P(M,N) = 2N \binom{N}{M} \int_0^1 d\bar{x}_- \, \bar{x}_-^M (1-\bar{x}_-)^{2N-M-1} \tag{27}$$

$$= \frac{N!(2N-M-1)!}{(N-M)!(2N-1)!} \, . \tag{28}$$

Note that the value $M=0$ is impossible as there is always at least one minimum present, and accordingly one has the normalization $\sum_{M=1}^{N} P(M,N) = 1$ as is easily checked.

There is in fact a simple counting argument that leads directly to the result (28). To generate the N random numbers in each box, we can first sample $2N$ random numbers $y_1, \ldots, y_{2N}$, again from the uniform distribution over $[0,1]$. We then take a bag containing N labels 'left' and N labels 'right' and, for each of the y_i, pull out one label from the bag and put y_i in the relevant box. Because the order in which we consider the different y_i for labelling is irrelevant, we can in particular take them to be ordered, $y_1 < \ldots < y_{2N}$. Then a configuration with M minima in the left box is one where y_1 to y_M have got labels 'left' and y_{M+1} the label 'right'. Keeping track how many 'left' and 'right' labels remain in the bag at each step of the labelling, and including the overall factor of 2 for the reverse situation where the M minima are in the right box gives

$$P(M,N) = 2\,\frac{N}{2N} \cdot \frac{N-1}{2N-1} \cdots \frac{N-(M-1)}{2N-(M-1)} \cdot \frac{N}{2N-M} \tag{29}$$

$$= 2\,\frac{N!}{(N-M)!}\,\frac{(2N-M)!}{(2N)!}\,\frac{N}{2N-M} \tag{30}$$

$$= 2\binom{2N-M-1}{N-1}\binom{2N}{N}^{-1} \, . \tag{31}$$

The second expression is the one most easily seen to agree with (28). The third one gives another way of thinking about the result: having M minima (in the left box) fixes the first $M+1$ labels, and the probability is then the number of arrangements of the remaining labels divided by the number of arrangements of *all* labels.

Returning now to our original aim of computing the generating function $G(z,N)$, it is in fact most convenient to employ the integral form (27) of $P(M,N)$. Then the sum over M in Eq. (1) can be evaluated to give

$$G(z,N) = 2N \int_0^1 d\bar{x}_- \,(1-\bar{x}_-)^{N-1} \left\{[1+(z-1)\bar{x}_-]^N - (1-\bar{x}_-)^N\right\} \, . \tag{32}$$

The following analysis shows that there is a phase transition, at $z_c = 2$, between a phase where the number of minima remains finite as $N \to \infty$ and a phase where the minima are extensive in number. We begin by setting $\bar{x}_- = v/N$ in Eq. (32), and taking the limit $N \to \infty$ at fixed v. In this limit the integrand becomes $\exp[-v(2-z)] - \exp(-2v)$ and the upper limit on v tends to infinity, to give

$$G(z, N\to\infty) = \frac{1}{2-z} - \frac{1}{2} = \frac{z}{2-z}, \quad z<2. \tag{33}$$

The same expression can be obtained by noting that, for finite M and large N, the number of minima has the geometric distribution $P(M, N\to\infty) = 2^{-M}$. The result (33) diverges at $z = 2$. This indicates that the underlying assumption – that values of $\bar{x}_-$ of order $1/N$ (corresponding to M of order unity) dominate the integral in (32) – no longer holds, and suggests a phase transition at $z_c = 2$.

For $z > 2$, the integral can be evaluated using the method of steepest descents. For this purpose we write the integral (32) in the form

$$G(z,N) = N \int_0^1 \frac{d\bar{x}_-}{1-\bar{x}_-} \{(1-\bar{x}_-)[1+(z-1)\bar{x}_-]\}^N \ . \tag{34}$$

We have discarded the second term in the integrand, which is exponentially subdominant except at $\bar{x}_- = 0$. For $z > 2$, the remaining integral is now dominated by values of $\bar{x}_-$ near the one that maximises the function $g(\bar{x}_-) = (1-\bar{x}_-)[1+\bar{x}_-(z-1)]$. This value is $x^* = (z-2)/[2(z-1)]$; the fact that $x^* > 0$ for $z > 2$ justifies a posteriori why we were able to discard the second term from (32). Inserting $\bar{x}_- = x^*$ into the integrand now gives

$$\ln G(z, N \to \infty) = N \ln \left(\frac{z^2}{4(z-1)} \right), \quad z > 2, \tag{35}$$

up to subextensive contributions. The value x^* has a natural interpretation: in the z-weighted ensemble it is the minimal value of the random numbers $\bar{x}_i$ in the box without the minima. The other numbers in this box are then distributed uniformly over $[x^*, 1]$; setting $z > 2$ is (for $N \to \infty$) sufficient to exclude any smaller values. In the box with the minima, on the other hand, random numbers from the whole interval $[0,1]$ occur, but the probability density is higher by a factor z on $[0, x^*]$ than on $[x^*, 1]$ (and uniform within these two intervals).

We can use the expressions for G in the two regimes to compute the expectation value, $\langle M \rangle$, of the number of minima using $\langle M \rangle = \sum_M M P(M,N) z^M / \sum_M P(M,N) z^M = d \ln G / d \ln z$ to obtain, in the limit $N \to \infty$,

$$\langle M \rangle = \begin{cases} \dfrac{2}{2-z}, & z < 2, \\ N\left(\dfrac{z-2}{z-1}\right), & z > 2. \end{cases} \tag{36}$$

3.2. Mean field theory: Minima and Maxima

Within the same mean field model, we can also compute the probability weights for configurations which contain maxima as well as minima. Call the number of minima M_1 and the number of maxima M_2, and the probability of such a configuration $P(M_1, M_2, N)$. The most direct way of obtaining this probability is from the counting argument outlined above. The result is

$$P(M_1, M_2, N) = 2 \, \frac{\binom{2N - M_1 - M_2 - 2}{N-2} + \binom{2N - M_1 - M_2 - 2}{N - M_1 - 1} + \delta_{M_1,N}\delta_{M_2,N}}{\binom{2N}{N}} \tag{37}$$

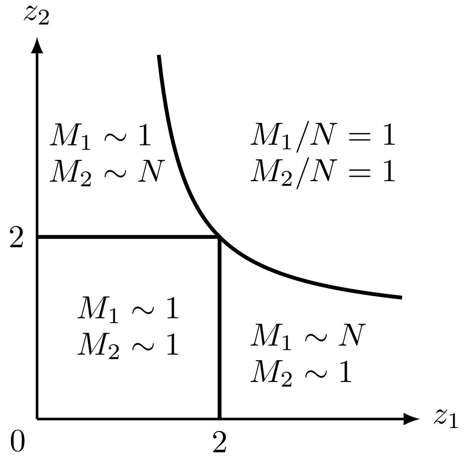

Figure 2. Sketch of the mean field phase diagram for the number of minima, M_1, and maxima, M_2, in the random energy landscape model.

and can be explained as follows. Take the case where the M_1 minima are in the left box; the prefactor 2 then accounts for the opposite case where they are in the right box. Now the M_2 maxima can either be in the left or the right box. Suppose they are in the left box. Then in our construction of first drawing $y_1, \ldots, y_{2N}$ and then labelling them, $y_1, \ldots, y_{M_1}$ and $y_{2N-M_2+1}, \ldots, y_{2N}$ need to have label 'left' while y_{M_1+1} and y_{2N-M_2} have label 'right'. The number of arranging the $2N - M_1 - M_2 - 2$ labels that remain in the bag, of which $N - 2$ are 'right', is given by the first binomial coefficient in the square brackets in (37). The second term is constructed in the same way but with the labels of $y_{2N-M_2}, \ldots, y_{2N}$ reversed: now $N - M_1 - 1$ 'left' and $N - M_2 - 1$ 'right' labels remain in the bag. This counting argument works while $M_1 \leq N - 1$ and $M_2 \leq N - 1$ (since we fix $M_1 + 1$ labels at the bottom and $M_2 + 1$ labels at the top). There is only one configuration that is not captured, namely, $M_1 = M_2 = N$, where all labels are fixed: the third term of (37) accounts for this.

For finite M_1 and M_2, where the third term of (37) is irrelevant, one easily sees that $P(M_1, M_2, N \to \infty) = 2^{-M_1-M_2}$: the populations of minima and maxima are uncorrelated. (Configurations with the minima and maxima in the same and in different boxes also have the same weight, each contributing half the result.) Defining a generating function

$$G(z_1, z_2, N) = \sum_{M_1, M_2} P(M_1, M_2, N) z_1^{M_1} z_2^{M_2} \ , \tag{38}$$

this implies $G(z_1, z_2, N \to \infty) = [z_1/(2 - z_1)][z_2/(2 - z_2)]$ for $z_1 < 2$ and $z_2 < 2$. In the ensemble weighted by z_1 and z_2 the average numbers of minima and maxima are then $\langle M_1 \rangle = 2/(2 - z_1)$ and $\langle M_2 \rangle = 2/(2 - z_2)$, respectively.

For larger z_1 or z_2 one can proceed using steepest descents as explained in Appendix A. We find that there is a first-order transition line at $1/z_1 + 1/z_2 = 1$. Beyond this (*i.e.* for $1/z_1 + 1/z_2 < 1$), the numbers in the two boxes separate essentially completely, with one containing only minima and the other maxima: $\langle M_1 \rangle / N = \langle M_2 \rangle / N = 1$. Between this transition and the other boundaries at $z_1 = 2$ and $z_2 = 2$ lie two regions where the number of minima is extensive but the number of maxima is not, and vice versa. E.g. when $z_1 > 2$ and $z_2 < z_1/(z_1 - 1)$ one finds

$$\langle M_1 \rangle = N \left(\frac{z_1 - 2}{z_1 - 1} \right) , \tag{39}$$

$$\langle M_2 \rangle = \frac{2(z_1^2 + z_2^2) - 2z_1 z_2 (z_1 + z_2) + z_1^2 z_2^2}{(2 - z_2)(z_1 - z_2)(z_1 + z_2 - z_1 z_2)} , \tag{40}$$

with an analogous result when the roles of z_1 and z_2 are swapped. The last factor in the denominator for $\langle M_2 \rangle$ diverges at $1/z_1 + 1/z_2 = 1$, signalling the transition to the regime where both M_1 and M_2 are extensive. Figure 2 shows a sketch of the overall phase diagram. The first order transition at $1/z_1 + 1/z_2 = 1$ has unusual features (see Appendix A): at the transition, an entire one-parameter family of phases becomes degenerate to leading order, i.e. has the same value of $N^{-1} \ln G$. This should produce unusual finite-size scaling effects which we have not yet explored.

3.3. Bethe lattice

Returning to the random minima problem, let us summarize the results so far. Within a two-box mean field theory, we found that the typical densities of minima in both boxes, $\rho = \langle M \rangle / N$ and $\bar{\rho} = \langle \bar{M} \rangle / N$, vanish for fugacities $z < z_{\rm c} = 2$. For higher fugacities, the system orders, with a nonzero density of minima $\rho = (z - 2)/(z - 1)$ in one box but a vanishing one in the other, $\bar{\rho} = 0$. A peculiar aspect of this behaviour is that the nonzero minima density increases *linearly* with $z - z_{\rm c} = z - 2$ around the transition, suggesting an order parameter exponent $\beta = 1$. For a mean field system this would be very unusual indeed as one would naively expect $\beta = 1/2$ in mean field theory. We therefore next consider the minima problem on a Bethe lattice of finite connectivity. While the large connectivity limit should then retrieve the mean field results, at finite connectivity we would hope that a standard mean field phase transition with $\beta = 1/2$ will reappear. This is indeed what we find.

We begin as in the hard particle scenario by considering a Cayley tree. The basic quantity of interest is $P^{(l)}(M^{(l)}, m^{(l)}, x^{(l)})$, the probability – under random sampling of the x_i – that the root node of a Cayley tree of depth l has function value $x^{(l)}$, that it is (or is not) a minimum as indicated by $m^{(l)} = 1$ ($m^{(l)} = 0$), and that there are a total number $M^{(l)}$ of minima in the tree. The basic recursion for this is

$$P^{(l+1)}(M^{(l+1)}, m^{(l+1)}, x^{(l+1)}) = Q(x^{(l+1)}) \prod_{i=1}^{\mu} \left(\sum_{M_i^{(l)}, m_i^{(l)}} \int dx_i^{(l)} P^{(l)}(M_i^{(l)}, m_i^{(l)}, x_i^{(l)}) \right) \times$$
$$\times\, \delta_{M^{(l+1)},\ldots} \delta_{m^{(l+1)},\ldots} \tag{41}$$

which expresses the fact that $x^{(l+1)}$ at the new root node is chosen independently of what happens in the μ different branches $i = 1, \ldots, \mu$ attached to it. Once $x^{(l+1)}$ and the properties of these branches are known, the values $M^{(l+1)}$ and $m^{(l+1)}$ for the new $(l + 1)$-level tree are fully determined as indicated schematically by the delta-functions.

Explicitly,

$$m^{(l+1)} = \begin{cases} 1 & \text{if } x^{(l+1)} < x_i^{(l)} \ \forall i = 1, \ldots, \mu \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

$$M^{(l+1)} = M^{(l)} + m^{(l+1)} - \sum_{i=1}^{\mu} m_i^{(l)} \Theta(x_i^{(l)} - x^{(l+1)}) \ . \tag{43}$$

The last sum runs over the μ nodes below the new root node as before. It expresses the fact that even if these nodes were minima within their own subtrees, once they are connected to the new root node they cease to be minima if they have function values $x_i^{(l)} > x^{(l+1)}$.

Introducing the generating functions for $P^{(l)}$,

$$G_m^{(l)}(x) = \sum_{M=0}^{\infty} z^M P^{(l)}(M, m, x) \tag{44}$$

where the fugacity z again acts on the number of minima, the recursion becomes (the restriction $0 \leq x \leq 1$ is understood for all x-variables and so in particular $Q(x^{(l+1)}) = 1$):

$$\begin{aligned} G_0^{(l+1)}(x) &= \left(\int_0^x dy \left[G_0^{(l)}(y) + G_1^{(l)}(y) \right] + \int_x^1 dy \left[G_0^{(l)}(y) + z^{-1} G_1^{(l)}(y) \right] \right)^{\mu} \\ &\quad - \left(\int_x^1 dy \left[G_0^{(l)}(y) + z^{-1} G_1^{(l)}(y) \right] \right)^{\mu} \end{aligned} \tag{45}$$

$$G_1^{(l+1)}(x) = z \left(\int_x^1 dy \left[G_0^{(l)}(y) + z^{-1} G_1^{(l)}(y) \right] \right)^{\mu} \ . \tag{46}$$

The second of these is easiest to explain: if the root node is a minimum with function value x, all μ nodes in the level below must have function values $x_i^{(l)} \equiv y > x$. The factor z^{-1} in front of $G_1^{(l)}(y)$ corresponds to the negative term in (43), *i.e.* the fact that none of these nodes can then be minima. The prefactor z accounts for the new minimum at the root. The recursion (45) works similarly: multiplying out the μ-th power in the first line and subtracting the term in the second line gives all the possible configurations where at least one of the nodes below the new root has a lower function value than the latter. The factor of z^{-1} is again for nodes which have higher function values than the new root and so cease to be minima if that is what they previously were.

As in the hard particle case, a simpler recursion is obtained by taking ratios of appropriate generating functions. Here, it turns out to be convenient to consider the ratio $S_m^{(l)}(x) = G_m^{(l)}(x)/G_0^{(l)}(1)$. We also abbreviate

$$H^{(l)}(x) = \int_x^1 dy \left[S_0^{(l)}(y) + z^{-1} S_1^{(l)}(y) \right] \tag{47}$$

$$I^{(l)}(x) = \int_0^x dy \left[S_0^{(l)}(y) + S_1^{(l)}(y) \right] + \int_x^1 dy \left[S_0^{(l)}(y) + z^{-1} S_1^{(l)}(y) \right] \ . \tag{48}$$

Then our recursions read simply:

$$(\lambda^{(l)})^{\mu} S_0^{(l+1)}(x) = I^{(l)}(x)^{\mu} - H^{(l)}(x)^{\mu} \tag{49}$$

$$(\lambda^{(l)})^{\mu} S_1^{(l+1)}(x) = z H^{(l)}(x)^{\mu} \tag{50}$$

where $(\lambda^{(l)})^\mu = I^{(l)}(1)^\mu$ is the normalizing coefficient that enforces $S_0^{(l+1)}(1) = 1$ for all l as it must be. The corresponding differential versions will be more useful for later: expressing the $S_m^{(l+1)}(x)$ as derivatives of $H^{(l+1)}(x)$ and $I^{(l+1)}(x)$ gives

$$(\lambda^{(l)})^\mu \partial_x H^{(l+1)}(x) = -I^{(l)}(x)^\mu \tag{51}$$

$$(\lambda^{(l)})^\mu \partial_x I^{(l+1)}(x) = (z-1) H^{(l)}(x)^\mu \tag{52}$$

with boundary conditions

$$H^{(l)}(1) = 0, \qquad H^{(l)}(0) = I^{(l)}(0) \ . \tag{53}$$

Once we have the functions $H^{(l)}(x)$ and $I^{(l)}(x)$, the density (*i.e.* the probability of having a minimum) at the central node of a Bethe lattice follows directly as

$$\rho^{(l+1)} = \frac{z \int dx\, H^{(l)}(x)^{\mu+1}}{\int dx \left[I^{(l)}(x)^{\mu+1} - H^{(l)}(x)^{\mu+1}\right] + z \int dx\, H^{(l)}(x)^{\mu+1}} \ . \tag{54}$$

We expect as in the hard particle case that iteration of the above recursion over l either gives l-independent values (disordered phase) or alternating layers (ordered phase): $H^{(2k)}(x) \to H(x)$, $H^{(2k+1)}(x) \to \bar{H}(x)$, and similarly for $I^{(l)}(x)$, $\lambda^{(l)}$ and $\rho^{(l)}$. It is convenient to study directly the ordered case since it includes the other. The fixed point equations that one needs to solve are then

$$\bar{\lambda}^\mu \partial_x H(x) = -\bar{I}(x)^\mu \tag{55}$$

$$\bar{\lambda}^\mu \partial_x I(x) = (z-1)\bar{H}(x)^\mu \tag{56}$$

$$\lambda^\mu \partial_x \bar{H}(x) = -I(x)^\mu \tag{57}$$

$$\lambda^\mu \partial_x \bar{I}(x) = (z-1) H(x)^\mu \tag{58}$$

with the boundary conditions (53) holding for the functions in both the even and odd layers, and with

$$\lambda = I(1), \qquad \bar{\lambda} = \bar{I}(1) \ . \tag{59}$$

A very useful property that follows by combining the fixed point conditions is

$$(z-1)\bar{\lambda}^\mu H(x)^{\mu+1} + \lambda^\mu \bar{I}(x)^{\mu+1} = \lambda^\mu \bar{\lambda}^{\mu+1} \tag{60}$$

independently of x: the x-derivative of the l.h.s. vanishes, and the value on the r.h.s. can be obtained by setting $x = 1$. This identity allows one to decouple the fixed point conditions for H and $\bar{I}$, giving for the former

$$\partial_x H = -\left[1 - (z-1)\lambda^{-\mu}\bar{\lambda}^{-1} H^{\mu+1}\right]^{\mu/(\mu+1)} \ . \tag{61}$$

Integrating by separation of variables and using the boundary condition $H(1) = 0$ yields the following implicit expression for $H(x)$

$$(\mu+1)(1-x)\left(\frac{z-1}{\lambda^\mu \bar{\lambda}}\right)^{1/(\mu+1)} = B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; (z-1)\lambda^{-\mu}\bar{\lambda}^{-1} H(x)^{\mu+1}\right) \tag{62}$$

with $B(p,q;a) = \int_0^a dt\, t^{p-1}(1-t)^{q-1}$ the incomplete Beta function. For $\bar{H}(x)$ one has the analogous result with λ and $\bar{\lambda}$ swapped.

It now remains to find λ and $\bar{\lambda}$. To this end one can exploit the remaining conditions $H(0) = I(0)$, $\bar{H}(0) = \bar{I}(0)$, from (53). Combining with (60) at $x = 0$ and the corresponding relation with even and odd layers swapped, we find

$$H(0)^{\mu+1} = I(0)^{\mu+1} = \lambda^{\mu+1}\frac{(z-1)(\bar{\lambda}/\lambda)-1}{z(z-2)} \tag{63}$$

and similarly for the odd layers. Inserting back into (62) for $x = 0$ gives

$$(\mu+1)\left(\frac{z-1}{\lambda^{\mu}\bar{\lambda}}\right)^{1/(\mu+1)} = B\left(\frac{1}{\mu+1},\frac{1}{\mu+1};(z-1)\frac{z-1-\lambda/\bar{\lambda}}{z(z-2)}\right) \tag{64}$$

The same relation again also holds with even and odd layers swapped. Together, these two conditions determine $\bar{\lambda}$ and λ. In the disordered phase, where $\bar{\lambda} = \lambda$, λ can be trivially found from (64) in closed form (and is equal to $\Lambda(0)$ as defined below).

For the densities, equation (54) suggests that one might need the explicit forms of $H(x)$, $I(x)$, $\bar{H}(x)$ and $\bar{I}(x)$. However, after some algebra one gets, by transforming integrals over x to integrals over H using (61) and similarly for integrals involving I, expressions that depend only on λ and $\bar{\lambda}$, and indeed only on their log-ratio $r = \ln(\lambda/\bar{\lambda})$:

$$\rho = \frac{\frac{z}{z-1}B\left(\frac{\mu+2}{\mu+1},\frac{1}{\mu+1};(z-1)\frac{z-1-e^{-r}}{z(z-2)}\right)}{B\left(\frac{\mu+2}{\mu+1},\frac{1}{\mu+1};(z-1)\frac{z-1-e^{-r}}{z(z-2)}\right)+e^{-2r/(\mu+1)}B\left(\frac{1}{\mu+1},\frac{\mu+2}{\mu+1};(z-1)\frac{z-1-e^{r}}{z(z-2)}\right)} \tag{65}$$

with an analogous expression for $\bar{\rho}$. In the disordered phase, where $r = 0$, the single density is then given by the relatively simple equation of state

$$\rho = \frac{z}{z-1}\,\frac{B\left(\frac{\mu+2}{\mu+1},\frac{1}{\mu+1};\frac{z-1}{z}\right)}{B\left(\frac{1}{\mu+1},\frac{1}{\mu+1};\frac{z-1}{z}\right)}\ . \tag{66}$$

To understand the solutions for λ, $\bar{\lambda}$ in the ordered phase, it is useful to have a single condition for r. Equation (64) gives $\lambda = \Lambda(r)$ with

$$\Lambda(r) = (\mu+1)\left[(z-1)e^{r}\right]^{1/(\mu+1)}B^{-1}\left(\frac{1}{\mu+1},\frac{1}{\mu+1};(z-1)\frac{z-1-e^{r}}{z(z-2)}\right)\ . \tag{67}$$

The swapped relation gives $\bar{\lambda} = \Lambda(-r)$ or $\lambda = e^{r}\Lambda(-r)$. Since the two expressions for λ have to agree, the desired condition on r is

$$e^{-r/2}\Lambda(r) - e^{r/2}\Lambda(-r) = 0\ . \tag{68}$$

The disordered phase has $r = 0$, which is the trivial solution. The bifurcation to the ordered phase takes place when the first r-derivative at $r = 0$ vanishes, *i.e.* when $(2\partial_r\Lambda - \Lambda)|_{r=0} = 0$. This gives the condition

$$B\left(\frac{1}{\mu+1},\frac{1}{\mu+1};\frac{z_{\rm c}-1}{z_{\rm c}}\right) = \frac{\mu+1}{\mu-1}\,\frac{2\left[(z_{\rm c}-1)z_{\rm c}^{\mu-1}\right]^{1/(\mu+1)}}{z_{\rm c}-2} \tag{69}$$

for the critical value $z_{\rm c}$ of the fugacity. It is easy to see that r initially departs from 0 as $(z-z_{\rm c})^{1/2}$ as z is increased to above $z_{\rm c}$; this follows because (68) is odd in r so when the first derivative vanishes the leading term is third order in r. The densities (65) then have the same leading-order square root singularity. At generic finite connectivity

$\mu+1$ (> 2) we therefore retrieve, as hoped, an ordering phase transition with a standard mean field order parameter exponent $\beta = 1/2$.

For generic z and μ one needs to solve numerically for r from (68) and then calculate the densities ρ and $\bar{\rho}$ from (65) and its analogue with even and odd layers swapped, *i.e.* $r \to -r$. Further analytical progress can again be made for large μ, however. We will need in particular the scaling of z_c for large μ. One uses that

$$B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; a\right) - (\mu+1) \to \ln[a/(1-a)] \tag{70}$$

for $\mu \to \infty$ and $a = \mathcal{O}(1)$ fixed (see (B.7)). The r.h.s. of (69) must then also diverge for $\mu \to \infty$, hence $z_c \to 2$. To leading order the l.h.s. is $\mu + \mathcal{O}(1)$ while the r.h.s. is $4/(z_c - 2)$; this forces

$$z_c = 2 + 4/\mu + \mathcal{O}(1/\mu^2) \tag{71}$$

for large μ.

3.3.1. Large μ, above the transition We need to find first how r scales for fixed $z > z_c$ and $\mu \to \infty$. Let us take $r > 0$ for definiteness since solutions come in pairs $(r, -r)$, and write (68) as

$$e^{r(1-\mu)/(\mu+1)} = \frac{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; (z-1)\frac{z-1-e^{r}}{z(z-2)}\right)}{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; (z-1)\frac{z-1-e^{-r}}{z(z-2)}\right)} \tag{72}$$

For large μ, the l.h.s. becomes e^{-r}; for this to be < 1, the third arguments of the Beta functions on the right cannot stay bounded away from 0 or 1 since otherwise their ratio would converge to unity from (70). The third argument of the numerator Beta function thus has to approach zero, *i.e.* $e^r = z - 1 - \delta r$ with $\delta r \to 0$. To leading order we have then

$$\frac{1}{z-1} = \frac{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; \frac{(z-1)\delta r}{z(z-2)}\right)}{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; 1 - \frac{\delta r}{(z-1)z(z-2)}\right)} \tag{73}$$

$$= \frac{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; \frac{(z-1)\delta r}{z(z-2)}\right)}{B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}\right) - B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; \frac{\delta r}{(z-1)z(z-2)}\right)}\ . \tag{74}$$

Now for a remaining finite or going to zero, $(\mu+1)^{-1}B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; a\right) \to a^{1/(\mu+1)}$ for $\mu \to \infty$ (see (B.4)). Bearing in mind that the complete Beta function obeys $(\mu+1)^{-1}B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}\right) \to 2$, we get to leading order

$$\frac{1}{z-1} = \frac{\delta r^{1/(\mu+1)}}{2 - \delta r^{1/(\mu+1)}} \tag{75}$$

or, calling α the limiting value of $\delta r^{1/(\mu+1)}$, $\alpha = 2/z$; δr thus decays exponentially with μ as $\delta r \sim (2/z)^\mu$.

The density in the even layers can now be worked out from (65). To leading order, using that $B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; 1-a\right) = B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}\right) - B\left(\frac{1}{\mu+1}, \frac{\mu+2}{\mu+1}; a\right) \to (\mu+1)(1 - a^{1/(\mu+1)})$

(see after (B.4))

$$\rho = \frac{\frac{z}{z-1}B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; 1-\frac{\delta r}{(z-1)z(z-2)}\right)}{B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; 1-\frac{\delta r}{(z-1)z(z-2)}\right) + B\left(\frac{1}{\mu+1}, \frac{\mu+2}{\mu+1}; \frac{(z-1)\delta r}{z(z-2)}\right)} \tag{76}$$

$$= \frac{z}{z-1}\,\frac{1-\alpha}{1-\alpha+\alpha} \;=\; \frac{z-2}{z-1}\,. \tag{77}$$

This agrees with the simple two-box mean field theory as we had hoped.

The density in the odd layers, on the other hand, goes to zero for large μ:

$$\bar\rho = \frac{\frac{z}{z-1}B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; \frac{(z-1)\delta r}{z(z-2)}\right)}{B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; \frac{(z-1)\delta r}{z(z-2)}\right) + B\left(\frac{1}{\mu+1}, \frac{\mu+2}{\mu+1}; 1-\frac{\delta r}{(z-1)z(z-2)}\right)} \tag{78}$$

$$\approx \frac{\delta r}{(z-2)(\mu+1)}\,. \tag{79}$$

This is exponentially small in μ because δr is.

We discuss briefly the behaviour for large μ of $S_0(x)$ and $S_1(x)$. Up to an overall normalization factor these give the probabilities, in the random landscape ensemble weighted by the fugacity z, that the root node of a Cayley tree has random function value x and is (for $S_1(x)$) or is not (for $S_0(x)$) a minimum. For large μ, the fact that the root node has μ rather than $\mu+1$ neighbours becomes unimportant and these probabilities also apply to an arbitrary node in the bulk of the Bethe lattice.

We find by constructing the explicit solutions for $H(x)$ etc. that there is a threshold value of x, $x^* = (z-2)/[2(z-1)]$ so that for $\mu \to \infty$ one has $S_0(x) = \bar S_0(x) = \Theta(x-x^*)$, $S_1(x) = z\Theta(x^*-x)$, $\bar S_1(x) = 0$. (Correspondingly, the functions H and I are piecewise linear below and above x^*.) So in the even layers, *i.e.* those with a nonzero density of minima, a site is a minimum if its value x_i is below x^*, and not a minimum otherwise. In the odd layers, no sites are minima, and the values x_i at all sites are above x^*. This is exactly the same phenomenology as in the two-box mean field theory, confirming again that the latter captures most of the physics of the large connectivity limit on the Bethe lattice. The exception is the region around the ordering transition where the square-root singularities and hence the order parameter exponent $\beta = 1/2$ are visible: this becomes vanishingly small as we will now see.

3.3.2. Large μ, around the transition From the large-μ expansion (71) we expect that the appropriate scaling for the fugacity in the region around the phase transition is $z = 2 + \tilde z/\mu$. As we will see, this corresponds to r being of order $1/\mu$, $r = \tilde r/\mu$. With these scalings, the third arguments of the Beta functions in (72) become $\frac{1}{2}(1 \mp \tilde r/\tilde z)$. Using from (B.7) that for $\mu \to \infty$, $B\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}; \frac{1}{2}(1+a)\right) - (\mu+1) \to \ln[(1+a)/(1-a)]$ and keeping only terms of $\mathcal{O}(1)$ and $\mathcal{O}(1/\mu)$ gives

$$1 - \frac{\tilde r}{\mu} = \frac{\mu+1+\ln[(1-\tilde r/\tilde z)/(1+\tilde r/\tilde z)]}{\mu+1+\ln[(1+\tilde r/\tilde z)/(1-\tilde r/\tilde z)]} = 1 + \frac{2}{\mu}\ln[(1-\tilde r/\tilde z)/(1+\tilde r/\tilde z)]\,. \tag{80}$$

Equating the $\mathcal{O}(1/\mu)$ terms shows

$$\tilde r = 2\ln[(1+\tilde r/\tilde z)/(1-\tilde r/\tilde z)] = 4\,\mathrm{artanh}(\tilde r/\tilde z) \tag{81}$$

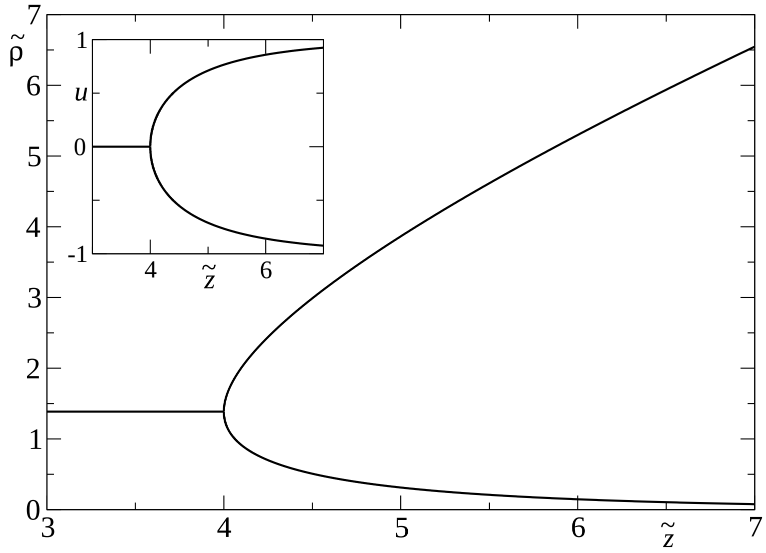

Figure 3. Equation of state of the random minima problem around the ordering transition, for the Bethe lattice in the limit of large connectivity $\mu+1$. Shown are the scaled densities of minima, $\tilde{\rho}$, in the two boxes, against the scaled fugacity $\tilde{z}$. Inset: After transforming nonlinearly to $u = 1 - 2e^{-\tilde{\rho}/2}$, the equation of state becomes that of a mean field ferromagnet at inverse temperature $\tilde{z}/4$.

or

$$\tilde{z} = \frac{\tilde{r}}{\tanh(\tilde{r}/4)} \ . \tag{82}$$

The critical point is reached for $\tilde{r} \to 0$, giving $\tilde{z}_{\rm c} = 4$ in agreement with (71).

It remains to work out the densities. In the expression (65) for the even layers, the arguments of the Beta functions simplify as before, and also $z/(z-1) \to 2$, so that

$$\rho = \frac{2B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; \frac{1}{2}\left(1+\frac{\tilde{r}}{\tilde{z}}\right)\right)}{B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; \frac{1}{2}\left(1+\frac{\tilde{r}}{\tilde{z}}\right)\right) + B\left(\frac{1}{\mu+1}, \frac{\mu+2}{\mu+1}; \frac{1}{2}\left(1-\frac{\tilde{r}}{\tilde{z}}\right)\right)} \ . \tag{83}$$

The second Beta function in the numerator equals $\mu+1$ to leading order while the other ones are $\mathcal{O}(1)$, $B\left(\frac{\mu+2}{\mu+1}, \frac{1}{\mu+1}; a\right) \to \int_0^a dt\,(1-t)^{-1} = -\ln(1-a)$, so that the scaled density $\tilde{\rho} = \rho\mu$ becomes for $\mu \to \infty$

$$\tilde{\rho} = -2\ln\left[\frac{1}{2}\left(1-\frac{\tilde{r}}{\tilde{z}}\right)\right] \tag{84}$$

and similarly in the odd layers, after swapping $\tilde{r} \to -\tilde{r}$,

$$\tilde{\bar{\rho}} = -2\ln\left[\frac{1}{2}\left(1+\frac{\tilde{r}}{\tilde{z}}\right)\right] \ . \tag{85}$$

The equations (82,84,85) give the equation of state in the phase transition region. The occurrence of the auxiliary parameter $\tilde{r}$ is a little awkward but can be eliminated if we transform the (scaled) densities nonlinearly as

$$u = 1 - 2e^{-\tilde{\rho}/2}, \qquad \bar{u} = 1 - 2e^{-\tilde{\bar{\rho}}/2} \tag{86}$$

so that, from (84,85), $u = -\bar{u}$ always. By combining (82,84) one then sees that

$$u = \frac{\tilde{r}}{\tilde{z}} = \tanh(\tilde{r}/4) \tag{87}$$

and so finally $\tilde{z} = \tilde{r}/u = 4\,\mathrm{artanh}(u)/u$ or

$$u = \tanh((\tilde{z}/4)u) \ . \tag{88}$$

Since $u = -\bar{u}$, the same equation also holds for the density in the odd layers. Remarkably, therefore, once the densities are nonlinearly transformed according to (86), they depend on the fugacity exactly as the magnetizations in a mean field ferromagnet with unit interaction strength and inverse temperature $\tilde{z}/4$. We show the equation of state in Fig. 3, both in terms of the (scaled) densities $\tilde{\rho}$ and, in the inset, the transformed variables u.

3.4. Mean field theory with soft minima

In this final subsection we ask whether the behaviour around the ordering transition that we found for a highly connected Bethe lattice can also be obtained directly within a mean field theory. It turns out that this is possible: drawing inspiration from our treatment of the hard particle model, we make the labelling of sites as minima 'soft'.

The two-box setup is initially the same as for hard particles, with generating function

$$G(z, N) = \left\langle z^{M+\bar{M}} \right\rangle = \left\langle z^{\sum_i (m_i+\bar{m}_i)} \right\rangle \tag{89}$$

Here the average is over our random landscape ensemble as before, while M and $\bar{M}$ label the total number of minima in the left and right box, respectively. For hard minima of course one and only one of these quantities is ever nonzero; for soft minima both M and $\bar{M}$ can be nonzero.

To define 'soft' minima, we first introduce an auxiliary variable $\tau_i \in \{0, 1\}$ at each site. We can obtain the usual generating function for hard minima by forcing this to be 0 if $m_i = 0$; otherwise we allow it to be 0 or 1. Assigning weight factors 1 and $z - 1$ to $\tau_i = 0$ and 1, respectively, we can then write the factor from each site in the generating function as

$$z^{m_i} = \sum_{\tau_i=0,1} (z-1)^{\tau_i} \delta_{\tau_i(1-m_i),0} \ . \tag{90}$$

(Indeed, for $m_i = 0$ only $\tau_i = 0$ is allowed and we get $z^0 = 1 = (z - 1)^0$; in the opposite case we have $z^1 = (z - 1)^0 + (z - 1)^1$.) Now to make the minima soft, we relax the constraint that $\tau_i = 0$ if $m_i = 0$, *i.e.* we replace $\delta_{\tau_i(1-m_i),0} \to \exp[-\alpha\tau_i(1 - \hat{m}_i)]$. Here

$$\hat{m}_i = N^{-1} \sum_{j=1}^{N} \Theta(\bar{x}_j - x_i) \tag{91}$$

is a soft version of m_i: it measures what fraction of numbers in the other box are above x_i, so that $\hat{m}_i = 0, 1/N, \ldots, 1 - 1/N$ corresponds to $m_i = 0$ and $\hat{m}_i = 1$ to $m_i = 1$. Thus, when $m_i = 0$ we have $1 - \hat{m}_i \geq 1/N$ and for $\alpha \to \infty$ at fixed N our soft minima weight $\exp[-\alpha\tau_i(1-\hat{m}_i)]$ reverts to $\delta_{\tau_i(1-m_i),0}$ as it should. As in the hard sphere case we in fact take $N \to \infty$ first and then $\alpha \to \infty$.

We summarize our starting point: the generating function for soft minima is

$$G(z,N) = \mathrm{Tr}_{\tau,\bar\tau}\,(z-1)^{\sum_{i=1}^{N}(\tau_i+\bar\tau_i)} e^{NA} \tag{92}$$

$$A = \frac{1}{N}\ln\left\langle \exp\left(-\alpha\sum_i[\tau_i(1-\hat{m}_i)+\bar\tau_i(1-\hat{\bar{m}}_i)]\right)\right\rangle \tag{93}$$

where $\mathrm{Tr}_{\tau,\bar\tau}$ abbreviates the sum over all τ_i and $\bar\tau_i$. The soft version of the minimum indicator variables $\bar{m}_i$ in the right box is defined in the obvious way by swapping the roles of x and $\bar{x}$ in (91), i.e. $\hat{\bar{m}}_i = N^{-1}\sum_{j=1}^{N}\Theta(x_j-\bar{x}_i)$.

To calculate G, consider first the average from (93). By permutation symmetry within each box, this can only depend on the numbers T, $\bar{T}$ of nonzero τ's in the two boxes. Writing out the definition of the $\hat{m}_i$ and $\hat{\bar{m}}_i$ in terms of sign functions, this gives

$$A = \frac{1}{N}\ln\left\langle \exp\left(-\frac{\alpha}{2N}\left[\sum_{i=1}^{T}\sum_{j=1}^{N}(1+\mathrm{sgn}(x_i-\bar{x}_j)) + \sum_{j=1}^{\bar{T}}\sum_{i=1}^{N}(1-\mathrm{sgn}(x_i-\bar{x}_j))\right]\right)\right\rangle \tag{94}$$

$$= -\,\alpha\frac{T+\bar{T}}{2N} + b(T,N-\bar{T}) + b(N-T,\bar{T}) \tag{95}$$

Here we have used that the sgn terms with $1 \le j \le \bar{T}$ in the first sum exactly cancel those with $1 \le i \le T$ in the second one, so that the remaining average factorizes into two independent terms of the form

$$e^{Nb(T_1,T_2)} = \left\langle \exp\left(-\frac{\alpha}{2N}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\mathrm{sgn}(x_i-\bar{x}_j)\right)\right\rangle \ . \tag{96}$$

The replacement $x_i \to 1-x_i$, $\bar{x}_j \to 1-\bar{x}_j$ leaves the distribution of these variables unchanged, hence b is symmetric under $\alpha \to -\alpha$, as well as under interchange of T_1 and T_2. To evaluate b, we can assume without loss of generality that the x_i are ordered. Using also that the average over the $\bar{x}_j$ factorizes,

$$e^{Nb(T_1,T_2)} = \Big\langle \Big[x_1 e^{-T_1\alpha/2N} + (x_2-x_1)e^{(2-T_1)\alpha/2N} + \ldots + (x_{T_1}-x_{T_1-1})e^{(T_1-2i)\alpha/2N} + (1-x_{T_1})e^{T_1\alpha/2N}\Big]^{T_2}\Big\rangle \tag{97}$$

where the remaining average is over the x_i. Denote the quantity raised to the power T_2 by y. Setting also $v_0 = x_1$, $v_1 = x_2 - x_1$, $\ldots$, $v_{T_1-1} = x_{T_1} - x_{T_1-1}$, $v_{T_1} = 1 - x_{T_1}$, the v_i are non-negative (because of the ordering of the x_i) and uniformly distributed apart from the constraint $\sum_{i=0}^{T_1} v_i = 1$, such that $P(\{v_i\}) = T_1!\,\delta(1-\sum_i v_i)$. The characteristic function of y is then

$$\left\langle e^{N\omega y}\right\rangle = T_1!\int\frac{Nd\lambda}{2\pi i}\int_0^\infty\prod_{i=0}^{T_1}dv_i\,\exp\left(N\lambda(1-\sum_i v_i) + N\omega\sum_i v_i e^{(2i-T_1)\alpha/2N}\right) \tag{98}$$

$$= T_1!\int\frac{Nd\lambda}{2\pi i}e^{N\lambda}\prod_{i=0}^{T_1}\left(N\lambda - N\omega e^{(i-T_1/2)\alpha/N}\right)^{-1} \ . \tag{99}$$

Reverse Fourier transforming now produces

$$e^{Nb(T_1,T_2)} = \left\langle y^{T_2} \right\rangle = T_1! \int dy \int \frac{Nd\omega}{2\pi i} \int \frac{Nd\lambda}{2\pi i} \exp\Big[T_2 \ln y - N\omega y + N\lambda - (T_1+1)\ln N - \sum_{i=0}^{T_1} \ln\left(\lambda - \omega e^{(i-T_1/2)\alpha/N}\right) \Big] . \tag{100}$$

Defining the intensive quantitites $t_1 = T_1/N$, $t_2 = T_2/N$, we can do the integral using steepest descents for $N \to \infty$:

$$b(t_1,t_2) = \max_{y,\omega,\lambda} \left\{ t_1 \ln(t_1/e) + t_2 \ln y - \omega y + \lambda - \int_{-t_1/2}^{t_1/2} du\, \ln(\lambda - \omega e^{\alpha u}) \right\} . \tag{101}$$

Setting the derivatives w.r.t. y, ω and λ to zero gives the saddle point equations

$$\omega = \frac{t_2}{y}, \qquad y = \int_{-t_1/2}^{t_1/2} du\, \frac{e^{\alpha u}}{\lambda - \omega e^{\alpha u}}, \qquad 1 = \int_{-t_1/2}^{t_1/2} du\, \frac{1}{\lambda - \omega e^{\alpha u}} . \tag{102}$$

Combining the last two we find

$$y = \frac{1}{\omega} \int_{-t_1/2}^{t_1/2} du \left(\frac{\lambda}{\lambda - \omega e^{\alpha u}} - 1 \right) = \frac{\lambda - t_1}{\omega} \tag{103}$$

and hence $\lambda = t_1 + t_2$. In the last saddle point equation we can perform the integral explicitly, yielding

$$1 = \frac{1}{\alpha\lambda} \ln\left(\frac{\lambda e^{\alpha t_1/2} - \omega}{\lambda e^{-\alpha t_1/2} - \omega} \right) \tag{104}$$

and we can solve for ω:

$$\omega = (t_1 + t_2) \frac{e^{\alpha t_2/2} - e^{-\alpha t_2/2}}{e^{\alpha(t_1+t_2)/2} - e^{-\alpha(t_1+t_2)/2}} \tag{105}$$

Together with $\lambda = t_1 + t_2$, $y = t_2/\omega$ we thus have all saddle point values explicitly. The derivatives of $b(t_1,t_2)$ that we will need become

$$\frac{\partial b}{\partial t_1} = \ln t_1 - \frac{1}{2} \ln\left[(\lambda - \omega e^{\alpha t_1/2})(\lambda - \omega e^{-\alpha t_1/2}) \right] \tag{106}$$

$$= \ln \frac{t_1}{t_1+t_2} + \frac{\alpha t_2}{2} + \ln\left(\frac{1 - e^{-\alpha(t_1+t_2)}}{1 - e^{-\alpha t_1}} \right) \tag{107}$$

and $\partial b/\partial t_2$ has the same form with t_1 and t_2 interchanged. This symmetry property is clear from the definition (96) but not so obvious from the saddle point representation (101). An explicitly symmetric expression can be obtained from (107) by integrating from $t_1 = 0$ (where $b = 0$); after a little algebra, this can be cast in the form

$$\alpha b(t_1,t_2) = F(\alpha(t_1+t_2)) - F(\alpha t_1) - F(\alpha t_2) \tag{108}$$

with $F(x) = \int_0^x du\, \ln[2\sinh(u/2)/u]$. One might hope that a derivation exists which directly reveals this simple structure, but so far we have been unable to find one.

Now we can finally write down the saddle point equations for the full generating function. In terms of the τ-densities $t = T/N$, $\bar{t} = \bar{T}/N$, one has from (92,93,95)

$$\frac{1}{N}\ln G = \max_{t,\bar{t}} \{\mathcal{H}(t) + \mathcal{H}(\bar{t}) + (t+\bar{t})\ln(z-1) + A\} \quad (109)$$

$$= \max_{t,\bar{t}} \{\mathcal{H}(t) + \mathcal{H}(\bar{t}) + (t+\bar{t})[\ln(z-1) - \alpha/2] + b(t, 1-\bar{t}) + b(1-t,\bar{t})\} \quad (110)$$

where $\mathcal{H}(t)$ and $\mathcal{H}(\bar{t})$ again account for the combinatorial (entropic) contributions. Then

$$\frac{\partial}{\partial t}\frac{1}{N}\ln G = \ln\frac{1-t}{t} + \ln(z-1) - \alpha/2 + \left.\frac{\partial b}{\partial t_1}\right|_{t_1=t,\, t_2=1-\bar{t}} - \left.\frac{\partial b}{\partial t_1}\right|_{t_1=1-t,\, t_2=\bar{t}} \quad (111)$$

$$= \ln(z-1) - \alpha\bar{t} + \ln\left(\frac{1-t+\bar{t}}{1+t-\bar{t}}\right) + \ln\left(\frac{1-e^{-\alpha(1+t-\bar{t})}}{1-e^{-\alpha(1-t+\bar{t})}}\,\frac{1-e^{-\alpha(1-t)}}{1-e^{-\alpha t}}\right) \quad (112)$$

and this must vanish at the saddle point. The corresponding equation for $\bar{t}$ just has t and $\bar{t}$ swapped.

In the disordered phase $t = \bar{t}$, the saddle point equation can be solved explicitly to get $t = \alpha^{-1}\ln[z/(1+e^{-\alpha})]$. One can then again ask about bifurcations to solutions where $t \neq \bar{t}$. The critical value of z can be got as follows: think of the first saddle point equation as defining implicitly $\bar{t}$ as a function of t; the second saddle point equation defines the inverse function, which graphically is flipped about the diagonal. The disordered fixed point on the diagonal becomes unstable when the slope $d\bar{t}/dt = -1$. The resulting condition on z_c looks complicated, but neglecting terms that are exponentially small in α one gets $z_c = 2(1-\alpha^{-1})^2/(1-4\alpha^{-1}) = 2 + 4\alpha^{-1} + \mathcal{O}(\alpha^{-2})$. This looks encouraging: with the identification $\mu \equiv \alpha$, it is identical to the result (71), suggesting that the soft minima mean field theory captures the large connectivity limit on the Bethe lattice.

We recall for the evaluation in the following subsections that $t = N^{-1}\sum_i \tau_i$, and similarly $\bar{t}$, are the densities of the τ-variables. For large enough α, we can have $\tau_i = 1$ only when there is genuinely a minimum at site i ($m_i = 1$); but even if $m_i = 1$ then $\tau_i = 0$ with probability $1/(1+z-1) = 1/z$. So for $\alpha \to \infty$ the τ-densities are related to the true densities of minima by $t = [(z-1)/z]\rho$, $\bar{t} = [(z-1)/z]\bar{\rho}$.

3.4.1. Large α, above the transition Here we expect that one of the two densities (say t) stays finite and < 1 while the other ($\bar{t}$) goes to zero. The saddle point equations are then, up to exponentially small terms:

$$0 = \ln(z-1) - \alpha\bar{t} + \ln\left(\frac{1-t+\bar{t}}{1+t-\bar{t}}\right) \quad (113)$$

$$0 = \ln(z-1) - \alpha t - \ln\left(\frac{1-t+\bar{t}}{1+t-\bar{t}}\right) - \ln(1-e^{-\alpha\bar{t}})\ . \quad (114)$$

In the second equation, the only way to balance the $-\alpha t$ term is to have $\bar{t}$ vanish faster than $1/\alpha$ so that the argument of the last log tends to zero; the log itself can then be approximated as $-\ln(\alpha\bar{t})$. This gives to leading order $\bar{t} \sim \alpha^{-1}e^{-\alpha t}$. Inserting into the first equation then leads to $(1+t)/(1-t) = z-1$ or $t = (z-2)/z$. The density of minima in this box is therefore $\rho = [z/(z-1)]t = (z-2)/(z-1)$, consistent with our direct calculation in the hard minima limit. The critical fugacity is $z_\mathrm{c} = 2$, also as expected. Overall the $\alpha \to \infty$ limit correctly reproduces the hard minima scenario as desired.

3.4.2. Large α, around the transition By analogy with the Bethe lattice calculation, we scale the τ-densities and the fugacity as $t = \tilde{t}/\alpha$, $\bar{t} = \tilde{\bar{t}}/\alpha$ and $z = 2 + \tilde{z}/\alpha$, respectively. The saddle point equations are then, again up to exponentially small terms,

$$0 = \ln(z-1) - \tilde{\bar{t}} + \ln\left(\frac{1-(\tilde{t}-\tilde{\bar{t}})/\alpha}{1+(\tilde{t}-\tilde{\bar{t}})/\alpha}\right) - \ln\left(1-e^{-\tilde{t}}\right) \tag{115}$$

$$0 = \ln(z-1) - \tilde{t} - \ln\left(\frac{1-(\tilde{t}-\tilde{\bar{t}})/\alpha}{1+(\tilde{t}-\tilde{\bar{t}})/\alpha}\right) - \ln\left(1-e^{-\tilde{\bar{t}}}\right) \tag{116}$$

It is again useful to make a nonlinear transformation from the τ-densities to

$$u = 1 - 2e^{-\tilde{t}}, \qquad \bar{u} = 1 - 2e^{-\tilde{\bar{t}}} \ . \tag{117}$$

The first saddle point equation then implicitly defines a function U via $\bar{u} = U(u)$. The second one gives $u = U(\bar{u})$. For these to be consistent with each other, we require

$$\alpha[U(u) - U^{-1}(u)] = 0 \ . \tag{118}$$

Here U^{-1} is the inverse function of U; the factor α will be useful shortly. To find the function U for large α, we expand the first saddle point equation, keeping terms of $\mathcal{O}(1)$ and $\mathcal{O}(1/\alpha)$:

$$0 = \frac{\tilde{z}}{\alpha} - \tilde{\bar{t}} - \frac{2(\tilde{t}-\tilde{\bar{t}})}{\alpha} - \ln\left(1-e^{-\tilde{t}}\right) = \ln\left(\frac{1-\bar{u}}{1+u}\right) + \frac{\tilde{z} + 2\ln[(1-u)/(1-\bar{u})]}{\alpha} \ . \tag{119}$$

To leading order this gives $\bar{u} = U(u) = -u$: the function U is identical to its inverse. This is why we need to go to $\mathcal{O}(1/\alpha)$ to get a nontrivial condition for u, as emphasized by the factor α in (118). Now insert the leading order relation $\bar{u} = -u$ into the $\mathcal{O}(1/\alpha)$ term above to get

$$\bar{u} = U(u) = 1 - (1+u)\left(1 - \frac{\tilde{z} + 2\ln[(1-u)/(1+u)]}{\alpha}\right) \tag{120}$$

$$= -u + (1+u)\frac{\tilde{z} + 2\ln[(1-u)/(1+u)]}{\alpha} \ . \tag{121}$$

The inverse function is obtained by solving (119) for u:

$$u = U^{-1}(\bar{u}) = -1 + (1-\bar{u})\left(1 + \frac{\tilde{z} + 2\ln[(1+\bar{u})/(1-\bar{u})]}{\alpha}\right) \tag{122}$$

$$= -\bar{u} + (1-\bar{u})\frac{\tilde{z} + 2\ln[(1+\bar{u})/(1-\bar{u})]}{\alpha} \ . \tag{123}$$

So the equation (118) determining u becomes for $\alpha \to \infty$

$$0 = (1+u)(\tilde{z} + 2\ln[(1-u)/(1+u)]) - (1-u)(\tilde{z} + 2\ln[(1+u)/(1-u)]) \tag{124}$$

$$= 2\tilde{z}u + 4\ln[(1-u)/(1+u)] = 2\tilde{z}u - 8\,\mathrm{artanh}(u) \tag{125}$$

or

$$u = \tanh((\tilde{z}/4)u) \; . \tag{126}$$

This is exactly as on the Bethe lattice around the transition, so the entire scaling behaviour in this region matches between the two cases, namely, the Bethe lattice in the limit of large connectivity μ and the soft minima problem in the nearly hard limit of large α. Notice that, while the definitions of the relevant nonlinear transformations (86) and (117) of the density variables look different, they are in fact identical because $\tilde{\rho} = [z/(z-1)]\tilde{t} = 2\tilde{t} + \mathcal{O}(1/\alpha)$.

4. Summary and outlook

In summary, we have analysed the number and distribution of minima in random landscapes defined on non-Euclidean lattices. Using an ensemble where random landscapes are reweighted by a fugacity factor z^M depending on the number of minima M, the simplest viable (two-box) mean field theory showed an ordering phase transition at $z_\mathrm{c} = 2$. For $z > z_\mathrm{c}$, one box contains an extensive number of minima with density $\rho = (z-2)/(z-1)$. The onset of order seemed to be governed by an unusual order parameter exponent $\beta = 1$, which motivated our study on the Bethe lattice.

Using recursion techniques, we found a full solution of the problem on the Bethe lattice which showed that for any finite connectivity $\mu + 1$ (> 2) there is indeed an ordering transition with a conventional mean field order parameter exponent $\beta = 1/2$. As μ becomes large, the region around the transition where this behaviour is visible shrinks as $1/\mu$. It disappears as $\mu \to \infty$ at fixed fugacity z, and this is what causes the unusual effective exponent in the two-box mean field theory. We analysed separately the scaling for large μ for fixed z above the transition and for z within $1/\mu$ of z_c. In the latter case, a nonlinear transformation turns out to map the equation of state neatly onto that of a mean field ferromagnet. Finally, we showed that the region around the phase transition can also be analysed directly within a mean field approach, by making the assignment of minima 'soft' and then taking the nearly hard limit ($\alpha \to \infty$). This was motivated by our analogous treatment of the hard sphere lattice gas, where a softening of the nearest neighbour exclusion revealed the ordering phase transition that remains entirely hidden within the two-box mean field theory.

In the mean field approach we also considered the joint distribution of minima and maxima of random landscapes. Here two fugacities enter, z_1 and z_2, and in addition to the phase transitions at $z_1 = 2$ and $z_2 = 2$ where the number of minima and maxima respectively first becomes extensive, there is a first-order transition on the line $1/z_1 + 1/z_2 = 1$. Beyond this line, essentially all points in the landscape are either minima and maxima; in our mean field setup, these sites are separated into the two boxes.

In future work, it should be possible to extend the analysis of joint distributions of minima and maxima to the Bethe lattice. This would presumably require three generating functions, for sites that are minima, maxima or neither. Generalizing the soft minima/maxima approach looks less easy because for 'soft' labels one would also have to consider sites that are labelled as both minima and maxima. It would also be interesting to generalize further, and consider not just minima but also nodes with fixed number $k = 1, 2, \ldots$ of lower-lying neighbours.

Finally, one would like to extend our calculation also to large random graphs with the same local structure as a Bethe lattice, i.e. regular graphs where all nodes have the same number $(\mu+1)$ of neighbours. Given that short loops are rare on such graphs, one might intuitively expect to see the same phenomenology. However, the strict sublattice ordering on the Bethe lattice cannot be maintained in the inevitable presence of at least some loops with an odd number of links, and so in actual fact it is likely that one would instead obtain glassy phases as in related hard particle models [34, 35]. Generalizing our approach to this scenario appears to be a challenging problem indeed.

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Appendix A. Minima and maxima for $z_1 > 2$ or $z_2 > 2$

We outline two methods for understanding the two-box problem in the case where we track both minima and maxima. The first one starts from the large-deviation form of (37). It is easy to see that the first and third terms can never be larger than the second. Taking N large at fixed densities $\rho_1 = M_1/N$ and $\rho_2 = M_2/N$ of the minima and maxima then gives up an irrelevant constant

$$N^{-1} \ln P(N\rho_1, N\rho_2, N) = (2-\rho_1-\rho_2)\mathcal{H}\left(\frac{1-\rho_1}{2-\rho_1-\rho_2}\right) . \tag{A.1}$$

If one rewrites the definition (38) of the generating function as an integral over ρ_1 and ρ_2, the latter will therefore be dominated by those values maximizing the function

$$\gamma(\rho_1,\rho_2) = (2-\rho_1-\rho_2)\mathcal{H}\left(\frac{1-\rho_1}{2-\rho_1-\rho_2}\right) + \rho_1 \ln z_1 + \rho_2 \ln z_2 . \tag{A.2}$$

Now take for definiteness $z_1 > z_2$, so that any maxima will obey $\rho_1 \geq \rho_2$. In this regime we can set $1-\rho_1 = \kappa(2-\rho_1-\rho_2)$ with $0 \leq \kappa \leq 1/2$ and have at fixed κ a *linear* variation with $1-\rho_2$:

$$\gamma(\rho_1,\rho_2) = \ln(z_1 z_2) + (1-\rho_2)s(\kappa) \tag{A.3}$$

$$s(\kappa) = -\frac{\kappa}{1-\kappa}\ln(\kappa z_1) - \ln[(1-\kappa)z_2] . \tag{A.4}$$

The slope function $s(\kappa)$ now tells us where the maxima of $\gamma(\rho_1,\rho_2)$ are. First we maximize over κ; if the maximum value of $s(\kappa)$ is positive, we get a maximum of $\gamma(\rho_1,\rho_2)$ at $\rho_2 = 0$ and hence $\rho_1 = (1-2\kappa)/(1-\kappa)$, otherwise a maximum at $\rho_2 = 1$ and $\rho_1 = 1$.

The derivative of $s(\kappa)$ is $s'(\kappa) = -\ln(\kappa z_1)/(1-\kappa)^2$. For $z_1 < 2$ this is always positive and the maximum is at $\kappa = 1/2$, where $s(1/2) = -\ln(z_1 z_2/4) > 0$ (given that $z_2 < z_1 < 2$). So $\gamma(\rho_1, \rho_2)$ is maximal at $\rho_1 = \rho_2 = 0$, consistent with the analysis in the main text that showed that in this regime minima and maxima are both intensive in number.

For $z_1 > 2$, the maximum of $s(\kappa)$ is at $\kappa = 1/z_1$, where $s(1/z_1) = -\ln[(1 - 1/z_1)/(1/z_2)]$. If $1/z_2 > 1 - 1/z_1$, this value is positive and $\gamma(\rho_1, \rho_2)$ has a maximum at $\rho_2 = 0$, $\rho_1 = (1-2\kappa)/(1-\kappa) = (z_1-2)/(z_1-1)$. This is the mixed regime, with M_1 extensive and M_2 intensive, see (40). For $1/z_2 < 1 - 1/z_1$, finally, the maximum value of $s(\kappa)$ is negative and $\gamma(\rho_1, \rho_2)$ has its maximum at $\rho_1 = \rho_2 = 1$. This is the fully separated regime, where one box contains essentially only minima and the other only maxima. Note that, as stated in the main text, at the first-order transition $1/z_1 + 1/z_2 = 1$, the entire line in the (ρ_1, ρ_2) plane corresponding to $\kappa = 1/z_1$ is degenerate, i.e. has the same value of $\gamma(\rho_1, \rho_2)$. A further peculiarity is that there is no metastability: neither of the phases persists as a local maximum of $\gamma(\rho_1, \rho_2)$ on the corresponding 'wrong' side of the transition line.

It remains to find the average number of maxima in the mixed regime ($z_1 > 2$, $1/z_1 + 1/z_2 > 1$). We already know that ρ_1 is nonzero then; on general grounds its fluctuations ($\sim 1/\sqrt{N}$) must become negligible for large N. We can then take the limit $N \to \infty$ in (37) at finite M_2 and $M_1 = N\rho_1$ to get for the distribution of M_2 at given ρ_1

$$P(M_2, N|\rho_1) = \frac{1}{2-\rho_1}\left(\frac{1-\rho_1}{2-\rho_1}\right)^{M_2} + \frac{1-\rho_1}{2-\rho_1}\left(\frac{1}{2-\rho_1}\right)^{M_2} . \tag{A.5}$$

Multiplying by $z_2^{M_2}$, normalizing and taking the average of M_2 then gives the result stated in (40). Note that (A.5) has a simple interpretation in the labelling picture: For the $M_1 = N\rho_1$ minima we have used up as many 'left' labels (and one 'right' label). The probability that the largest number y_{2N} will be labelled 'left' is then $(N - N\rho_1)/(2N - N\rho_1 - 1) \to (1-\rho_1)/(2-\rho_1)$ for large N. The first term in (A.5) thus gives the probability that $y_{2N}, y_{2N-1}, \ldots, y_{2N-M_2+1}$ are all labelled 'left' and the next number down, y_{2N-M_2}, is labelled 'right'; the second term gives the analogous contribution from the reverse labelling. The respective probabilities $(1-\rho_1)/(2-\rho_1)$ and $1/(2-\rho_1)$ for a 'left' and 'right' label remain the same throughout as we are only labelling finitely many (M_2+1) numbers and so for large N the fraction of labels in the bag of either kind only changes negligibly.

The second approach parallels more closely the one taken for the minima problem. We first calculate the joint probability distribution, $P(x_-, x_+)$ of the smallest and largest number in the left box. The probability that all the x_i are greater that some value x_- and smaller than some other value x_+ is $\mathcal{P} = (x_+ - x_-)^N$. This is also the probability that the minimum of these numbers is larger than x_-, and the maximum smaller than x_+, so the joint probability density of the minimum and maximum is obtained by differentiation as

$$P(x_-, x_+) = -\frac{\partial^2 \mathcal{P}}{\partial x_- \, \partial x_+} = N(N-1)(x_+ - x_-)^{N-2}. \tag{A.6}$$

There is an analogous expression for $P(\bar{x}_-, \bar{x}_+)$. The remaining $N-2$ numbers in each box are then distributed uniformly between x_- and x_+, and $\bar{x}_-$ and $\bar{x}_+$, respectively.

One can now represent the probability of getting M_1 minima and M_2 maxima in terms of averages over this distribution. As before we assume that the minima are in the left box, *i.e.* $x_- < \bar{x}_-$, and multiply the probability by a factor 2 to cover the opposite case:

$$\begin{aligned}
\frac{1}{2}P(M_1, M_2, N) = & \left\langle \Theta(\bar{x}_- - x_-)\Theta(x_+ - \bar{x}_+)\frac{(N-2)!}{(M_2-1)!(N-M_1-M_2)!(M_1-1)!} \right. \\
& \left. \times \left(\frac{x_+ - \bar{x}_+}{x_+ - x_-}\right)^{M_2-1} \left(\frac{\bar{x}_+ - \bar{x}_-}{x_+ - x_-}\right)^{N-M_1-M_2} \left(\frac{\bar{x}_- - x_-}{x_+ - x_-}\right)^{M_1-1} \right\rangle \\
& + \left\langle \Theta(\bar{x}_- - x_-)\Theta(x_+ - \bar{x}_-)\Theta(\bar{x}_+ - x_+) \right. \\
& \times \binom{N-2}{M_1-1} \left(\frac{x_+ - \bar{x}_-}{x_+ - x_-}\right)^{N-M_1-1} \left(\frac{\bar{x}_- - x_-}{x_+ - x_-}\right)^{M_1-1} \\
& \left. \times \binom{N-2}{M_2-1} \left(\frac{\bar{x}_+ - x_+}{\bar{x}_+ - \bar{x}_-}\right)^{M_2-1} \left(\frac{x_+ - \bar{x}_-}{\bar{x}_+ - \bar{x}_-}\right)^{N-M_2-1} \right\rangle \\
& + \delta_{M_1,N}\delta_{M_2,N} \left\langle \Theta(\bar{x}_- - x_-)\Theta(\bar{x}_+ - x_+)\Theta(\bar{x}_- - x_+) \right\rangle \; . \qquad \text{(A.7)}
\end{aligned}$$

The three terms on the r.h.s. are arranged in the same order as in (37), and represent different orderings of x_-, $\bar{x}_-$, x_+ and $\bar{x}_+$. In the first term, the Theta functions and the constraint $\bar{x}_- < \bar{x}_+$ enforce the ordering $x_- < \bar{x}_- < \bar{x}_+ < x_+$, so that the left box contains both minima and maxima and the right box neither. The remaining factors in this term give the probability that out of the $N-2$ numbers in the left box (other than x_- and x_+) exactly M_1-1 are below $\bar{x}_-$ and hence minima, and M_2-1 are above $\bar{x}_+$ and therefore maxima. The second term corresponds to the ordering $x_- < \bar{x}_- < x_+ < \bar{x}_+$, where the maxima are in the right box but the ranges of numbers in the two boxes still overlap. In this case we need to find M_1-1 numbers in the left box (in addition to x_-) that are below $\bar{x}_-$, and M_2-1 numbers in the right box (in addition to $\bar{x}_+$) that are above x_+. Finally, the last term is for the ordering $x_- < x_+ < \bar{x}_- < \bar{x}_+$. All numbers in the left box are then smaller than in the right one, and we have $M_1 = N$ minima on the left and $M_2 = N$ maxima on the right.

In the representation (A.7) one can easily perform the sums defining the generating function (38) to get

$$\begin{aligned}
\frac{1}{2}G(z_1, z_2, N) \; = & \left\langle \Theta(\bar{x}_- - x_-)\Theta(x_+ - \bar{x}_+) \right. \\
& \left. \times z_1 z_2 \left(\frac{z_2(x_+ - \bar{x}_+) + \bar{x}_+ - \bar{x}_- + z_1(\bar{x}_- - x_-)}{x_+ - x_-}\right)^{N-2} \right\rangle \\
& + \left\langle \Theta(\bar{x}_- - x_-)\Theta(x_+ - \bar{x}_-)\Theta(\bar{x}_+ - x_+) \right. \\
& \left. \times z_1 z_2 \left(\frac{x_+ - \bar{x}_- + z_1(\bar{x}_- - x_-)}{x_+ - x_-}\right)^{N-2} \left(\frac{z_2(\bar{x}_+ - x_+) + x_+ - \bar{x}_-}{\bar{x}_+ - \bar{x}_-}\right)^{N-2} \right\rangle
\end{aligned}$$

$$+ z_1^N z_2^N \langle \Theta(\bar{x}_- - x_-)\Theta(\bar{x}_+ - x_+)\Theta(\bar{x}_- - x_+)\rangle \ . \tag{A.8}$$

To carry out the averages one inserts (A.6) and its analogue for the right box and integrates over x_-, $\bar{x}_-$, x_+ and $\bar{x}_+$. In each term two of the integrals can be done directly and one is left with

$$\begin{aligned}\frac{1}{2}G(z_1, z_2, N) = & \int_0^1 d\bar{x}_- \int_{\bar{x}_-}^1 d\bar{x}_+ N(N-1)(\bar{x}_+ - \bar{x}_-)^{N-2} \Big\{ [z_2(1-\bar{x}_+) + \bar{x}_+ - \bar{x}_- + z_1\bar{x}_-]^N \\ & - [z_2(1-\bar{x}_+) + \bar{x}_+ - \bar{x}_-]^N - [\bar{x}_+ - \bar{x}_- + z_1\bar{x}_-]^N + [\bar{x}_+ - \bar{x}_-]^N \Big\} \\ & + \int_0^1 d\bar{x}_- \int_{\bar{x}_-}^1 dx_+ \, N^2 \Big\{ [x_+ - \bar{x}_- + z_1\bar{x}_-]^{N-1} - [x_+ - \bar{x}_-]^{N-1} \Big\} \\ & \times \Big\{ [z_2(1-x_+) + x_+ - \bar{x}_-]^{N-1} - [x_+ - \bar{x}_-]^{N-1} \Big\} \\ & + z_1^N z_2^N \int_0^1 dx_+ \int_{x_+}^1 d\bar{x}_- \, N^2 x_+^{N-1} (1-\bar{x}_-)^{N-1} \ . \end{aligned} \tag{A.9}$$

The remaining integrals in the last line can of course also be done and give $[(2N)!/N!^2]^{-1}$ as expected from (37).

From here on one can proceed as in the minima-only case. If both z_1 and z_2 are below 2, one rescales $\bar{x}_- = u/N$, $\bar{x}_+ = 1 - v/N$ in the first integral and similarly for the other terms; this gives $G(z_1, z_2, N \to \infty) = [z_1/(2-z_1)][z_2/(2-z_2)]$ as derived by a different route in the main text. For larger z_1 or z_2 one uses steepest descents again. The functions to be maximized always have negative Hessian determinants so the maxima are on the boundary. We illustrate only the mixed case $z_1 > z_2$ at $1/z_1 + 1/z_2 > 1$. Here the relevant saddle point in the first integral is $\bar{x}_- = (z_1 - 2)/[2(z_1 - 1)]$ and $\bar{x}_+ = 1$. Because this is at the upper extreme of the integration range of $\bar{x}_+$, however, one needs to rescale $\bar{x}_+ = 1 - v/N$ to treat the near-cancellation of the first and third terms in the integrand explicitly. (The second and fourth terms make exponentially subleading contributions.) In the second integral one needs to set similarly $x_+ = 1 - v/N$ to capture the near-cancellation in the second factor. One thus gets for large N, after neglecting the exponentially subdominant third term of (A.9):

$$\begin{aligned}\frac{1}{2}G(z_1, z_2, N) = & \int_0^1 d\bar{x}_- \int_0^\infty dv \, (N-1)(1-\bar{x}_-)^{N-2} e^{-v/(1-\bar{x}_-)} \\ & \times [1 + (z_1 - 1)\bar{x}_-]^N \Big\{ e^{(z_2-1)v/[1+(z_1-1)\bar{x}_-]} - e^{-v/[1+(z_1-1)\bar{x}_-]} \Big\} \\ & + \int_0^1 d\bar{x}_- \int_0^\infty dv \, N[1 + (z_1-1)\bar{x}_-]^{N-1} e^{-v/[1+(z_1-1)\bar{x}_-]} \\ & \times (1-\bar{x}_-)^{N-1} \Big\{ e^{(z_2-1)v/(1-\bar{x}_-)} - e^{-v/(1-\bar{x}_-)} \Big\} \ . \end{aligned} \tag{A.10}$$

The common exponential factor $\{(1-\bar{x}_-)[1 + (z_1-1)\bar{x}_-]\}^N$ means that for large N we can replace $\bar{x}_- = (z_1-2)/[2(z_1-1)]$ in all other, slowly varying, terms to obtain

$$\frac{1}{2}G(z_1, z_2, N) = \frac{2N(z_1-1)^2 z_2(z_2-2)}{z_1(z_1-z_2)(z_1z_2 - z_1 - z_2)} \int_0^1 d\bar{x}_- \{(1-\bar{x}_-)[1 + (z_1-1)\bar{x}_-]\}^N \ . \tag{A.11}$$

This result is asymptotically exact for $N \to \infty$. It depends on z_2 only through subexponential factors, which is why the cancellations referred to above have to be treated so carefully. Using $\langle M_2 \rangle = \partial \ln G / \partial \ln z_2$ then retrieves after a little algebra the result for the number of maxima in the mixed phase stated in (40).

Appendix B. Beta function asymptotics

Here we gather the asymptotic properties of the incomplete Beta function $B(p, q; a) = \int_0^a dt\, t^{p-1}(1-t)^{q-1}$ that we need in Sec. 3.3. Specifically, setting $p = 1/(\mu+1)$, we require the behaviour of $B(p, p; a)$, $B(p, p+1; a)$ and $B(p+1, p; a)$ in the limit $p \to 0$. Directly from the definition one sees that these three functions are linked by the simple sum rule

$$B(p, p; a) = B(p, p+1; a) + B(p+1, p; a) \ . \tag{B.1}$$

We will always keep $a < 1$, with a either fixed as $p \to 0$ or itself going to zero.

The last function in (B.1) is simplest as it remains non-singular:

$$B(p+1, p; a) = \int_0^a dt\, t^p(1-t)^{p-1} \to \int_0^a dt\, (1-t)^{-1} = -\ln(1-a) \ . \tag{B.2}$$

The other two functions, on the other hand, diverge as $p \to 0$. With the variable transformation $s = t^p$ one gets

$$pB(p, p; a) = p\int_0^a dt\, t^{p-1}(1-t)^{p-1} = \int_0^{a^p} ds\, \left(1 - s^{1/p}\right)^{p-1} \ . \tag{B.3}$$

Now if a vanishes quickly enough (exponentially in $1/p$) when $p \to 0$ for a^p to stay bounded below 1, we can exploit the fact that the integrand approaches unity for all $s < 1$ to get

$$pB(p, p; a) \to a^p \ . \tag{B.4}$$

From (B.1) and (B.2), which shows that $B(p+1, p; a)$ stays finite, the same limit applies to $B(p, p+1; a)$.

The result (B.4) does in fact extend also to a that vanish more slowly or stay finite, so that $a^p \to 1$. One can see this by subtracting off the leading term:

$$B(p, p; a) - \frac{1}{p}a^p = \int_0^a dt\, t^{p-1}[(1-t)^{p-1} - 1] \ . \tag{B.5}$$

The integrand now remains non-singular for $p \to 0$ and approaches $1/(1-t)$, so that

$$B(p, p; a) - \frac{1}{p}a^p \to -\ln(1-a) \ . \tag{B.6}$$

Multiplying by p gives back (B.4) as claimed. For fixed a it is more useful to rewrite the last relation, using $a^p = 1 + p\ln a + \mathcal{O}(p^2)$, as

$$B(p, p; a) - \frac{1}{p} \to \ln[a/(1-a)] \ . \tag{B.7}$$

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