Phase Transitions in the Distribution of the Andreev Conductance of Superconductor-Metal Junctions with Multiple Transverse Modes

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We compute analytically the full distribution of Andreev conductance $G_{\rm NS}$ of a metal-superconductor interface with a large number N_c of transverse modes, using a random matrix approach. The probability distribution $\mathcal{P}(G_{\rm NS}, N_c)$ in the limit of large N_c displays a Gaussian behavior near the average value $\langle G_{\rm NS}\rangle = (2-\sqrt{2})N_c$ and asymmetric power-law tails in the two limits of very small and very large $G_{\rm NS}$. In addition, we find a novel third regime sandwiched between the central Gaussian peak and the power-law tail for large $G_{\rm NS}$. Weakly nonanalytic points separate these four regimes—these are shown to be consequences of three phase transitions in an associated Coulomb gas problem.

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Introduction.—Advances in fabrication of mesoscopic structures has led to a great deal of interest in their electrical and thermal transport properties, from the point of view of both fundamental questions in the quantum theory of transport, and of device applications [1]. When the devices are disordered or chaotic, a statistical approach in which one characterizes the phase-coherent motion of electrons in terms of an ensemble of unitary scattering matrices **S** [2–7] and uses Landauer's description [8,9] of transport in terms of the corresponding transmission eigenvalues $\{T_n\}$, has proved very successful. Among the early successes of this approach was a general and transparent explanation [2–4] for the phenomenon of universal conductance fluctuations [1,10,11]: the variance var(G) corresponding to sample-to-sample fluctuations of the conductance G (measured in units of the conductance quantum $G_0 = 2e^2/h$) of disordered mesoscopic structures is independent of their size and the disorder strength, and is determined solely by whether or not time-reversal (TR) and other symmetries are present.

Within this random matrix approach, the conductance G of a structure with N_c transverse channels is given as $G = \sum_{n=1}^{N_c} T_n$, and the fact that its variance var(G) is a universal $\mathcal{O}(1)$ number is then seen to be a natural consequence of strong correlation between the $\{T_n\}$—the precise nature of these correlations is determined only by the symmetry properties of the relevant ensemble of scattering matrices. These correlations cause var(G) to become independent of N_c at large N_c, contrary to expectations from the usual "central limit considerations" for sums of a large number of independent random variables.

How do these strong correlations affect the form of the full probability distributions of various transport properties, including their large deviations from the mean? This question is interesting not only because recent experimental advances may make it possible to measure such distribution functions in some cases [12,13], but also because similar questions about the behavior of correlated random variables have recently surfaced in many disparate fields with a large number of applications [14]. In spite of this broad interest, there are few results available along these lines—notable among these are the recent calculations for the full distribution of the conductance and shot noise of mesoscopic structures in their normal metallic state [15–18], and for chaotic structures with one or two superconducting outgoing channels [19].

In this Letter, we have obtained the full distribution of the conductance $G_{\rm NS}$ of a time-reversal symmetric normal metal-superconductor (NS) junction in the limit of large N_c. Transport across an NS junction is particularly interesting because an electron incident from the normal side can be reflected as a hole, with the injection of a Cooper pair into the superconducting condensate [20]. Incorporating the effects of such processes in the presence of TR symmetry allows one to write the conductance $G_{\rm NS}$ (measured in units of G_0) of such junctions as $G_{\rm NS} = 2\sum_{n=1}^{N_c}(\frac{T_n}{2-T_n})^2$, where $\{T_n\}$ are the transmission eigenvalues of the same junction in its putative normal state [21]. The conductance $G_{\rm NS}$ thus ranges from 0 to $2N_c$, and its average $\langle G_{\rm NS}\rangle = (2-\sqrt{2})N_c$, and variance var$(G_{\rm NS}) = 9/16 \simeq 0.563$ are well-known in this TR symmetric case [6] (see also [22]).

Here we show that $\mathcal{P}(G_{\rm NS}, N_c)$ for large N_c has the scaling form [23]:

$$\mathcal{P}(G_{\rm NS}, N_c) \approx \exp[-N_c^2 \mathcal{R}(g_{\rm NS})], \tag{1}$$

where the large-deviation function $\mathcal{R}(g_{\rm NS})$ is plotted in Fig. 1 and $g_{\rm NS} \in [0, 2]$ is the dimensionless conductance per channel, $g_{\rm NS} = G_{\rm NS}/N_c$. A striking consequence of our exact computation of $\mathcal{R}$ is the prediction of a marked asymmetry in the large-deviation asymptotics near $G_{\rm NS} \to 0$ where $\mathcal{P}(G_{\rm NS}, N_c) \sim g_{\rm NS}^{N_c^2/4}$ and near $G_{\rm NS} \to 2N_c$ where $\mathcal{P}(G_{\rm NS}, N_c) \sim (2-g_{\rm NS})^{N_c^2/2}$.

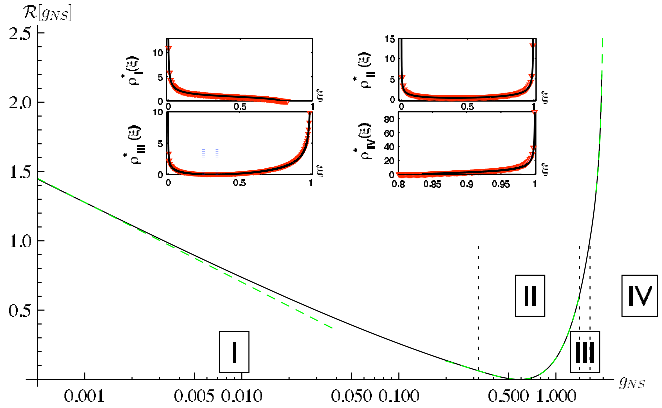

FIG. 1 (color online). The rate function $\mathcal{R}(g_{\rm NS})$ obtained from our large N_c solution is shown along with an inset that displays the form of the equilibrium density of the Coulomb gas in each regime (analytical formulae in solid black lines and Monte Carlo simulations in red triangles, see [24] for details). The green dashed lines are fit to the asymptotic forms for the left and the right tails and the central Gaussian region mentioned in the text. The vertical black dashed lines correspond to the critical points g_1, g_2, and g_3.

Another interesting feature is that the rate function $\mathcal{R}(g_{\rm NS})$ is piecewise smooth over a domain consisting of four distinct regions $g_{\rm NS} \in \bigcup_{j=0}^{3}[g_j, g_{j+1}]$ glued together via weak nonanalytic points: apart from the asymmetric large-deviation tails displayed above for $g_{\rm NS}$ near 0 and 2, respectively, and the universal Gaussian behavior

$$\mathcal{P}(G_{\rm NS}, N_c) \sim \exp[-(G_{\rm NS} - \langle G_{\rm NS}\rangle)^2/2\sigma^2], \qquad (2)$$

with dimensionless variance $\sigma^2 = 9/16$ around the mean $\langle G_{\rm NS}\rangle = (2-\sqrt{2})N_c$, there is a fourth tiny region that separates the Gaussian central region from the large-deviation tail near $G_{\rm NS} = 2N_c$. As we shall demonstrate below, this is a direct consequence of three phase transitions in an associated Coulomb gas problem.

The Coulomb gas problem.—The N_c transmission eigenvalues $T_n \in [0,1]$ are distributed according to the Jacobi orthogonal random matrix ensemble [7]:

$$\mathcal{P}_{\mathbf{T}}(\{T_n\}) = A_{N_c}\prod_{n<m}|T_n - T_m|\prod_n T_n^{-1/2}, \qquad (3)$$

with A_{N_c} ensuring normalization. The probability distribution of $G_{\rm NS}$ is given by

$$\mathcal{P}(G_{\rm NS}, N_c) = \int_{[0,1]^{N_c}}\prod_i dT_i\,\delta\Big(g_{\rm NS}N_c - 2\sum_{n=1}^{N_c}\frac{T_n^2}{(2-T_n)^2}\Big) \times \mathcal{P}_{\mathbf{T}}(\{T_n\}). \qquad (4)$$

Changing variables $\xi_n = T_n/(2-T_n)$ and exponentiating the δ function [24] leads to

$$\mathcal{P}(G_{\rm NS}, N_c) = \frac{N_c}{2}\int\frac{d\kappa}{2\pi}\int_{[0,1]^{N_c}}\prod_i d\xi_i \times e^{iN_c^2\kappa[(1/N_c)\sum_{n=1}^{N_c}\xi_n^2-(g_{\rm NS}/2)]}\mathcal{P}_{\xi}(\{\xi_n\}), \qquad (5)$$

where

$$\mathcal{P}_{\xi}(\{\xi_n\}) = \tilde{A}_{N_c}\prod_{n<m}|\xi_n - \xi_m|\prod_n\frac{\xi_n^{-1/2}}{(1+\xi_n)^{N_c+1/2}} \qquad (6)$$

and κ is constrained by the saddle-point condition to be purely imaginary [24]. While the N_c-fold $\{\xi\}$ integral (5) can be computed for any finite N_c in terms of Pfaffians [25], for large enough N_c one can map (5) to a continuum Coulomb gas problem. To make this connection, we represent a particular realization of ξ in terms of a continuum density function $\rho(\xi) = \frac{1}{N_c}\sum_{n=1}^{N_c}\delta(\xi-\xi_n)$ obeying the normalization condition $\int_0^1 d\xi\rho(\xi) = 1$. Originally introduced by Dyson [26], this procedure has recently been successfully used in a number of different contexts [27–29].

We may now write the probability distribution $\mathcal{P}(G_{\rm NS}, N_c)$ in this large N_c limit as a functional integral over the normalized density field ρ, supplemented by two additional integrals enforcing two constraints

$$\mathcal{P}(G_{\rm NS}, N_c) = \mathcal{A}_{N_c}\int dC_0\int dC_1\int\mathcal{D}\rho\exp(-N_c^2\mathcal{S}[\rho]), \qquad (7)$$

where the action $\mathcal{S}$ is given by

$$\mathcal{S}[\rho] = C_1\Big(\int d\xi\xi^2\rho(\xi) - \frac{g_{\rm NS}}{2}\Big) + C_0\Big(\int d\xi\rho(\xi) - 1\Big) + \int d\xi\rho(\xi)\ln(1+\xi) - \frac{1}{2}\iint d\xi d\xi'\rho(\xi)\rho(\xi') \times \ln|\xi-\xi'|. \qquad (8)$$

Here, $\mathcal{A}_{N_c} \sim \exp(N_c^2\Omega_0)$, with $\Omega_0 = (3/2)\ln 2$ is the overall normalization factor in this large N_c limit. The two variables C_0 and $C_1 = -i\kappa$ represent the integral representations of the two delta functions enforcing, respectively, the normalization condition $\int_0^1 d\xi\rho(\xi) = 1$ and $\int d\xi\xi^2\rho(\xi) = \frac{g_{\rm NS}}{2}$. We have also dropped contributions to the action $\mathcal{S}$ that are subdominant in the large N_c limit. For notational convenience, we have also suppressed the C_0 and C_1 dependence of the action $S[\rho]$.

Clearly, (7) can be viewed as the partition function of a 2D gas of particles confined on the segment [0,1], subject to an all-to-all Coulomb repulsion and sitting in an external potential $V(\xi) = \ln(1+\xi) + C_1\xi^2 + C_0$ at inverse temperature N_c^2. In this large N_c limit, equilibrium properties of this Coulomb gas are clearly determined by the saddle point of the functional integral (7), that corresponds to the minimum energy configuration of the fluid. We have three saddle point equations. Varying $\mathcal{S}[\rho]$ over C_0 and C_1 just gives the two constraints mentioned above. The third equation $\frac{\delta\mathcal{S}[\rho]}{\delta\rho} = 0$, gives the minimum energy density configuration $\rho^\star$ which satisfies the integral equation

$$\ln(1+\xi) + C_0 + C_1\xi^2 = \int \rho^\star(\xi')\ln|\xi - \xi'|d\xi' \quad (9)$$

for all ξ in the support of $\rho^\star$. Differentiating (9) with respect to ξ we get

$$2C_1\xi + \frac{1}{1+\xi} = \mathrm{Pr}\int \frac{\rho^\star(\xi')}{\xi - \xi'}d\xi' \quad (10)$$

for all ξ in the support of $\rho^\star$, where Pr stands for Cauchy's principal part.

Finding the solution $\rho^\star(\xi)$ of (10) with the constraints $\int_0^1 d\xi\rho^\star(\xi) = 1$ and $\int_0^1 d\xi\xi^2\rho^\star(\xi) = g_{\mathrm{NS}}/2$ is the main technical challenge. The saddle-point density $\rho^\star(\xi)$ obtained in this manner then depends parametrically only on $g_{\mathrm{NS}} \in [0, 2]$, and the required result for the probability distribution in the large N_c limit is finally given in terms of the action $\mathcal{S}$ evaluated on $\rho^\star$,

$$\mathcal{P}(G_{\mathrm{NS}}, N_c) \approx \exp\left[-N_c^2\underbrace{(\mathcal{S}[\rho^\star] - \Omega_0)}_{\mathcal{R}(g_{\mathrm{NS}})}\right]. \quad (11)$$

Solution of (10) and phase transitions for $\rho^\star$.—Singular integral equations of the type (10) can be solved in closed form using either Tricomi's theorem [30] when $\rho^\star$ has support on a single interval $[L_1, L_2]$, or a more general scalar Riemann-Hilbert method [18,31] if this assumption is not valid.

We find that [25]

$$\rho^\star(\xi) = \begin{cases} \rho_I^\star(\xi) & \text{for } g_0 = 0 \le g_{\mathrm{NS}} \le g_1, \\ \rho_{II}^\star(\xi) & \text{for } g_1 \le g_{\mathrm{NS}} \le g_2, \\ \rho_{III}^\star(\xi) & \text{for } g_2 \le g_{\mathrm{NS}} \le g_3, \\ \rho_{IV}^\star(\xi) & \text{for } g_3 \le g_{\mathrm{NS}} \le g_4 = 2, \end{cases}$$

[see (13), (14), (15), (16), respectively] where $g_1 \equiv 2 - 19/8\sqrt{2} = 0.320621\ldots$, $g_2 \equiv (968 - 499\sqrt{2} + 102\sqrt{17})/484 = 1.41088\ldots$, and $g_3 \equiv 2 - (9 - \sqrt{21})/\sqrt{15(6+\sqrt{21})} = 1.64939\ldots$. The emerging physical picture is as follows. Since $2\int d\xi\xi^2\rho^\star(\xi) = g_{\mathrm{NS}}$, small values of g_{NS} are expected to correspond to a large value of C_1 (the strength of the quadratic part of the confining potential $V(\xi)$) and a resulting $\rho^\star(\xi)$ that is concentrated near the left edge $\xi = 0$. Making the ansatz that the density has support on the interval $[0, L_1]$ we determine it by using Tricomi's formula:

$$\rho_I^\star(\xi) = \frac{\left[\frac{\sqrt{L_1+1}}{\xi+1} + \frac{C_1}{4}(L_1^2 + 4L_1\xi - 8\xi^2) + a_I\right]}{\pi\sqrt{\xi(L_1 - \xi)}}, \quad (12)$$

where a_I is a constant of integration. We now fix C_1, L_1, and a_I by requiring that $\rho_I^\star(\xi = L_1) = 0$, it is normalized to 1, and has a second moment equal to $g_{\mathrm{NS}}/2$. We obtain

$$\rho_I^\star(\xi) = \frac{\sqrt{L_1 - \xi}}{\pi\sqrt{\xi}}\left(\frac{1}{(\xi+1)\sqrt{L_1+1}} + C_1(L_1 + 2\xi)\right), \quad (13)$$

where $C_1 = \frac{4}{3L_1^2\sqrt{L_1+1}}$ and $1 + \frac{5L_1^2 - 8L_1 - 16}{16\sqrt{L_1+1}} = g_{\mathrm{NS}}/2$.

For $g_{\mathrm{NS}} > g_1$, L_1 becomes greater than 1, invalidating the solution. This corresponds to a phase transition in the Coulomb gas: the external potential becomes weak enough that the density is spread out over the entire available space to minimize the effects of the interparticle repulsion. In this extended phase, $\rho^\star$ has support over the entire interval $\xi \in [0, 1]$ and is obtained by simply setting $L_1 = 1$ in (12). Fixing the integration constant and C_1, we obtain

$$\rho_{II}^\star(\xi) = \frac{1}{\pi\sqrt{\xi(1-\xi)}}\left(\frac{\sqrt{2}}{\xi+1} + \frac{C_1}{4}(1 + 4\xi - 8\xi^2)\right), \quad (14)$$

where now $C_1 = \frac{32}{9}(2 - \sqrt{2} - g_{\mathrm{NS}})$. For $g_{\mathrm{NS}} > g_2$, $\rho_{II}^\star$ goes negative in the middle of its support, thereby invalidating this solution.

For $g_2 < g_{\mathrm{NS}} < g_3$, we find that no single-support solution is able to satisfy all the constraints on the equilibrium density. In this narrow region, the external potential pushes the Coulomb fluid to the right edge $\xi = 1$ (C_1 is negative for these values of g_{NS}) but cannot fully overcome the effects of the interparticle Coulomb repulsion. As a result the Coulomb gas breaks up in this novel intermediate phase into two spatially disjoint fluids separated by an empty region in the middle. More precisely, we find using a more general Riemann-Hilbert ansatz [25] that the solution in the regime $g_2 < g_{\mathrm{NS}} < g_3$ has two supports, the first on the interval $[0, L_2]$, and the second on the interval $[L_3, 1]$, with $L_3 > L_2$, with the equilibrium density in these two intervals being given by the formula

$$\rho_{III}^\star(\xi) = \frac{-2C_1\sqrt{(\xi - L_2)(\xi - L_3)^3}(\xi + \frac{(L_2 + 3L_3 + 1)}{2})}{\pi\sqrt{\xi(1-\xi)}(1+\xi)}, \quad (15)$$

with L_3 related to L_2 via the constraint $5 - 2L_2 - 6L_3 - 3L_2^2 - 6L_2L_3 - 15L_3^2 = 0$, and L_2 and C_1 being fixed by normalization and second moment equal to $g_{\mathrm{NS}}/2$ (see [24] for details).

Finally, as $g_{\mathrm{NS}} \to g_3$, $L_2 \to 0$, and C_1 is now large enough in magnitude and negative in sign, giving way to a conventional single-support solution on $[L_4, 1]$ when $g_{\mathrm{NS}} > g_3$. In this case, Tricomi's formula along with normalization condition yields

$$\rho_{IV}^\star(\xi) = \frac{\sqrt{2}}{\pi}\frac{\sqrt{\xi - L_4}}{\sqrt{1 + L_4}}\frac{1}{\sqrt{1-\xi}} \times\left(\frac{4(2\xi + L_4 - 1)}{(1 - L_4)(1 + 3L_4)} - \frac{1}{1+\xi}\right), \quad (16)$$

where L_4 is determined by

$$\frac{\sqrt{2}(1-L_4)(1-18L_4-15L_4^2)}{16\sqrt{1+L_4}(1+3L_4)} = \frac{g_{NS}}{2} - 1. \quad (17)$$

Inserting the analytical expressions of the densities in the four phases into the action (8), the rate function $\mathcal{R}(g_{NS})$ can now be evaluated in terms of elementary integrals [25]. This is shown in Fig. 1, where we display $\mathcal{R}(g_{NS})$, along with an inset showing the analytically calculated curves and Monte Carlo data for the typical form of the equilibrium density in each of the four phases. Finally, a straightforward asymptotic expansion of these results allows us to obtain closed form expressions for the power-law asymptotics of $\mathcal{R}(g_{NS})$ as detailed in the introduction.

Summary.—In summary, the Coulomb gas formulation of the problem of Andreev conductance distribution reveals a rich thermodynamic behavior which can be addressed analytically. Four zero-temperature phases in the associated Coulomb fluid, dictated by the precise value of $g_{NS} \in [0, 2]$ correspond to as many regions in the rate function domain within which $\mathcal{R}(g_{NS})$ is smooth. The central Gaussian region is flanked by long-power-law tails with a novel intermediate regime corresponding to a disconnected support in the Coulomb fluid density. Our result for the full probability distribution of the Andreev conductance, besides solving a challenging problem, has clear physical and experimental significance. Such rate functions in related Coulomb gas systems have been recently measured experimentally [13]. A direct experimental confirmation of our predictions in the Andreev case may be within reach with existing device setups. Extensions to the case of broken TR appear very challenging and are left as an open question.

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