Nonlocal Resistivity in the Vortex Liquid Regime of Type-II Superconductors

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We develop a phenomenological hydrodynamic description of the transport properties of a viscous vortex liquid in realistic geometries with nonuniform currents, including the effects of the boundaries of the sample. This approach should be useful for modeling multiterminal transport measurements in the vortex liquid regime of, for example, YBCO.

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One of the interesting thermal fluctuation effects that has been observed in the cuprate superconductors is the existence of the vortex liquid regime over a substantial range of temperature and magnetic field [1]. The vortex liquid regime is where *local* superconductivity and vortices occur, since the system is well below its mean-field transition temperature, but the thermal fluctuations are so strong that the vortices form a disordered and mobile liquid instead of the Abrikosov vortex lattice or pinned, immobile vortex glass that they freeze into at lower temperature [1]. For a bulk, three-dimensional type-II superconductor, the vortices may form vortex lines that extend over large distances as well-connected extended objects. As has been stressed by Marchetti and Nelson [2], such a liquid of vortex *lines* is rather analogous to a liquid of polymers and may have a large viscosity. Here we wish to explore the consequences of the viscosity of the vortex liquid for the transport properties within a simple hydrodynamic model. This work is motivated by recent and ongoing multiterminal transport measurements in the vortex liquid regime of the cuprate superconductors [3–5].

The essence of why the vortex liquid has nonlocal resistivity is the following: A current density $\mathbf{j}(\mathbf{r})$ flowing at position $\mathbf{r}$ pushes on the vortices at $\mathbf{r}$ with the Magnus (or Lorentz) force. The vortices at $\mathbf{r}$ move in response to this force, but they are connected to (and perhaps entangled with [2]) vortices extending far away from $\mathbf{r}$. Therefore the force applied locally by the current at $\mathbf{r}$ may induce vortex motion, and thus electric fields [6], far away from $\mathbf{r}$.

Here we examine a simple hydrodynamic description of steady-state (dc) transport in the vortex liquid. We work in the Ohmic regime only, where the behavior is linear in the current and the voltage. We look at length scales well in excess of any microscopic scales such as the microscopic magnetic penetration length and the typical spacing between vortices. The vortex liquid is treated as incompressible, which is a reasonable approximation for strongly type-II (large κ) superconductors in magnetic fields well above the lower critical field: $H \gg H_{c1}$. The description is solely in terms of the current density $\mathbf{j}(\mathbf{r})$ and the voltage $V(\mathbf{r})$. The steady-state voltage difference between two points, $V(\mathbf{r})-V(\mathbf{r}')$, is, by Josephson's relation, proportional to the rate of phase slip between the points, which in turn is proportional to the rate at which vortex lines cross the line connecting the two points. Thus the electric field, $\mathbf{E}(\mathbf{r})=\nabla V(\mathbf{r})$, is a measure of the average local velocity of the vortices [6]. A more detailed description might consider the distribution of local vortex orientations in the liquid at $\mathbf{r}$ and the vortex velocity as a function of this orientation; however, we will study the simpler description where one just keeps track of the local electric field.

The force on the vortices at position $\mathbf{r}$ is a sum of (i) the current-induced forces, such as the Magnus force, which are proportional to the local current density; (ii) viscous drag forces, which are proportional to second spatial derivatives of the local vortex velocity; and (iii) frictional drag forces [6] (e.g., Bardeen-Stephen drag) which are proportional to the local vortex velocity and thus $\mathbf{E}$. In the steady state these forces must add to zero. The proportionality coefficients are all tensors; following Marchetti and Nelson [2], we may write this zero total force condition as

$$j_\alpha(\mathbf{r})+\eta_{\alpha\beta\gamma\delta}\partial_\beta\partial_\gamma E_\delta(\mathbf{r})=\sigma_{\alpha\beta}E_\beta(\mathbf{r}) , \tag{1}$$

where the subscripts take on the values x, y, and z, denoting the components of the vectors or tensors, and repeated indices are summed over. Under conditions of uniform $\mathbf{j}$ and $\mathbf{E}$, $\sigma_{\alpha\beta}$ is the usual conductivity tensor. Note we have assumed the material is uniform, so the tensors σ and η do not depend on $\mathbf{r}$. We also assume that higher-order gradients than those in (1) do not play an important role. The viscosity of the vortex liquid is encoded in the tensor η. This tensor is defined so that the viscous drag force is equal to the force that would be produced by an additional current density of $\hat{\mathbf{e}}_\alpha\eta_{\alpha\beta\gamma\delta}\partial_\beta\partial_\gamma E_\delta$, where $\hat{\mathbf{e}}_\alpha$ is the unit vector in the α direction. We will treat the problem for general σ and η first, and look at a specific simple example later.

For an infinite material (1) may be simply solved in momentum (i.e., Fourier) space, yielding a momentum-dependent (thus nonlocal) conductivity:

$$\hat{\sigma}^{(nl)}_{\alpha\beta}(\mathbf{k})=\sigma_{\alpha\beta}+\eta_{\alpha\gamma\delta\beta}k_\gamma k_\delta . \tag{2}$$

One may also define the momentum-dependent nonlocal *resistivity* tensor $\hat{\rho}^{(nl)}(\mathbf{k})$, which is the inverse of $\hat{\sigma}^{(nl)}(\mathbf{k})$.

In real space we then have

$$E_\alpha(\mathbf{r})=\int d\mathbf{r}'\rho^{(nl)}_{\alpha\beta}(\mathbf{r}-\mathbf{r}')j_\beta(\mathbf{r}')\,, \tag{3}$$

where the tensor nonlocal resistivity kernel, $\rho^{(nl)}(\mathbf{r})$, is the Fourier transform of $\hat{\rho}^{(nl)}(\mathbf{k})$.

How does the resistivity kernel fall off with distance? The nonlocal conductivity, $\hat{\sigma}^{(nl)}(\mathbf{k})$, is nonzero and analytic at small $\mathbf{k}$, so $\hat{\rho}^{(nl)}(\mathbf{k})$ is also. The nearest zero of $\hat{\sigma}^{(nl)}(\mathbf{k})$ to $\mathbf{k}=0$ is at a complex momentum of order $\sqrt{\sigma/\eta}$. This implies that $\rho^{(nl)}(\mathbf{r})$ falls off exponentially for large r, with the decay lengths for its various components being of order $\sqrt{\eta/\sigma}$. These decay lengths are the viscous penetration lengths for the vortex liquid. A particular case of these viscous screening lengths was introduced by Marchetti and Nelson [2] in analyzing the effects of strong pins such as twin boundaries on the flow of the vortex liquid. They focus on the "shear" component of the viscosity which arises due to vortex-vortex interactions or vortex entanglement [7].

A real piece of material is always finite, with boundaries, and it is at the boundaries (the contacts) that the current is injected and withdrawn from the material. In the bulk of the sample in steady state the current must be divergence-free, $\nabla\cdot\mathbf{j}=0$. This means that away from the boundaries of the material, the voltage obeys

$$(\sigma_{\alpha\beta}\partial_\alpha\partial_\beta-\eta_{\alpha\beta\gamma\delta}\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta)V(\mathbf{r})=0\,. \tag{4}$$

The bulk equations (1) and (4) must be supplemented with boundary conditions to obtain the steady-state current and voltage patterns for a given sample geometry.

We consider an ideal surface in the following sense: The vortices at the surface feel viscous drag due to their velocity relative to neighboring vortices inside the sample, but none from outside the sample, where there are no vortices. Thus the viscous drag these surface vortices feel is just as if the local vortex velocity, and thus $\mathbf{E}$, is unchanged as one moves from the surface outward. Thus $\mathbf{E}(\mathbf{r})$ may effectively have discontinuous first spatial derivatives at the surface and, by (1), the current may therefore contain a delta-function part flowing in the surface. Physically, this delta function of current density is presumably spread over some microscopic distance. If we consider a local coordinate system where the z axis is normal to the surface and pointed inwards and the surface is at $z=0$, the current in the sample near and at that point on the surface is

$$j_\alpha(\mathbf{r})=\sigma_{\alpha\beta}E_\beta(\mathbf{r})-\eta_{\alpha\beta\gamma\delta}\partial_\beta\partial_\gamma E_\delta(\mathbf{r}) -\delta(z)\eta_{\alpha zz\beta}(\partial_z E_\beta(\mathbf{r})|_{z\to 0^+})\,, \tag{5}$$

where the subscript z is *not* summed over.

One boundary condition is that the delta-function component of the current must be flowing parallel to (thus in) the surface. In terms of the voltage this condition is

$$\eta_{zzz\alpha}\partial_z\partial_\alpha V(\mathbf{r})|_{z\to 0^+}=0\,, \tag{6}$$

where, again, z is the coordinate normal to the surface. The other boundary condition is that the delta-function part of the divergence of the current at the surface is equal to the current injected at that point on the surface, $I(\mathbf{s})$, where $\mathbf{s}$ denotes a point on the surface. At the contacts $I(\mathbf{s})$ may be nonzero, but the integral of $I(\mathbf{s})$ over the entire surface must vanish. This boundary condition is

$$[\sigma_{z\alpha}\partial_\alpha V(\mathbf{r})-\eta_{z\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma V(\mathbf{r})-\eta_{\alpha' zz\beta}\partial_{\alpha'}\partial_z\partial_\beta V(\mathbf{r})]|_{z\to 0^+} =I(\mathbf{s})\,, \tag{7}$$

where the dummy index α' is summed only over the coordinates (x and y) within the local surface, while the other dummy indices are summed over x, y, and z.

There are a number of cases where one can solve (4), (6), and (7) to obtain the voltage and current patterns. Here we will present one simple example. Consider a layered superconductor with a uniform magnetic field oriented normal to the layers. Let us assume we are in a regime where the interactions between "pancake" vortices [8] just above one another in adjacent layers are the only important vortex-vortex interactions. Thus the component of the viscosity we will include is that involving vortices moving within the layers but with different velocities in adjacent layers. It may be called the "tilt viscosity." This component of the viscosity couples to derivatives normal to the layers of the components of the electric field oriented parallel to the layers. To simplify, let us neglect the Hall effect and assume the system and boundary conditions are translationally invariant along one direction parallel to the layers. Thus we have an effectively two-dimensional problem. We label the axis parallel to the layers as x, and the axis normal to the layers and parallel to the field as y. We thus consider the situation where the only elements of σ and η that are nonzero are σ_{xx}, σ_{yy}, and η_{xyyx}. Let us drop most of the subscripts and call these elements σ_x, σ_y, and η, respectively.

In this simple case we have

$$j_y=\sigma_y E_y\,, \tag{8}$$

and

$$j_x=\sigma_x E_x-\eta\frac{\partial^2 E_x}{\partial y^2}\,. \tag{9}$$

Only the x component of the resistivity is nonlocal, with

$$\hat{\rho}^{(nl)}_{xx}(\mathbf{k})=\frac{1}{\sigma_x+\eta k_y^2}\,. \tag{10}$$

Let us consider a simple rectangular sample oriented parallel to the x and y axes, with contacts only on the top and bottom of the sample (the x axis is horizontal), as studied in Refs. [3–5] and illustrated in Fig. 1. In this sample geometry, Eq. (4) for voltage reads

$$\sigma_x\frac{\partial^2 V}{\partial x^2}+\sigma_y\frac{\partial^2 V}{\partial y^2}-\eta\frac{\partial^4 V}{\partial x^2\partial y^2}=0\,, \tag{11}$$

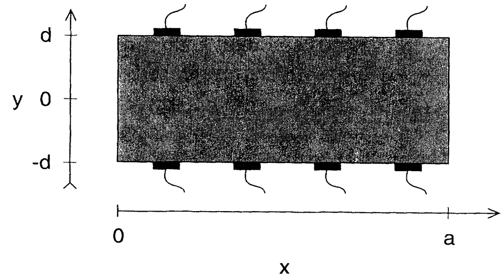

FIG. 1. The sample geometry considered in the text. The magnetic field is parallel to the y axis.

with the boundary conditions (7);

$$\sigma_x \frac{\partial V}{\partial x} - \eta \frac{\partial^3 V}{\partial y^2 \partial x}\bigg|_{x=0,a} = 0\,, \tag{12}$$

$$\sigma_y \frac{\partial V}{\partial y} - \eta \frac{\partial^3 V}{\partial x^2 \partial y}\bigg|_{y=\pm d} = \mp I(x, \pm d)\,, \tag{13}$$

where $I(x, \pm d)$ are the patterns of external current sources and sinks on the top $(y=d)$ and the bottom $(y=-d)$ of the sample. We may write the voltage as

$$V(x,y) = \sum_{n=0}^{\infty} U_n(y)\cos(n\pi x/a)\,, \tag{14}$$

where a is the sample length along the x direction. From (11), we then find for $n>0$

$$U_n(y) = C_n \cosh(K_n y) + S_n \sinh(K_n y) \tag{15}$$

with

$$K_n = \frac{n\pi}{a}\left(\frac{\sigma_x}{\sigma_y + \eta n^2\pi^2/a^2}\right)^{1/2} \tag{16}$$

and for $n=0$

$$U_0(y) = C_0 + S_0 y\,, \tag{17}$$

where C_0 sets an arbitrary zero for the voltage. We may write the current sources and sinks at the contacts as

$$I(x, \pm d) = \sum_{n=0}^{\infty} i_n^{\pm} \cos(n\pi x/a)\,, \tag{18}$$

where $i_0^+ = -i_0^-$, from the condition that the integral of the net current injected over the entire surface must vanish. Solving Eqs. (11)-(13), we get

$$S_0 = i_0^-/\sigma_y\,,$$

$$S_n = \frac{i_n^- - i_n^+}{2(n\pi/a)\cosh(K_n d)[\sigma_x(\sigma_y + \eta n^2\pi^2/a^2)]^{1/2}}\,, \quad n>0\,, \tag{19}$$

$$C_n = -\frac{i_n^- + i_n^+}{2(n\pi/a)\sinh(K_n d)[\sigma_x(\sigma_y + \eta n^2\pi^2/a^2)]^{1/2}}\,, \quad n>0\,.$$

This determines completely the voltage pattern $V(x,y)$ given the input and output current patterns in terms of $i_n^{\pm}$'s. The delta-function sheet of current flowing on the top and bottom of the sample is easily computed to be

$$I_\delta(x, \pm d) = \pm \eta \frac{\partial^2 V}{\partial x \partial y}\bigg|_{y=\pm d} = \eta \sum_{n=1}^{\infty} \frac{i_n^{\pm}(n\pi/a)\sin(n\pi x/a)}{\sigma_y + \eta n^2\pi^2/a^2}\,. \tag{20}$$

To see how the effect of viscosity shows up in transport experiments, we consider two simple experiments. First, consider the experiment done in Refs. [3] and [4] where the current is injected at the upper left contact and withdrawn at the upper right contact in a sample as in Fig. 1. Let $V_{\rm top}$ and $V_{\rm bottom}$ denote the voltage drops measured across the two central contacts on the top surface and the two opposite central contacts at the bottom surface, respectively. For a given σ_x, σ_y, and η, we first define the "apparent" conductivities, $\sigma_x^{(a)}$ and $\sigma_y^{(a)}$. These are the conductivities calculated from the measured $V_{\rm top}$ and $V_{\rm bottom}$ using Eqs. (15)-(19), assuming $\eta=0$, i.e., assuming that the material is simply an anisotropic *local* conductor. Of course, in the normal state, where $\eta=0$, the apparent conductivities are the same as the true conductivities σ_x and σ_y. However, they start to differ in the liquid regime where η is nonzero. Let us then ask, for the first experiment, how the apparent conductivity ratio $\sigma_y^{(a)}/\sigma_x^{(a)}$ behaves as a function of η. It can be demonstrated from Eqs. (15)-(19) that the ratio $\sigma_y^{(a)}/\sigma_x^{(a)}$ is a monotonically decreasing function of the voltage ratio $V_{\rm top}/V_{\rm bottom}$. Also, for a given pair of real σ_x and σ_y, the voltage ratio $V_{\rm top}/V_{\rm bottom}$ is a decreasing function of η in the large η regime, where the ratio of the viscous length $\sqrt{\eta/\sigma_x}$ to the sample thickness exceeds a geometry-dependent number. These two facts together indicate that for this first experiment $\sigma_y^{(a)}/\sigma_x^{(a)}$ is an *increasing* function of η at fixed σ_x and σ_y in this large η regime. This can be most easily understood in the large viscosity *limit* $(\eta \to \infty)$. For the experiment we are now considering, the current flowing in the x direction pushes the vortices in the direction perpendicular to the x-y plane. In the large viscosity limit, the vortex velocity is the same at the top and the bottom of the sample, so $V_{\rm top} = V_{\rm bottom}$. For a local conductor, identical voltage drops at the top and bottom $(V_{\rm top} = V_{\rm bottom})$ occur in this experiment only when $\sigma_y^{(a)}/\sigma_x^{(a)}$ diverges. Thus when η diverges, here it causes the "apparent" conductivity along the y direction, $\sigma_y^{(a)}$, to diverge. The vortices do move in response to the current so $\sigma_x^{(a)}$ remains finite. In this $\eta \to \infty$ limit the current actually flows only in the top surface of the sample.

Let us also consider a second experiment, where we instead withdraw the current from the lower left contact (still injecting it at the upper left contact). Now we measure $V_{\rm left}$ and $V_{\rm right}$, which are the voltage differences between opposite contacts on the top and bottom surfaces at the two central contact positions. In this experiment the apparent conductivity ratio $\sigma_y^{(a)}/\sigma_x^{(a)}$ is a monotonically *increasing* function of $V_{\rm left}/V_{\rm right}$. For fixed real σ_x and σ_y, the measured voltage ratio $V_{\rm left}/V_{\rm right}$, however, remains a monotonically decreasing function of the viscosity η. Thus, in this second experiment, an increase in the viscosity leads to a *decrease* in the apparent conductivity ratio $\sigma_y^{(a)}/\sigma_x^{(a)}$; in the large η regime this is precisely opposite to the first experiment. In the large viscosity limit ($\eta \rightarrow \infty$) for this second experiment, the currents flowing in the top and bottom surfaces are equal and opposite, and their contribution to the x component of the electric field cancels, leading to no voltage drop along the x direction, but the voltage shows a drop in the y direction due to the uniform current flowing across the sample from top to bottom. So, in this second experiment, the apparent conductivity remains finite along the y direction but diverges along the x direction as the viscosity diverges.

Thus the effect of nonlocal resistivity is to give an increase in the apparent $\sigma_y^{(a)}/\sigma_x^{(a)}$ in the first experiment where the net current flows in the x direction, while it gives a decrease in the apparent $\sigma_y^{(a)}/\sigma_x^{(a)}$ in the second experiment when the net current flows in the y direction. Note that in both experiments the increased viscosity causes the strongest enhancement in the component of the apparent conductivity in the direction perpendicular to the net current flow. Recent unpublished measurements [5] on YBCO crystals in the vortex liquid regime do observe such behavior as the temperature is reduced. The observed behavior [5] is quite inconsistent with that of an anisotropic local conductor ($\eta = 0$).

In summary, we have explored, within a simple hydrodynamic model, how viscosity of the vortex liquid affects the transport properties of type-II superconductors in the vortex liquid regime. This effect of viscosity is reflected in a macroscopic way as a nonlocal resistivity and should be detectable in the voltage patterns seen in multiterminal transport experiments. Explicit calculation of the current and voltage pattern is sample specific and in general hard as it requires solving a fourth-order partial differential equation with complicated boundary conditions. However, it can be worked out explicitly in a few special cases and, in this paper, we have presented a detailed calculation and some discussion of one such specific case.

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[4] R. Busch *et al.*, Phys. Rev. Lett. **69**, 522 (1992).

[5] H. Safar *et al.* (unpublished).

[6] See, e.g., Y. B. Kim and M. J. Stephen, in *Superconductivity*, edited by R. D. Parks (Dekker, New York, 1969), Vol. 2.

[7] We note that the flow of a vortex liquid with a "shear" viscosity between two large twin boundaries as studied by Marchetti and Nelson (Ref. [2]) is different from our case. They consider a geometry where the liquid flows between two infinite twin boundaries situated at $x = \pm W/2$, with the magnetic field $\mathbf{H} \parallel \hat{\mathbf{z}}$ and a uniform current $\mathbf{j} = j\hat{\mathbf{x}}$ along the x direction. The vortices then tend to move in the $\hat{\mathbf{y}}$ direction creating an electric field $\mathbf{E} = E_x\hat{\mathbf{x}}$. However, the vortices are pinned at the twin boundaries, yielding the boundary condition $E_x = 0$ at $x = \pm W/2$. Thus, their equations read, in our notation, $\sigma_{xx}E_x - \eta_{xxxx}(\partial^2 E_x/\partial x^2) = j$ where j is a constant and with boundary conditions $E_x = 0$ at $x = \pm W/2$. This, then, can be simply solved to give the electric field profile, $E_x(x)$. Their boundary condition is different from (and much simpler than) ours, and they analyze a situation that is nonuniform in only one direction.

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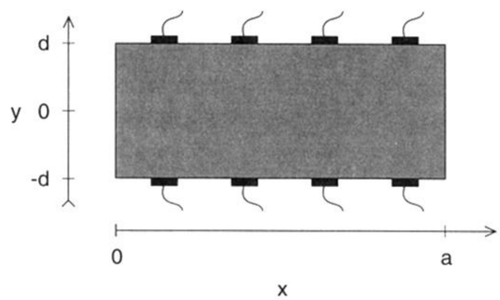

FIG. 1. The sample geometry considered in the text. The magnetic field is parallel to the y axis.